RECIPROCAL STOCHASTIC MECHANICS MODEL OF QUANTUM INTERFERENCE AND MEASUREMENT*

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Abstract

A new stochastic description of quantum mechanics based on a class of reciprocal diffusions, called the quantum diffusions, was proposed recently in [20, 21]. The quantum diffusions differ from the class of Markov diffusions employed in earlier formulations of stochastic mechanics. They admit a set of closure rules which ensure that the infinite chain of conservation laws satisfied by general reciprocal diffusions reduces to the first two laws, which are equivalent to Schrödinger's equation. These diffusions are employed here to develop a stochastic model of the two-slit quantum interference experiment, and of quantum measurements. Only position measurements are considered. It is shown that measurements can be broken into two phases: an interaction phase, during which the observed system and measurement device evolve together as a single quantum diffusion, and a conditioning phase, where after the measurement has been recorded, Bayes' rule is employed to evaluate the conditional statistics of the observed system, which still specify a quantum diffusion.

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1 Introduction

In 1931, Schrödinger [26, 27] proposed an interpretation of quantum mechanics in terms of Markov diffusions, which associates to a quantum process process $x(t)$ with density $\rho(x,t)$ a Markov diffusion with identical probability density. The Markov diffusions employed by Schrödinger are slightly unusual, since instead of propagating all the state information either forward or backward in time through Fokker-Planck equations, as in standard forward or backward Markovian models, they use a mixed model, where part of the information is propagated forward, and part of it backward. Unfortunately, as shown by Zambrini [32], the two Fokker-Planck equations satisfied by the forward and backward information components of the Markov diffusion are not completely equivalent to the Schrödinger equation of the matching wave function. This equivalence can be achieved only if one adds to the physical potentials of the system a quantum potential of the same type as arising in Bohm's hidden variables formulation [5, 15] of quantum mechanics. This potential is nonlocal, in the sense that it couples instantaneously widely separated particles.

Relying on early work of Fenyes [9], a slightly different approach was followed by Nelson [23, 24] and other researchers [3, 13] to formulate quantum mechanics in terms of Markov diffusions. The key aspect of this work is that it relies on Itô's stochastic calculus to give a precise meaning to kinematic and dynamic quantities such as velocity and acceleration. Through the use of an elaborate definition of the acceleration, this theory gives the impression of being strictly equivalent to Schrödinger's equation. Nevertheless, as shown in [24, 25], the resulting Markovian stochastic mechanics is also nonlocal, and differs in significant ways from standard quantum theory.

A new form of stochastic mechanics was proposed recently in [21]. It relies on a subclass of reciprocal diffusions, called the quantum diffusions. Reciprocal processes were introduced by Bernstein [2], and studied in detail by Jamison [16]. In modern terminology, a reciprocal process is a Markov random field on a one-dimensional parameter space. Reciprocal processes contain Markov processes as a subclass, and in particular, when a reciprocal process is pinned at one end, it becomes Markov. In [17, 18], Krener showed that reciprocal diffusions satisfy a stochastic form of Newton's law, which in the Gaussian case [19] can be used to express such diffusions as solutions of second-order stochastic differential equations. A stochastic quantization procedure was developed in [20] which associates a class of reciprocal diffusions to a dynamic system precribed by its Hamiltonian. The reciprocal diffusions usually satisfy an infinite chain of conservation laws. These conservation laws close after the first two laws for two disjoint subclasses of reciprocal diffusions, the Markov and quantum diffusions. In [21], it was shown that the closure rules satisfied by the quantum diffusions are in essence an expression of Heisenberg's uncertainty principle.

The version of stochastic mechanics proposed in [21] presents a number of advantages with respect to earlier Markovian formulations. First, the two conservation laws of quantum diffusions are strictly equivalent to Schrödinger's equation, i.e., no quantum potential needs to be introduced. Second, the uncertainty principle is built in the closure rules. Finally, as proved in [21], the resulting stochastic mechanics is local in the sense that for two dynamically uncoupled but statistically coupled particles, the parameters of the potentials affecting one particle do not appear in the marginal statistics of the second particle. This property holds in standard quantum mechanics, but is violated by Markovian stochastic mechanics. The only drawback of the reciprocal form of stochastic mechanics described in [21], is that,

like Wigner's joint distribution for the position and momentum of a particle [14], the finite joint probability densities for the position of a system at multiple times may take negative values. However, this is only a drawback if one holds onto the unrealistic expectation that all Borel cylinder sets should be probabilizable. The sigma field of observable events is severely restricted in quantum mechanics by the commutation requirement for all observed variables, and the stochastic mechanics proposed in [21] appears to yield positive values for all such events.

In this paper, we extend the work started in [21] by proposing quantum diffusions models of the two-slit quantum interference experiment and of quantum measurements. As noted by Feynman in [12], the two-slit quantum diffraction experiment "... has in it the heart of quantum mechanics. In reality it contains the only mystery." The two key features of this experiment are as follows. First, if both slits are open, an interference pattern appears on the screen used to record the positions of arriving particles, thus revealing the so-called "wave-particle" duality of matter. Second, if a measurement device is employed to determine through which of the two slits each particle passes, the interference pattern vanishes from the recording screen, which now displays coherent arrival patterns located about the positions of the top and the bottom slits. The stochastic model obtained by applying the construction technique of [21] is a compound process, where the diffusion $x(t)$ modelling the quantum interference process is a mixture of four Gaussian wavepackets. These wavepackets are centered around four possible classical trajectories: two of them are real and correspond to the trajectories through the top and bottom slit, and two are imaginary, and can be viewed as interference trajectories. From a stochastic point of view, $x(t)$ is obtained by randomly selecting one of four possible classical trajectories according to certain a priori probabilities, and once a trajectory has been picked, $x(t)$ evolves about it as a single free Gaussian wavepacket. This model accounts for the presence of interference fringes on the recording screen. Furthermore, the a priori probabilities of the interference trajectories are so small that, when the position of a particle is measured as it crosses the screen, and we attempt to decide which of the four trajectories it follows, only the two classical trajectories are selected by a maximum a posteriori hypothesis testing scheme, so that the interference pattern vanishes from the a posteriori density function of $x(t)$.

The stochastic model of quantum measurements generated by the procedure of [21] is relatively close to the description of the measurement process in standard quantum theory [4, 31]. Only position measurements are considered. The measurement process can be broken into two phases. First, the observed system and and measurement device, which were originally statistically independent, interact and evolve together as a single quantum diffusion. The interaction then ceases, leaving the observed system and measurement device dynamically uncoupled, but statistically coupled. This means they can no longer be analyzed independently of one another, and in particular although the marginal statistics of the observed system and measurement device still specify reciprocal diffusions, these are no longer of quantum type. The second phase of the measurement process consists merely in recording the measurement value, and applying Bayes rule to evaluate the conditional statistics of the observed system. It turns out that these statistics form a quantum diffusion, so that measurements transform quantum diffusions into quantum diffusions. The Bayesian conditioning phase is the statistical analogue of the "collapse of the wavefunction" in Von Neumann's model of quantum measurements [31]. The only difference is that whereas quantum theories of experiments seem to address primarily the case of perfect measurements,

because of the statistical approach adopted here, all measurements are viewed as imperfect.

The paper is organized as follows. The stochastic mechanics of reciprocal diffusions is reviewed in Section 2. The quantum diffusion describing the minimum uncertainty free wavepacket, which appears as a component of several stochastic models considered here, is described in Section 3. Section 4 constructs the compound quantum diffusion model for the two-slit diffraction experiment, and the reduction of the wavefunction associated to the observation of the path taken by each particle is analyzed in Section 5. Finally, the stochastic description and properties of quantum measurements are presented in Section 6.

