Channel Gain Power Distribution using AR Channel Model Derivation

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I. Objective

We aim to derive the channel gain power distribution in two main cases, case I when the channel gain power information at $t-1$ is available and case II when the overheard CQI is the available information instead of the channel gain.

In order to make the derivation easier to follow, we will divide it into two stages: Stage 1 starts with the channel gain distribution and the CQI information at time $t-1$ and derive the distribution of the channel gain power at $t-1$ when given the CQI information, while in Stage 2, we use the channel evolution over time described by the AR channel model and the channel gain power distribution at $t-1$ (Stage 1 result) to develop the channel gain power distribution at time t in case I and II.

II. Derivation Body

A. Stage 1

We assume that the channel is a Rayleigh fading channel and consequently the channel gain power is exponentially distributed (a priori information) with mean $2\lambda^2$

$$f_G(g) = \frac{1}{2\lambda^2} e^{-\frac{g}{2\lambda^2}}. \tag{1}$$

At $t-1$, combining the a priori information and the overheard CQI, we can evaluate the SINR distribution given the CQI using Bayes rule

$$\mathbf{Pr}(\gamma | CQI = q) = \frac{\mathbf{Pr}(\gamma)\mathbf{Pr}(CQI|\gamma)}{\pi_q}, \tag{2}$$

where π_q is the probability that the $CQI = q$ and since the CQI is considered to be the quantized version of the SINR (γ), therefore at given γ

$$\mathbf{Pr}(CQI|\gamma) = \begin{cases} 1 & , CQI = Q(\gamma) \\ 0 & , \text{Otherwise} \end{cases}, \tag{3}$$

where $Q()$ is the quantization function (lookup table) which specifies the SINR quantization intervals

$$\mathbf{Pr}(\gamma|CQI) = \begin{cases} \frac{f_\Gamma(\gamma)}{\pi_q} & , \gamma \in [a_q \ b_q] \\ 0 & , \text{Otherwise} \end{cases}, \tag{4}$$

where a_q and b_q are the boundaries defined by the quantization value. π_q is computed using the SINR a priori probability such that

$$\pi_q = \int_{a_q}^{b_q} f_\Gamma(\gamma) d\gamma, \tag{5}$$

where $f_\Gamma(\gamma)$ is the SINR PDF, and since the SINR at $t-1$ is defined as

$$\gamma(t-1) = \frac{P_M(t-1)}{P_F(t-1)G_F(t-1) + N_0} G(t-1), \tag{6}$$

$$\gamma(t-1) = c_{t-1} G(t-1), \tag{7}$$

from which we can calculate the PDF of $\gamma(t-1)$ based on the channel distribution

$$f_\Gamma(\gamma) = \frac{1}{c_{t-1}} f_G(\frac{\gamma}{c_{t-1}}), \tag{8}$$

then by substituting in (5)

$$\pi_q = e^{-\frac{a_q}{2\lambda^2 c_{t-1}}} - e^{-\frac{b_q}{2\lambda^2 c_{t-1}}}. \tag{9}$$

Define k such that

$$k = \left(e^{-\frac{a_q}{2\lambda^2 c_{t-1}}} - e^{-\frac{b_q}{2\lambda^2 c_{t-1}}}\right)^{-1}, \tag{10}$$

then the PDF of the SINR given the CQI will be

$$\mathbf{Pr}(\gamma|CQI) = \begin{cases} \frac{k}{c_{t-1}} f_G(\frac{\gamma}{c_{t-1}}) & , \gamma \in [a_q \ b_q] \\ 0 & , \text{Otherwise} \end{cases}, \tag{11}$$

from which we can derive the channel gain PDF given the CQI to be

$$f_{G_{t-1}|CQI}(\gamma|q) = c_{t-1} f_{\Gamma|CQI}(c_{t-1}\gamma|q) \tag{12}$$

and finally the channel gain PDF at $t-1$ given the overheard CQI can be expressed as

$$f_{G_{t-1}|CQI}(\gamma|q) = \begin{cases} \frac{k}{2\lambda^2} e^{\frac{\gamma}{2\lambda^2}} & , \gamma \in [\frac{a_q}{c_{t-1}} \ \frac{b_q}{c_{t-1}}] \\ 0 & , \text{Otherwise} \end{cases}, \tag{13}$$

where $c_{t-1} = P_M(t-1)[P_F(t-1)G_F(t-1) + N_0]^{-1}$, k is the constant defined in equation (10). The result shown in equation (13) is the objective of Stage 1 in our derivation, and we are ready to move to the second stage.