2 Reciprocal Stochastic Mechanics

Let $\mathbb{R}^n$ be the standard n-dimensional Euclidean space with metric $\delta_{ij} = 1$ for $i = j$, and $= 0$ otherwise. Consider a dynamical system with Hamiltonian

$$H(x,p,t) = \frac{1}{2}(p^i - A^i(x,t))(p_i - A_i(x,t)) + \phi(x,t)\,, \tag{2.1}$$

where $\{\phi, A_i\}$ denotes a scalar and vector potential pair, and where we use the standard tensor contraction convention with repeated upper and lower indices corresponding to a summation. Following [20, 21], we can associate a class of reciprocal diffusions to this system by replacing p_j by $-\nabla_j$ inside the Hamiltonian $H(x,p,t)$, where ∇_j denotes the differentiation with respect to x^j. This correspondence rule is the stochastic analogue of the quantization rule $p_j \leftrightarrow -i\nabla_j$ of quantum mechanics. This yields the elliptic operator

$$\begin{aligned} \mathbf{H} &= \frac{1}{2}(\nabla^i + A^i)(\nabla_i + A_i) + \phi & \text{(2.2a)} \\ &= \frac{1}{2}\Delta + A^i\nabla_i + \frac{1}{2}(\nabla^i A_i + A^i A_i) + \phi\,, & \text{(2.2b)} \end{aligned}$$

where Δ denotes the Laplacian. Then, the generalized heat operator

$$\mathbf{L} = \mathbf{H} - \frac{\partial}{\partial t} \tag{2.3}$$

is the forward operator of a general Markov diffusion with diffusion matrix δ_{ij}, drift $b^i(x,t) = -A^i(x,t)$ and creation/killing rate

$$c(x,t) \stackrel{\triangle}{=} \frac{1}{2}(A^i A_i - \nabla^i A_i) + \phi(x,t)\,. \tag{2.4}$$

See [22] for a study of Markov diffusions with creation or killing. The Green's function associated to $\mathbf{L}$ is given by

$$\begin{aligned} \mathbf{L}_{y,t} G(x,s;y,t) &= 0\,,\ t \geq s & \text{(2.5a)} \\ G(x,s;y,s) &= \delta(x-y) & \text{(2.5b)} \end{aligned}$$

where the subscripts $\{y,t\}$ specify the variables upon which the operator $\mathbf{L}$ is acting. To ensure that $G(x,s;y,t)$ represents the transition density of a general Markov diffusion, $G(x,s;y,t)$ is required to decay as $|y| \to \infty$.

Then $x(\cdot)$ belongs to the class of reciprocal diffusions associated to the Hamiltonian $H(x,p,t)$ if the joint probability density of $x(t_0),\ x(t_1),\ \cdots, x(t_N)$ for $t_0 = 0 < t_1 < \cdots < t_N = T$ can be expressed as

$$p(x_0,t_0;x_1,t_1;\cdots;x_N,t_N) = q(x_0,t_0;x_N,t_N)\prod_{k=0}^{N-1} G(x_k,t_k;x_{k+1},t_{k+1})\,, \tag{2.6}$$

where the function $q(x_0,0;x_T,T)$ satisfies the normalization condition

$$\int\int q(x_0,0;x_T,T)G(x_0,0;x_T,T)dx_0dx_T = 1\,. \tag{2.7}$$

If the densities (2.6) are used to specify a probability measure over the Borel cylinder sets, the end-point density $q(x_0,0;x_T,T)$ must be positive. However, if the sigma field $\mathcal{E}$ of events that need to be probabilized is much smaller, as is the case in quantum mechanics, where $\mathcal{E}$ corresponds to the set of "observable" events, then q may take negative values, as long as for all events $E \in \mathcal{E}$, we have $0 \le P(E) \le 1$. See [10, 29, 21] for a discussion of the use of negative probabilities to model quantum phenomena.

Since the Green's function $G(x,s;y,t)$ is specified by the Hamiltonian $H(x,p,t)$, the expression (2.6) shows that all reciprocal diffusions associated to the same physical system differ only by the choice of end-point density $q(x_0,0;x_T,T)$. Diffusions which differ only by the choice of q are said to belong to the same reciprocal class, since as shown by Jamison [16] they have the same three-point transition density $r(x,s;t,y;z,u)$. For $s < t < u$, r is defined as the conditional density of $x(t) = y$ given that $x(s) = x$ and $x(u) = z$, and using the expression (2.6) for the finite joint densities, it is easy to verify it can be expressed as

$$r(x,s;y,t;z,u) = \frac{G(x,s;y,t)G(y,y;z,u)}{G(x,s;z,u)}\,, \tag{2.8}$$

which is clearly independent of q.

This property implies that all diffusions in the same reciprocal class have the same dynamics. Specifically, let

$$\bar{x}(t,h) = \frac{1}{2}(x(t+h)+x(t-h)) \tag{2.9a}$$

$$d^1x(t,h) = \frac{1}{2}(x(t+h)-x(t-h) \tag{2.9b}$$

$$d^2x(t,h) = x(t+h)+x(t-h)-2x(t) \tag{2.9c}$$

denote respectively the mean position and the centered first- and second-order differences of the process $x(t)$. It is shown in [20, 18] that the three-point transition density $r(\bar{x}-uh, t-h; x, t; \bar{x}+uh, t+h)$ of $x(t) = x$ given $x(t\pm h) = \bar{x} \pm uh$ is locally Gaussian, where d^2x has for mean

$$E[d^2x^i \mid x(t\pm h) = \bar{x} \pm uh] = F^i(\bar{x},u,t)h^2 + O(h^{5/2})\,, \tag{2.10}$$

and variance

$$E[d^2x^i d^2x^j \mid x(t\pm h) = \bar{x} \pm uh] = 2h\delta^{ij} + O(h^{5/2})\,. \tag{2.11}$$

In identity (2.10), if

$$dA_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \tag{2.12}$$

denotes the exterior derivative of A_i,

$$F_i(x,u,t) = dA_{ij}(x,t)u^j - \left(\frac{\partial \phi}{\partial x^i} + \frac{\partial A_i}{\partial t}\right)(x,t) \tag{2.13}$$

is the force applied to a particle with position x and velocity u due to the potentials $\{A_i, \phi\}$. Thus, (2.10) represents a stochastic form of Newton's law, since it equates the conditional mean acceleration

$$a^i = E[\frac{d^2x^i}{h^2} \mid x(t \pm h) = \bar{x} \pm uh] \tag{2.14}$$

of the process at time t to the force based on the mean position $\bar{x}(t,h)$ and empirical velocity $u(t,h) = d^1x(t,h)/h$. This Newton law differs from the one proposed by Nelson [23, 24] for Markov diffusions, because the acceleration (2.14) does not coincide with Nelson's definition of the acceleration (see [30] for a study of the differences existing between the two concepts of acceleration). Also, since the evaluation of the conditional moments (2.10)–(2.11) is based exclusively on three-point transition density r, all processes in the same reciprocal class obey the same Newton law.

If the interval of definition $[0,T]$ of a reciprocal diffusion is changed to a subinterval $[s,t]$ with $0 < s \leq t < T$, the end-point density that needs to be applied to the new interval is given by

$$q(x,s;y,t) = \int\int G(x_0,0;x,s)G(y,y;x_T,T)q(x_0,0;x_T,T)dx_0dx_T\ , \tag{2.15}$$

from which we deduce that $q(x,s;y,t)$ with $s \leq t$ obeys the forward and backward heat equations

$$\mathbf{L}_{x,s}q(x,s;y,t) = 0 \tag{2.16a}$$

$$\mathbf{L}^*_{y,t}q(x,s;y,t) = 0 \tag{2.16b}$$

where $\mathbf{L}^*$ denotes the adjoint operator of $\mathbf{L}$. In particular, the function $q(x,t;y,t)$ obtained by letting the length of the interval of definition shrink to zero plays an important role in our analysis. Its restriction to $y = x$ gives the probability density

$$\rho(x,t) = q(x,t;x,t) \tag{2.17}$$

of $x(t)$. From (2.16a)–(2.16b), we see that q satisfies the evolution equation

$$\frac{\partial q}{\partial t}(x,t;y,t) = (\mathbf{H}_{x,t} - \mathbf{H}^*_{y,t})q(x,t;y,t)\ . \tag{2.18}$$

The function

$$m(a,b,t) \stackrel{\triangle}{=} q(a-b,t;a+b,t) \tag{2.19}$$

plays the role of generating function for the conservation laws of reciprocal diffusions. Specifically, by performing a Taylor series expansion of $m(a,b,t)$ in the vicinity of $b = 0$, we find it admits the power series representation

$$m(a,b,t) = \rho(a,t)\left(1 + 2w_i(a,t)b^i + 4(\pi_{ij} + w_iw_j)\frac{b^ib^j}{2} + O(b^3)\right) \tag{2.20}$$

for small b, where the mean momentum w_i and stress tensor π_{ij} are defined in terms of

$$M(a,b,t) \stackrel{\triangle}{=} \ln m(a,b,t) \tag{2.21}$$

as

$$w_i(a,t) = \frac{1}{2}\frac{\partial M}{\partial b^i}(a,0,t) \tag{2.22a}$$

$$\pi_{ij}(a,t) = \frac{1}{4}\frac{\partial^2 M}{\partial b^i \partial b^j}(a,0,t)\ . \tag{2.22b}$$

Then, to generate the chain of conservation laws of reciprocal diffusions, we only need to substitute the expansion (2.20) inside the evolution equation (2.18), perform the change of coordinates $x = a - b$ and $y = a + b$, and match like powers of b on both sides of this equation. The first two conservation laws obtained by this procedure take the form

$$\frac{\partial \rho}{\partial t} + \nabla^i(\rho v_i) = 0 \tag{2.23}$$

$$\frac{\partial}{\partial t}(\rho v_j) + \nabla^i(\rho P_{ij}) = \rho F_j(x,v,t) \tag{2.24}$$

where

$$v^i(x,t) \stackrel{\triangle}{=} w^i(x,t) - A^i(x,t) \tag{2.25}$$

denotes the mean velocity of the diffusion, and

$$P_{ij}(x,t) \stackrel{\triangle}{=} (\pi_{ij} + v_i v_j)(x,t) \tag{2.26}$$

is the flux of momentum tensor. The identities (2.23) and (2.24) express respectively the conservation of mass and of momentum, and are only the first two of what constitutes usually an infinite chain of conservation laws.

The mean velocity v^i and stress tensor π_{ij} appearing in the above expressions have a simple kinematic interpretation. specifically, it is shown in [20, 18] that the conditional density of the first difference $d^1x(t,h)$ given the mean position $\bar{x}(t,h)$ is locally Gaussian with mean

$$E[d^1x^i(t,h) \mid \bar{x}(t,h) = x] = v^i(x,t)h + O(h^2) \tag{2.27a}$$

and variance

$$\begin{aligned} &E[(d^1x^i - v^i(x,t)h)(d^1x^j - v^j(x,t)h) \mid \bar{x}(t,h) = x] \\ &\qquad = \delta^{ij}h/2 + \pi_{ij}(x,t)h^2 + o(h^2)\ . \end{aligned} \tag{2.27b}$$

For the case of Markov diffusions with forward and backward drifts $b^{\pm}(x,t)$, the mean velocity $v(x,t)$ coincides with the current velocity

$$v(x,t) = \frac{1}{2}(b^+(x,t) + b^-(x,t)) \tag{2.28}$$

of Markovian stochastic mechanics [23, 24]. Note however, that neither the mean velocity $v(x,t)$ or stres tensor $\pi(x,t)$ are reciprocal invariants, i.e., they change when we consider different diffusions within a reciprocal class.

An interesting feature of the conservation laws of reciprocal diffusions is that if we denote

$$R(x,t) = \frac{1}{2}\ln\rho(x,t) \tag{2.29}$$

and assume that the two closure rules

$$w_i(x,t) = \nabla_i S(x,t) \tag{2.30a}$$

$$\pi_{ij}(x,t) = -\frac{1}{4}\nabla_i\nabla_j \ln\rho(x,t) \tag{2.30b}$$

hold, the $n+1$ conservation laws (2.23)–(2.24) reduce to the two scalar equations

$$\frac{\partial R}{\partial t} + (\nabla^i - A^i)\nabla_i R + \frac{1}{2}\nabla^i(\nabla_i S - A_i) = 0 \tag{2.31}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla^i S - A^i)(\nabla_i S - A_i) + \phi - \frac{1}{2}(\nabla^i R\nabla_i R + \Delta R) = 0\,. \tag{2.32}$$

Note that the closure rule (2.30a) specifies $S(x,t)$ only up to function of t. However, as shown in [21], this degree of freedom is exhausted when setting the right-hand side of (2.32) equal to zero.

Condider now the Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \mathbf{H}_Q\psi(x,t) \tag{2.33}$$

where

$$\mathbf{H}_Q = (-i\nabla^j - A^j)(-i\nabla_j - A_j) + \phi \tag{2.34}$$

is the Hermitian operator obtained by replacing p_j by $-i\nabla_j$ inside the Hamiltonian $H(x,p,t)$, and where we have set Planck's constant $\hbar = 1$. By parametrizing the wave function as

$$\psi(x,t) = \exp(R(x,t) + iS(x,t)) \tag{2.35}$$

and separating the real and imaginary parts of Schrödinger'e equation (2.33), it is easy to verify it reduces to (2.31) and (2.32). This establishes a one-to-one correspondence between quantum processes and the class of reciprocal diffusions satisfying the closure rules (2.30a)–(2.30b). Because of this equivalence, the reciprocal diffusions which satisfy (2.30a)–(2.30b) are called the quantum diffusions. This class of diffusions does not overlap with the class of Markov diffusions, which also satisfy (2.30a), but for which the sign is reversed in the relation (2.30b) between the stress tensor π_{ij} and the Hessian of the log-density $\ln\rho$.

It is shown in [21] that $x(t)$ is a quantum diffusion if the function $M(a,b,t)$ given by (2.21) is the real part of an analytic function $F(z,t)$ of $z = a + ib \in \mathbb{C}^n$. Furthermore, the end-point density $q(x,t;y,t)$ corresponding to a wave function $\psi(x,t)$ takes the form

$$q(x,t;y,t) = |\psi(\frac{y+x}{2} - i\frac{y-x}{2})|^2\,. \tag{2.36}$$

This specifies the end-point density for an interval of zero length. To obtain the density $q(x,s;y,t)$ for a finite interval $[s,t]$ with $s < t$, one must either propagate (2.16a) backward in time with initial condition $q(x,t;y,t)$, or propagate (2.16b) forward in time, starting from $q(x,s;y,s)$. In both cases, this corresponds to prapagating the heat equation in an unstable direction, i.e., we are trying to undo diffusion effects. Furthermore, if the function $q(x,t;y,t)$ admits nodes, i.e. values (x_0,y_0) for which q is zero, then the solution $q(x,s;y,t)$ admits negative values as soon as $s < t$.