B. Stage 2

We will start by assuming that the channel gain time evolution follow the 1^{st} order autoregressive (AR) channel model such that

$$h_t = \mu h_{t-1} + w_t, \tag{14}$$

where w_t is a complex Gaussian distributed random variable with zero mean, $\sigma_w^2/2$ variance for the imaginary and real parts and independent on h_{t-1}, h_{t-1} is the channel gain at time $t-1$ and represented by a complex Gaussian distributed random variable with zero mean, λ^2 variance for the imaginary and real parts and μ is the correlation coefficient.

From equation (14), the channel gain power evolution over time can be described as follows

$$G_t = \eta G_{t-1} + v_t, \tag{15}$$

where $\eta = \mu^2$, and $G_t = |h_t|^2$, while

$$v_t = |w_t|^2 + 2\mathbf{Re}(\mu h_{t-1} w_t^*), \tag{16}$$

for $\lambda^2 \ggg \sigma_w^2$

$$v_t \approx 2\mathbf{Re}(\mu h_{t-1} w_t^*), \tag{17}$$

where w_t^* is the conjugate of w_t. Now, we need to derive the distribution of v_t given the channel gain power (G_{t-1})

$$v_t = 2\mu\big(w_t^{re}|h_{t-1}|\cos\theta + w_t^{imag}|h_{t-1}|\sin\theta\big), \tag{18}$$

where w_t^{re}, w_t^{imag} are the real and imaginary parts of w_t, both are normally distributed with zero mean and variance $\sigma_w^2/2$ and θ is the phase of h_{t-1},

$$v_t = 2\sqrt{\eta G_{t-1}}\big(w_t^{re}\cos\theta + w_t^{imag}\sin\theta\big), \tag{19}$$

in order to evaluate $\mathbf{Pr}(v_t|G_{t-1})$, we will evaluate the joint conditional distribution $\mathbf{Pr}(v_t,\theta|G_{t-1})$ and then marginalize over θ

$$\mathbf{Pr}(v_t,\theta|G_{t-1}) = \mathbf{Pr}(v_t|G_{t-1},\theta)\mathbf{Pr}(\theta|G_{t-1}), \tag{20}$$

assuming that θ is uniformly distributed between $[0\ 2\pi]$ and independent on G_{t-1} then

$$\mathbf{Pr}(v_t,\theta|G_{t-1}) = \frac{1}{2\pi}\mathbf{Pr}(v_t|G_{t-1},\theta), \tag{21}$$

and since w_t^{re}, w_t^{imag} are normally distributed with zero mean and variance $\sigma_w^2/2$, then $\mathbf{Pr}(v_t|G_{t-1},\theta)$ is gaussian distributed with zero mean and $2\eta G_{t-1}\sigma_w^2$ as a variance, and this is directly concluded from the fact that the sum of two normally distributed RVs is normally distributed with the sum of the means as a mean and the sum of the variances as the variance.

In order to get $\mathbf{Pr}(v_t|G_{t-1})$, we will need to marginalize with respect to θ which will result in

$$\mathbf{Pr}(v_t|G_{t-1}) = \frac{1}{2\pi}\int_0^{2\pi}\mathbf{Pr}(v_t|G_{t-1},\theta)d\theta, \tag{22}$$

$$\mathbf{Pr}(v_t|G_{t-1}) = \mathbf{Pr}(v_t|G_{t-1},\theta), \tag{23}$$

and therefore we can see that $\mathbf{Pr}(v_t|G_{t-1})$ is normally distributed with zero mean and variance $2\eta G_{t-1}\sigma_w^2$.

Combining this conclusion with equation (15), we can derive the distribution of the channel gain power at time t conditioned on known channel gain power at time $t-1$ (case I), which will turn out to be Gaussian distributed with mean ($m_g = \eta g_{t-1}$) and variance ($\sigma_g^2 = 2\eta g_{t-1}\sigma_w^2$)

$$f_{G_t|G_{t-1}}(g_t|g_{t-1}) = \frac{1}{\sqrt{2\pi}\sigma_g}e^{-\frac{(g_t-m_g)^2}{2\sigma_g^2}}. \tag{24}$$

Deriving the channel gain power distribution in case II can be done by evaluating the joint distribution of the channel gain at time t and $t-1$ given the CQI and then marginalize for the channel gain at time $t-1$ to get our final distribution,

$$f_{G_t;G_{t-1}|CQI}(g_t;g_{t-1}|q) = f_{G_t|G_{t-1};CQI}(g_t|g_{t-1};q)f_{G_{t-1}|CQI}(g_{t-1}|q), \tag{25}$$

and since that having the channel gain and the CQI at time $t-1$ is the same as just having the channel gain then

$$f_{G_t|CQI}(g_t|q) = \int f_{G_t|G_{t-1}}(g_t|g_{t-1})f_{G_{t-1}|CQI}(g_{t-1}|q)dg_{t-1}, \tag{26}$$

substituting with (13) and (24)