3 Minimum Uncertainty Wavepacket

The above procedure was employed in [21] to construct reciprocal diffusions corresponding to several quantum processes such as the coherent state of the harmonic oscillator, and the minimum uncertainty free wavepacket. Since this last process plays an important role in the analysis of following sections, we describe briefly the model obtained in [21]. Consider the Gaussian wave function

$$\psi(x,t) = \frac{1}{\pi^{1/4}(r+it/r)^{1/2}} \exp(-\frac{1}{2}\frac{(x-x_C(t))^2}{r^2+it}) \,. \tag{3.1}$$

For $t \geq 0$, it solves the Schrödinger equation (2.33) with

$$A(x,t) = \phi(x,t) \equiv 0 \,. \tag{3.2}$$

In (3.1), $x_C(t) = x_0 + v_0 t$ specifies a reference classical trajectory for a free particle with initial position x_0 and velocity v_0, and $\psi(x,t)$ represents the evolution of a quantum wavepacket about this trajectory. The standard deviations of the position and momentum distributions are given by

$$\sigma_x(t) = \frac{1}{\sqrt{2}}(r^2 + t^2/r^2)^{1/2} \qquad \sigma_p(t) = \frac{1}{r\sqrt{2}} \,. \tag{3.3}$$

The position momentum uncertainty product $\sigma_x(t)\sigma_p(t)$ equals the Heisenberg lower bound of $1/2$ at $t = 0$, which explains why this process is called a minimum uncertainty wavepacket. The parameter r can be used to adjust the relative width of the position and momentum distributions. It was shown in [21] that the corresponding quantum diffusion is a Gaussian process with mean $x_C(t)$ and covariance

$$K(t,s) = \frac{1}{2}(r^2 + st/r^2 - |t-s|) \,. \tag{3.4}$$

This diffusion has for mean velocity and stress tensor

$$v(x,t) = v_0 + \frac{t(x-x_C(t))}{r^4+t^2} \tag{3.5a}$$

$$\pi(x,t) = \frac{r^2}{2(r^4+t^2)} \,. \tag{3.5b}$$

Furthermore, using a characterization of Gaussian reciprocal diffusions given in [19] (see also [18, 8]), it was shown that over a finite interval $[0,T]$, the deviation $z(t) = x(t) - x_C(t)$ of $x(t)$ from its classical trajectory $x_C(t)$ satisfies the second-order stochastic differential equation

$$\mathcal{L}_F z(t) = \xi(t) \tag{3.6a}$$

$$\mathcal{L}_F \stackrel{\triangle}{=} -d^2/dt^2 \tag{3.6b}$$

with Dirichlet boundary conditions

$$\begin{bmatrix} z(0) \\ z(T) \end{bmatrix} \sim \mathcal{N}(0, \mathbf{P}) \,. \tag{3.7}$$

The equation (3.6a) indicates that the motion of $z(t)$ is due entirely to the random fluctuations $\xi(t)$, where $\xi(t)$, which is usually called the dual or conjugate proces of $z(t)$, is a generalized Gaussian process independent of $z(0)$ and $z(T)$, with zero-mean and autocorrelation

$$E[\xi(t)\xi(s)] = \mathcal{L}_F \delta(t-s) \; . \tag{3.8}$$

In the boundary conditions (3.7), the matrix $\mathbf{P}$ is the covariance matrix of $x(0)$ and $x(T)$, so that

$$\mathbf{P} = \begin{bmatrix} K(0,0) & K(0,T) \\ K(T,0) & K(T,T) \end{bmatrix} \; . \tag{3.9}$$

Since the Green's function of the operator $\mathcal{L}_F$ with homogeneous Dirichlet conditions at $t=0$ and $t=T$ is given by

$$\Gamma_F(t,s) = \begin{cases} (1-\frac{t}{T})s & \text{for} \;\; t \geq s \\ t(1-\frac{s}{T}) & \text{for} \;\; t \geq s \; , \end{cases} \tag{3.10}$$

the solution of (3.6)–(3.7) takes the form

$$z(t) = B(t) + (1-\frac{t}{T})z(0) + \frac{t}{T}z(T) \; , \tag{3.11a}$$

where

$$B(t) = \int_0^T \Gamma_F(t,s)\xi(s)ds \; . \tag{3.11b}$$

has covariance $\Gamma_F(t,s)$, which implies it is a Brownian bridge process. Thus, it can be written as

$$B(t) = W(t) - \frac{t}{T}W(T) \; , \tag{3.12}$$

where $W(t)$ is a Wiener process independent of $z(0)$ and $z(T)$. Since the interval length T is arbitrary, the quantum diffusion corresponding to the wavepacket (3.1) has an infinite lifetime.

Using results of [1, §5] on the representation of scalar Gaussian reciprocal processes in terms of the Wiener process, it can also be shown that $z(t)$ admits the representation

$$z(t) = W(t) + z(0)(1 - t/r^2) \tag{3.13}$$

for $t \geq 0$, where $W(t)$ is a Wiener process independent of $z(0)$, which does not bear any relation with the Wiener process of identity (3.12).

4 Interference

The following expressions will be useful in the analysis of interference processes. Let

$$\psi_j = \exp(R_j + iS_j) \tag{4.1}$$

with $j = 1,\ 2$ be two wavefunctions satisfying Schrödinger's equation. By linearity,

$$\psi = \exp(R + iS) = \psi_1 + \psi_2 \tag{4.2}$$

also solves this equation. To express the real and imaginary parts R and S of $\ln\psi$ in terms of the corresponding components R_j and S_j of $\ln\psi_j$ for $j = 1,\ 2$, note that

$$\begin{aligned} \mathrm{Re}\,\psi &= \exp(R_1)\cos(S_1) + \exp(R_2)\cos(S_2) && \text{(4.3a)} \\ \mathrm{Im}\,\psi &= \exp(R_1)\sin(S_1) + \exp(R_2)\sin(S_2)\,. && \text{(4.3b)} \end{aligned}$$

This implies

$$\begin{aligned} \rho &= \exp(2R) = \psi\psi^* \\ &= 2\exp(R_1 + R_2)[\cosh(R_1 - R_2) + \cos(S_1 - S_2)] && \text{(4.4)} \end{aligned}$$

and

$$S = \arctan\left(\frac{\exp(R_1)\sin(S_1) + \exp(R_2)\sin(S_2)}{\exp(R_1)\cos(S_1) + \exp(R_2)\cos(S_2)}\right)\,. \qquad \text{(4.5)}$$

Then consider the two-slit diffraction experiment described in [11, §3.2]. As illustrated in Fig. 1, a plane particle wave originating from a distant source is incident upon a screen with two slits located at $x = \pm\ell$. For ease of computation, the two slits are assumed to be Gaussian. The particle crosses the diffracting screen at $t = 0$ and later hits a detector screen located at a distance $y = d$ from the diffracting screen. Only the x-motion of the particle is considered.

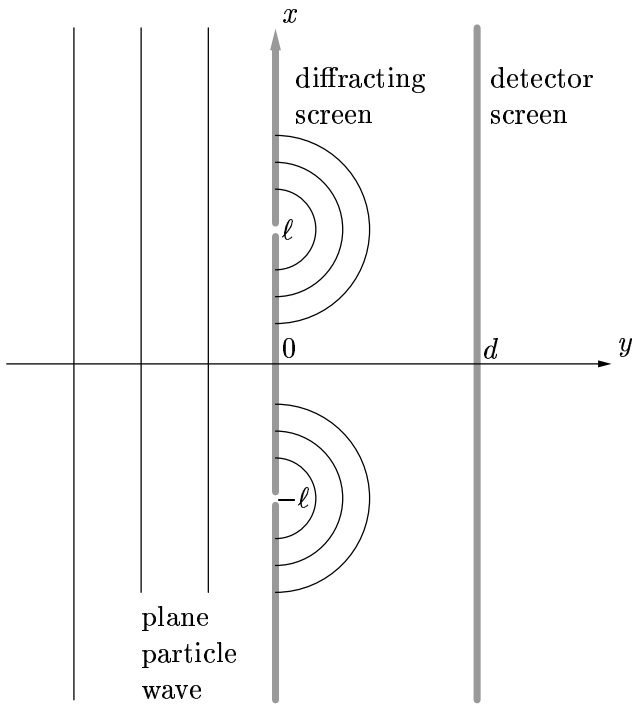

Figure 1: Two-slit particle diffraction experiment.

The wavefunction $\psi(x,t)$ associated to a particle after it has passed through one of the two slits takes the form (4.2), where ψ_1 and ψ_2 are minimum uncertainty Gaussian wavepackets centered about the two classical trajectories $x_{C1}(t) = \ell$ and $x_{C2}(t) = -\ell$, i.e.,

$$\psi_j(x,t) = \frac{B}{\pi^{1/4}(r+it/r)^{1/2}} \exp(-\frac{1}{2}\frac{(x-x_{Cj})^2}{r^2+it}) \,, \tag{4.6}$$

with $j = 1,\ 2$. B denotes here a normalization constant which is selected such that the total probability mass obtained by integrating (4.4) equals 1. The standard deviation of each wavepacket as it emerges from its slit is $r/\sqrt{2}$, so that the parameter ℓ/r represents approximately the ratio of the slit separation to their width. For the wavepackets given by (4.6), we have

$$R_j(x,t) = -\frac{1}{2}R(t)(x-x_{Cj})^2 + \ln B + \frac{1}{4}\ln(\frac{R(t)}{\pi}) \tag{4.7a}$$

$$S_j(x,t) = \frac{1}{2}S(t)(x-x_{Cj})^2 - \frac{1}{2}\arctan(\frac{t}{r}) \,, \tag{4.7b}$$

with

$$R(t) = \frac{r^2}{r^4+t^2} \qquad S(t) = \frac{t}{r^4+t^2} \,. \tag{4.8}$$

Then, by using the expression (4.4) for the density $\rho(x,t)$, one finds that it can be expressed as a superposition of four Gaussian distributions. Specifically, if $N(m,K)$ denotes a Gaussian density with mean m and variance K,

$$\rho(x,t) = \sum_{j=1}^{4} \rho_j(x,t) P_j \tag{4.9a}$$

with

$$\rho_j(x,t) = N(x_{Cj}(t), (2R(t))^{-1}) \,. \tag{4.9b}$$

Here $x_{C1}(t) = \ell$ and $x_{C2}(t) = -\ell$ correspond to the classical paths through each of the two slits, and

$$x_{C3}(t) = i\ell t/r^2 \qquad x_{C4}(t) = -i\ell t/r^2 \tag{4.10}$$

are fictitious imaginary paths modelling the quantum interference of the two wavepackets.

The probabilities of each path are given respectively by

$$P_1 = P_2 = B^2 \tag{4.11a}$$

$$P_3 = P_4 = B^2 \exp(-\ell^2/r^2) \tag{4.11b}$$

and the requirement that the total probability mass equals 1 implies

$$P_1 + P_2 + P_3 + P_4 = 1 \,, \tag{4.12}$$

which gives

$$B^2 = \frac{1}{2(1+\exp(-\ell^2/r^2))} \,. \tag{4.13}$$

In particular, when the ratio ℓ/r of the slit separation to the slit width is large, so that the wavepackets do not interact significantly, the probabilities P_3 and P_4 of the two imaginary interference paths are very small, and $B \approx 1/\sqrt{2}$.

Thus, the density $\rho(x,t)$ for $x(t)$ can be viewed as a marginal probability density obtained from the joint distribution of two random variables $(x(t),\ J)$. Here J is discrete valued with

$$P[J=j] = P_j \tag{4.14}$$

for $1 \le j \le 4$. The random variable K selects one reference trajectory among the two paths $x_{C1}(t)$ and $x_{C2}(t)$ corresponding to the particle going through the $+\ell$ and $-\ell$ slits, and the two fictitious paths $x_{C3}(t)$ and $x_{C4}(t)$ modelling quantum interference. Then, given $J=j$, the particle's position $x(t)$ admits the conditional density

$$p_{x(t)|J}(x \mid J=j) = N(x_{Cj}(t), (2R(t))^{-1}) \,, \tag{4.15}$$

which corresponds to the density of a standard minimum uncertainty wavepacket for a free particle moving about the classical trajectory $x_{Cj}(t)$. Although this model may appear artificial, since it relies on the imaginary classical trajectories $x_{C3}(t)$ and $x_{C4}(t)$, it will be useful for interpreting the reduction of the wavefunction after the path followed by the particle has been determined.

Note that $\rho(x,t)$ can be rewritten as

$$\begin{aligned} \rho(x,t) \;&=\; A^2 \left(\frac{R(t)}{\pi}\right)^{1/2} \exp -R(t)(x^2+\ell^2) \\ &\qquad [\,2\cosh(2R(t)\ell x) + 2\cos(2S(t)\ell x)\,] \end{aligned} \tag{4.16}$$

so that although the representation (4.9a) of ρ as a superposition of Gaussians gives the impression that the density remains coherent, the use of complex reference trajectories induces the usual interference pattern. As noted in [24, §17], the process does not have nodes, i.e., values of x for which $\rho(x,t)=0$, but it comes close to having nodes when

$$S(t)\ell x = (n+\frac{1}{2})\pi \tag{4.17}$$

with n integer, and $R(t)\ell x$ is small. Note also that even if we are careful to eliminate events with complex probabilities, the joint probability distribution for $x(t)$ and J specified by (4.14) and (4.15) may take negative values. For example, conditioning $x(t)$ with respect to the event $E = \{J = 3 \text{ or } 4\}$ gives the conditional density

$$p_{x(t)|E}\,(x \mid E) = \left(\frac{R(t)}{\pi}\right)^{1/2} \exp[-R(t)(x^2 - \ell^2 t^2/r^4)]\cos(2S(t)\ell x) \,, \tag{4.18}$$

which takes negative values. Although the appearance of negative values may seem surprising, it is consistent with interpretations of the two-slit diffraction experiment given recently in [10, 29].