$$f_{G_t|CQI}(g_t|q)=\int_{\frac{a_q}{c_{t-1}}}^{\frac{b_q}{c_{t-1}}}\big(\frac{1}{\sqrt{2\pi}\sigma_g}e^{-\frac{(g_t-m_g)^2}{2\sigma_g^2}}\big)\big(\frac{k}{2\lambda^2}e^{\frac{g_{t-1}}{2\lambda^2}}\big)dg_{t-1}, \tag{27}$$

let $g=\sqrt{g_{t-1}}$ and $dg=\frac{1}{2\sqrt{g_{t-1}}}dg_{t-1}$

$$f_{G_t|CQI}(g_t|q)=\frac{k}{2\lambda^2}\frac{e^{\frac{g_t}{2\sigma_w^2}}}{\sqrt{\pi\eta}\sigma_w}\int_{\sqrt{\frac{a_q}{c_{t-1}}}}^{\sqrt{\frac{b_q}{c_{t-1}}}}e^{-\frac{g_t^2}{4\sigma_w^2\eta g^2}}e^{-\frac{(\lambda^2\eta+2\sigma_w^2)g^2}{4\lambda^2\sigma_w^2}}dg. \tag{28}$$

Define the integration $I(a,b,L_1,L_2)$ such that

$$I(a,b,L_1,L_2)=\int_{L_1}^{L_2}e^{-\frac{a}{x^2}}e^{-bx^2}dx, \tag{29}$$

$$\begin{aligned}I(a,b,L_1,L_2)=&\frac{1}{4}\sqrt{\frac{\pi}{b}}\Big[e^{-2\sqrt{ab}}\big(\text{erf}(\sqrt{b}L_2-\frac{\sqrt{a}}{L_2})\\&-\text{erf}(\sqrt{b}L_1-\frac{\sqrt{a}}{L_1})\big)+e^{2\sqrt{ab}}\big(\text{erf}(\sqrt{b}L_2+\frac{\sqrt{a}}{L_2})\\&-\text{erf}(\sqrt{b}L_1+\frac{\sqrt{a}}{L_1})\big)\Big],\end{aligned} \tag{30}$$

and since

$$f_{G_t|CQI}(g_t|q)=\frac{k}{2\lambda^2}\frac{e^{\frac{g_t}{2\sigma_w^2}}}{\sqrt{\pi\eta}\sigma_w}I(a,b,L_1,L_2), \tag{31}$$

where $a=\frac{g_t^2}{4\sigma_w^2\eta}, b=\frac{\lambda^2\eta+2\sigma_w^2}{4\lambda^2\sigma_w^2}, L_1=\sqrt{\frac{a_q}{c_{t-1}}}$ and $L_2=\sqrt{\frac{b_q}{c_{t-1}}}$, then finally we can get our final form from substituting in (31) by (30)

$$f_{G_v^c|CQI_v^c}(g_t|q)=\zeta(\mathcal{A}(g_t)e^{\mathcal{C}_1g_t}+\mathcal{B}(g_t)e^{\mathcal{C}_2g_t}), \tag{32}$$

where

$$\begin{aligned}\mathcal{A}(g_t)&=\text{erf}\big[\sqrt{\frac{c_2b_q}{c_{t-1}}}-g_t\sqrt{\frac{\bar{c}_1c_{t-1}}{b_q}}\big]-\text{erf}\big[\sqrt{\frac{c_2a_q}{c_{t-1}}}-g_t\sqrt{\frac{\bar{c}_1c_{t-1}}{a_q}}\big],\\ \mathcal{B}(g_t)&=\text{erf}\big[\sqrt{\frac{c_2b_q}{c_{t-1}}}+g_t\sqrt{\frac{\bar{c}_1c_{t-1}}{b_q}}\big]-\text{erf}\big[\sqrt{\frac{c_2a_q}{c_{t-1}}}+g_t\sqrt{\frac{\bar{c}_1c_{t-1}}{a_q}}\big],\\ \mathcal{C}_1&=\frac{\lambda-\sqrt{\lambda^2+\frac{2\sigma_w^2}{\eta}}}{2\sigma_w^2\lambda},\qquad \mathcal{C}_2=\frac{\lambda+\sqrt{\lambda^2+\frac{2\sigma_w^2}{\eta}}}{2\sigma_w^2\lambda}\\ \zeta&=\frac{k}{4\lambda\sqrt{\eta(\eta\lambda^2+2\sigma_w^2)}},\quad \bar{c}_1=\frac{1}{4\sigma_w^2\eta},\quad c_2=\frac{\eta}{4\sigma_w^2}+\frac{1}{2\lambda^2}.\end{aligned}$$

In conclusion we derived the distribution of the channel gain at time t conditioned on given channel gain at time $t-1$ (shown in equation (24)), while equation (32) present the channel gain at time t conditioned on given CQI at time $t-1$.