Up to this point, we have examined only the density of $x(t)$ at a single time. To specify the finite joint densities of the quantum diffusion which models the two-slit interference

process, we must construct its end-point density $q(x,s;y,t)$. For $s = t$, the expression (2.36) gives

$$q(x,t;y,t) = \sum_{j=1}^{4} q_j(x,t;y,t)P_j \,, \tag{4.19}$$

where

$$\begin{aligned} \ln q_j(x,t;y,t) &= -\frac{1}{2}\left[\begin{array}{cc} x-x_{Cj}(t) & y-x_{Cj}(t) \end{array}\right] Q(t,t) \left[\begin{array}{c} x-x_{Cj}(t) \\ y-x_{Cj}(t) \end{array}\right] \\ &\quad - \left[\begin{array}{cc} x-x_{Cj}(t) & y-x_{Cj}(t) \end{array}\right] \left[\begin{array}{c} p_{Cj}(t) \\ -p_{Cj}(t) \end{array}\right] + f(t) \,, \end{aligned} \tag{4.20}$$

with

$$Q(t,t) = \left[\begin{array}{cc} S(t) & R(t) \\ R(t) & -S(t) \end{array}\right] . \tag{4.21}$$

Here $f(t)$ is a function of t only, and $p_{Cj} = \dot{x}_{Cj}$ is the momentum process corresponding to the j-th reference trajectory, so that

$$p_{C1} = p_{C2} = 0 \tag{4.22a}$$

$$p_{C3} = il/r^2 \quad , \quad p_{C4} = -il/r^2 \,. \tag{4.22b}$$

Note that the use of imaginary momenta has a precedent in quantum mechanics, where plane waves with an imaginary momentum are often used to analyze the tunnel effect for a particle penetrating a square potential barrier [6, pp. 65–67]. Thus, the end-point density $q(x,t;y,t)$ is again the superposition of four Gaussian functions q_j, where each q_j represents the end-point density corresponding to a minimum uncertainty wavepacket centered about the reference trajectory $(x_{Cj},\ p_{Cj})$. By superposition, the solution of the backward heat equation

$$(\frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial s})q(x,s;y,t) = 0 \quad s \le t \tag{4.23}$$

takes therefore the form

$$q(x,s;y,t) = \sum_{j=1}^{4} q_j(x,s;y,t)P_j \,, \tag{4.24}$$

where each function $q_j(x,s;y,t)$ satisfies (4.23) with initial condition $q_j(x,t;y,t)$. This representation reduces the evaluation of the end-point density for a mixture of Gaussian wavepackets to its evaluation for a single wavepacket. This computation was performed in [21], where it is shown that $q_j(x,s;y,t)$ can be expressed as

$$\begin{aligned} \ln q_j(x,s;y,t) &= -\frac{1}{2}\left[\begin{array}{cc} x-x_{Cj}(s) & y-x_{Cj}(t) \end{array}\right] Q(s,t) \left[\begin{array}{c} x-x_{Cj}(s) \\ y-x_{Cj}(t) \end{array}\right] \\ &\quad - \left[\begin{array}{cc} x-x_{Cj}(s) & y-x_{Cj}(t) \end{array}\right] \left[\begin{array}{c} p_{Cj}(s) \\ -p_{Cj}(t) \end{array}\right] + f(s,t) \,, \end{aligned} \tag{4.25}$$

where $f(s,t)$ is a function of s and t only, and the 2×2 matrix $Q(s,t)$ satisfies the Riccati equation

$$-\frac{dQ}{ds}(s,t) = Q \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] Q \tag{4.26}$$

for $s \leq t$ with initial condition (4.21). With $R(t)$ and $S(t)$ given by (4.8), it was found in [21] that

$$Q(s,t) = \frac{1}{r^4 + st} \begin{bmatrix} t & r^2 \\ r^2 & -s \end{bmatrix} . \tag{4.27}$$

The Green's function for a free particle is

$$G(x,s;y,t) = \frac{1}{(2\pi(t-s))^{1/2}} \exp -\frac{(y-x)^2}{2(t-s)} \tag{4.28}$$

for $t \geq s$. Substituting this expression and the identities (4.24), (4.25) and (4.27) for q inside the form (2.6) for the finite joint densities of a reciprocal diffusion, we find that the finite joint densities of the quantum diffusion modelling the two-slit interference process can be written as

$$p(x_0,t_0;x_1,t_1;\cdots;x_N,t_N) = \sum_{j=1}^{4} p_j(x_0,t_0;x_1,t_1;\cdots;x_N,t_N)P_j . \tag{4.29}$$

In this expression, each density $p_j(x_0,t_0;x_1,t_1;\cdots;x_N,t_N)$ with $1 \leq j \leq 4$ corresponds to the quantum diffusion modelling a minimum uncertainty wavepacket centered about $x_{Cj}(t)$. Thus, the finite joint densities for $x(\cdot)$ can be viewed as generated from a compound process $(x(\cdot), J)$, where given $J = j$, $x(\cdot)$ is Gaussian, with mean

$$E[x(t) \mid J = j] = x_{Cj}(t) \tag{4.30a}$$

and covariance

$$E[(x(t) - x_{CJ}(t))(x(s) - x_{CJ}(s)) \mid J = j] = K(t,s) , \tag{4.30b}$$

where $K(t,s)$ is given by (3.4) and does not depend on j. Equivalently, if we introduce the deviation process $z(t) = x(t) - x_{CJ}(t)$, $z(t)$ is independent on J and admits the representations (3.11) and (3.13). Thus $x(t)$ can be viewed as generated by selecting a reference trajectory $x_{CJ}(t)$ at random according to the probabilities P_j, $1 \leq j \leq 4$, and superposing a deviation process $z(t)$ independent of J.

The expression (4.19)–(4.21) for the the end-point density $q(x,t;y,t)$ for an interval of zero length can be used to evaluate the mean velocity and stress tensor of the quantum diffusion modeling the two-slit interference process. Note first that since the particle is free, $v(x,t) = w(x,t)$ (the covector potential $A(x,t) = 0$). Using the identities (2.22a) and (2.22b), for $w(x,t)$ and $\pi(x,t)$, we find

$$v(x,t) = \frac{1}{\rho(x,t)} \sum_{j=1}^{4} v_j(x,t)\rho_j(x,t)P_j \tag{4.31}$$

$$\pi(x,t) + v^2(x,t) = \frac{1}{\rho(x,t)} \sum_{j=1}^{4} (\pi_j(x,t) + v_j^2(x,t))\rho_j(x,t)P_j , \tag{4.32}$$

where v_j and π_j are obtained by setting $v_0 = p_{Cj}$ and $x_C(t) = x_{Cj}(t)$ in the expressions (3.5a)–(3.5b) for the mean velocity and stress-tensor of the quantum diffusion representing a minimum uncertainty wavepacket. Note that this implies π_j does not depend on j.

From (4.31), we obtain after some algebra

$$v(x,t) = S(t)x - \frac{S(t)\sinh(2R(t)\ell x) + R(t)\sin(2S(t)\ell x)}{\cosh(2R(t)\ell x) + \cos(2S(t)\ell x)}\,\ell\,, \tag{4.33}$$

which can also be evaluated directly by differentiating the expression (4.5) for the wavefunction phase $S(x,t)$.

5 Path Observation and Wavefunction Reduction

A key feature of the two-slit diffraction experiment is that the interference fringes disappear when an observation is performed to determine through which slit the particle has passed. For the diffraction of electrons, Feynman proposed in [11, pp. 7–9] to place a light source behind the diffracting screen and use the light scattered by each electron to determine through which of the two slits it crossed. However, light scattering affects the momentum of the electrons, and thus their trajectory. More sophisticated quantum optics experiments have been proposed recently [28] where the determination of the path taken by the particle does not affect its motion. The disappearance of the interference fringes is then attributed to the fact that the wavefunction of a diffracting particle become entangled with that of the measuring device.

In the discussion below we ignore all physical aspects of "which way" experiments and focus exclusively on their statistical interpretation. We need to explain why in the absence of path information the probability density obtained from recording the positions on the detector screen of a large number of particles is given by (4.16), whereas if we observe the path taken, and record the positions on the detector screen of the particles which have crossed through the $+\ell$ (resp. $-\ell$) slit, we obtain the Gaussian wavepacket $\rho_1(x,t)$ (resp. $\rho_2(x,t)$). To do so, we employ the model

$$x(t) = z(t) + x_{CJ}(t) \tag{5.1}$$

of the two-slit interference process, where the zero-mean Gaussian deviation process $z(\cdot)$ is independent of J, and with covariance (3.4).

Suppose that just as the particle has crossed through the diffracting screen, we collect a measurement

$$y = x(0) + n = x_J(0) + z(0) + n \tag{5.2}$$

of the position in the presence of a zero-mean Gaussian random variable n with variance σ_n^2, which represents the measurement error, and is independent of $z(\cdot)$ and J. Since the variance of $z(0)$ is $r^2/2$, given that $J = j$, the probability density of the observation y is

$$p(y|J=j) = N(x_{Cj}(0), \frac{r^2}{2} + \sigma_n^2) \tag{5.3}$$

where

$$x_{C1}(0) = \ell \quad x_{C2}(0) = -\ell \quad x_{C3}(0) = x_{C4}(0) = 0 \tag{5.4}$$

are all real. Suppose that after observing y we seek to determine which trajectory the particle has taken. The trajectory estimate $\hat{J}(y)$ that we select is the one which maximizes the a posteriori probability

$$P(J=j|y) = \frac{p(y|J=j)P_j}{p(y)} \tag{5.5}$$

for $j = 1, \ldots 4$. This is equivalent to maximizing the logarithm of the a-posteriori probability, so that ignoring terms independent of j,

$$\begin{aligned} \hat{J}(y) &= \arg \max_{1 \le j \le 4} \ln p(y|J=j) + \ln P_j \\ &= \arg \min_{1 \le j \le 4} \frac{(y - x_{Cj}(0))^2}{r^2 + 2\sigma_n^2} + \epsilon_j \frac{\ell^2}{r^2} \end{aligned} \tag{5.6}$$

with

$$\epsilon_j = \begin{cases} 0 & \text{for } j = 1,\ 2 \\ 1 & \text{for } j = 3,\ 4\,. \end{cases} \tag{5.7}$$

This gives

$$\hat{J}(y) = \begin{cases} 1 & \text{for } y \ge 0 \\ 2 & \text{for } y \le 0\,, \end{cases} \tag{5.8}$$

so that if the position measurement y is positive we decide the particle has crossed through the top slit, and if $y < 0$, we decide it went through the bottom slit. The suprising aspect of this result is that even though particles following the two interference trajectories $J = 3,\ 4$ contribute to the probability density $\rho(x, 0)$ of $x(0)$, the a-priori probabilities $P_3 = P_4$ of these trajectories are so small that the presence of particles following these two trajectories cannot be detected from the position measurement y, which automatically assigns all such particles to either the top or bottom slit. Note also that this remains true even if the noise variance σ_n^2 is zero, in which case the position observation becomes perfect.

Then, once $\hat{J}(y)$ has been evaluated, the finite joint densities of $x(\cdot)$ are described by the likelihood function $p_{\hat{J}(y)}(x_0, t_0; x_1, t_1; \cdots; x_N, t_N)$, so that $x(\cdot)$ admits the a posteriori model

$$x(t) = \begin{cases} z(t) + x_{C1}(t) & \text{for } \hat{J}(y) = 1 \\ z(t) + x_{C2}(t) & \text{for } \hat{J}(y) = 2\,. \end{cases} \tag{5.9}$$

Thus, given $\hat{J}(y)$, the quantum diffusion describing $x(\cdot)$ is a minimum uncertainty wavepacket centered about either the top or bottom slit, depending on whether $\hat{J}(y) = 1$ or 2. This provides a statistical model explaining the disappearance of the interference fringes after the path followed by each particle has been determined. However, from a physical point of view, the disappearance of the interference fringes remains mysterious. Specifically, we have seen that in the absence of a position measurement, the particles following the interference paths $J = 3,\ 4$ contribute in a crucial way to the appearance of the fringes through the introduction of a negative probability density which destroys the coherent structure of the density components ρ_1 and ρ_2 contributed by the particles going through the two slits. On the other hand, the a-posteriori density (5.9) indicates that the measurement apparatus has the effect of blocking the formation of interference paths, and forces all particles to go through one of the two slits.

6 Quantum Measurements

The description of the measurement process in the previous section was of an ad-hoc nature, and in this section we give a more precise stochastic model of the interaction between a quantum system S and a measuring device M. We only consider the case of position

measurements, where the position $x_S(0)$ is estimated by letting S interact over a brief time interval $[0,\delta]$ with the measurement apparatus M, whose position $x_M(\delta)$ at the end of the interaction is then recorded. The measurement model we employ is similar to the one described in [4, §22]. Prior to $t=0$, it is assumed that the quantum diffusions $x_S(t)$ and $x_M(t)$ describing S and M are independent, so that if

$$x(t) = \begin{bmatrix} x_S(t) \\ x_M(t) \end{bmatrix} \tag{6.1}$$

represents the combined diffusion, its end-point density for $s=t=0^-$ can be factored as

$$q(x,0^-;y,0^-) = q_S(x_S,y_S)q_M(x_M,y_M)\,, \tag{6.2}$$

where q_S and q_M represent respectively the end-point densities of the system S and measurement apparatus M at $s=t=0^-$. We assume for simplicity that x_S and x_M are scalar. Over the interval $[0,\delta]$, an intense interaction takes place between the position variable x_S of S and momentum variable p_M of M. Thus, for $0\le t\le\delta$, the combined Hamiltonian $H=H_S+H_M+H_I$ of S and M is dominated by the interaction term

$$H_I = ax_Sp_M\,, \tag{6.3}$$

where a is a constant. Replacing p_M by $-\nabla_M$ to generate the corresponding operator, we find from (2.18) that over the interval $0\le t\le\delta$, q satisfies the evolution equation

$$\frac{\partial q}{\partial t} = -a(x_S\frac{\partial}{\partial x_M} + y_S\frac{\partial}{\partial y_M})\,q(x,t;y,t)\,, \tag{6.4}$$

where the contributions due to the Hamiltonians H_S and H_M have been neglected.

The solution of (6.4) is given by

$$q(x,t;y,t) = q_S(x_s,y_S)q_M(x_M-atx_S,y_M-aty_S) \tag{6.5}$$

for $0\le t\le\delta$. For simplicity, we assume that $a\delta=1$, and the interaction time δ is very short, so that we may set $\delta=0^+$. Thus, after the interaction between S and M has taken place, their joint end-point density takes the form

$$q(x,0^+;y,0^+) = q_S(x_S,y_S)q_M(x_M-x_S,y_M-y_S)\,, \tag{6.6}$$

which now couples S and M. Setting $x=y$ gives the probability density

$$\rho(x,0^+) = \rho_S(x_S)\rho_M(x_M-x_S) \tag{6.7}$$

of the joint process at $t=0^+$. Similarly, by observing that the function M defined in (2.21) can be decomposed as

$$M(a,b,0^+) = M_S(a_S,b_S) + M_M(a_M-a_S,b_M-b_S)\,, \tag{6.8}$$

we find the mean momentum and stress tensor of the combined process at $t=0^+$ can be expressed in terms of the corresponding quantities for S and M at $t=0^-$ as

$$w(x,0^+) = \begin{bmatrix} w_S(x_S)-w_M(x_M-x_S) \\ w_M(x_M-x_S) \end{bmatrix} \tag{6.9a}$$

$$\pi(x,0^+) = \begin{bmatrix} \pi_S(x_S)+\pi_M(x_M-x_S) & -\pi_M(x_M-x_S) \\ -\pi_M(x_M-s_S) & \pi_M(x_M-x_S) \end{bmatrix}. \tag{6.9b}$$

Furthermore, if the quantum closure rules (2.30a)–(2.30b) are satisfied separately by S and M at $t = 0^-$, and

$$S(x, 0^+) = S_S(x_S) + S_M(x_M - x_S) \tag{6.10}$$

it is easy to verify that the expressions (6.9a)–(6.9b) for the mean momentum and stress tensor can be rewritten as

$$w(x, 0^+) = \begin{bmatrix} \nabla_S \\ \nabla_M \end{bmatrix} S(x, 0^+) \tag{6.11a}$$

$$\pi(x, 0^+) = -\frac{1}{4} \begin{bmatrix} \nabla_S \\ \nabla_M \end{bmatrix} \begin{bmatrix} \nabla_S & \nabla_M \end{bmatrix} \ln \rho(x, 0^+) \,, \tag{6.11b}$$

so that after the interaction between the system S and measurement apparatus M has taken place, their joint evolution still forms a quantum diffusion. Note that the relations (6.7) and (6.10) could have been obtained directly by observing that if

$$\psi_S(x_S) = \exp(R_S + iS_S)(x_M) \tag{6.12a}$$

$$\psi_M(x_M) = \exp(R_M + iS_M)(x_M) \tag{6.12b}$$

denote the wave functions of the system S and measuring device M before their interaction, their joint wave function after the observation takes the form [4]

$$\psi(x) = \psi(x_S)\psi(x_M - x_S) \,. \tag{6.13}$$

For $t \geq 0^+$, the system S and M are dynamically decoupled, and their evolution is governed by the Hamiltonian

$$H(x, p, t) = H_S(x_S, p_S, t) + H_M(x_M, p_M, t) \,, \tag{6.14}$$

so that the Green's function associated to the corresponding operator $\mathbf{H}$ can be factored as

$$G(x, s; y, t) = G_S(x_S, s; y_S, t) G_M(x_M, s; y_M, t) \,. \tag{6.15}$$

However, the result of the interaction between S and M is that these two systems are coupled through the end-point density $q(x, 0^+, y, 0^+)$. This coupling is of course preserved when the end-point density $q(x, 0^+, y, T)$ of the joint system is computed by propagating the reverse heat equation (2.16b) over a finite interval $(0^+, T]$. Thus, the components $x_S(t)$ and $x_M(t)$ of the combined quantum diffusion $x(t)$ are statistically correlated.

According to Theorem 6.1 of [21], the correlation existing between $x_S(t)$ and $x_M(t)$ implies each of them is a reciprocal diffusion, but not of quantum type. Specifically, for $t_0 = 0^+ < t_1 < \ldots < t_N = T$, the marginal finite joint densities of $x_S(t_0), \ \ldots, \ x_S(t_N)$ (resp. $x_M(t_0), \ \ldots, \ x_M(t_N)$) can be expressed as

$$p_S(x_{S0}, t_0; \ldots; x_{SN}, t_N) = q_S(x_{S0}, t_0; x_{SN}, t_N) \prod_{k=0}^{N-1} G_S(x_{Sk}, t_k; x_{Sk+1}, t_{k+1}) \tag{6.16a}$$

$$\begin{aligned} &p_M(x_{M0}, t_0; \ldots; x_{MN}, t_N) \\ &\qquad = q_M(x_{M0}, t_0; x_{MN}, t_N) \prod_{k=0}^{N-1} G_M(x_{Mk}, t_k; x_{Mk+1}, t_{k+1}) \,, \end{aligned} \tag{6.16b}$$

where the end-point densities q_S and q_M are given by

$$q_S(x_{S0}, 0^+; y_{ST}, T) = \int G_M(x_{M0}, 0^+; y_{MT}, T) q(x, 0^+; y, T) dx_{M0} dy_{MT} \tag{6.17a}$$

$$q_M(x_{M0}, 0^+; y_{MT}, T) = \int G_S(x_{S0}, 0^+; y_{ST}, T) q(x, 0^+; y, T) dx_{S0} dy_{ST} \, . \tag{6.17b}$$

However, the reciprocal diffusions specified by the above statistics do not obey the closure rules (2.30a)–(2.30b). This indicates that after the system S has interacted with M, the two become inexorably intertwined, and one cannot be analyzed without the other. However, it would be rather complicated if, to analyze S after a measurement takes place, we needed to consider its joint evolution with M, even though the two systems have no further interaction for $t \geq 0^+$. It turns out that, if the initial wavefunction of the measurement device M is a minimum uncertainty Gaussian wavepacket, only the measurement result $x_M(0^+)$ is needed to analyze $x_S(\cdot)$ for $t \geq 0^+$.

Theorem 6.1 *Assume that the initial wavefunction of M is a minimum uncertainty Gaussian wavepacket, so that in (6.12b), we have*

$$R_M(x_M) = -\frac{R_M}{2}(x_M - \alpha_M)^2 + \beta_M \tag{6.18a}$$

$$S_M(x_M) = \gamma_M x_M \, , \tag{6.18b}$$

where α_M, β_M, γ_M and R_M are constants. Then, for $t \geq 0^+$ $x_S(\cdot)$ and $x_M(\cdot)$ are conditionally independent given the initial value $x(0^+)$ of the combined process. Furthermore, conditioned on the measurement $x_M(0^+)$, $x_S(\cdot)$ is a quantum diffusion.

Proof: The end-point density q_M corresponding to the Gaussian wavepacket (6.18a)–(6.18b) takes the form

$$q_M(x_M, y_M) = \exp\Big(-R_M(x_M - \alpha_M)(y_M - \alpha_M) - \gamma_M(y_M - x_M) + \beta_Q\Big) \, , \tag{6.19}$$

with β_Q constant. This implies that the joint end-point density (6.6) for S and M at $t = 0^+$, can be factored as

$$q(x, 0^+; y, 0^+) = f_S(x, 0^+; y_S, 0^+) f_M(x, 0^+; y_M, 0^+) \, , \tag{6.20}$$

with

$$f_S(x, 0^+; y_S, 0^+) = q_S(x_S, y_S) \exp\Big(R_M(x_M - x_S - \alpha_M) y_S + \gamma_M(y_S - x_S)\Big) \tag{6.21a}$$

$$\begin{aligned} &f_M(x, 0^+; y_M, 0^+) \\ &\quad = \exp\Big(-R_M(x_M - x_S - b_M)(y_M - \alpha_M) - \gamma_M(y_M - x_M) + \beta_Q\Big) \, . \end{aligned} \tag{6.21b}$$

Since the dynamics of S and M are decoupled for $t \geq 0^+$, the end-point density $q(x, 0^+; y, T)$ obtained by propagating (2.16b) with initial condition (6.20) can also be decomposed as

$$q(x, 0^+; y, T) = f_S(x, 0^+; y_S, T) f_M(x, 0^+; y_M, T) \, . \tag{6.22}$$

Substituting this expression and the factored form (6.15) for the Green's function of the joint system formed by S and M for $t \geq 0^+$, it is easy to verify that for $t_0 = 0^+ < t_1 < \ldots < t_N = T$, the finite joint densities of $x(t_0),\ x(t_1),\ \ldots,\ x(t_N)$ can be factored as

$$\begin{aligned} p(x_0, t_0; \ldots; x_N, t_N) &= p_{S \mid x(0^+)}(x_{S1}, t_1; \ldots; x_{SN}, t_N \mid x_0) \\ & \quad p_{M \mid x(0^+)}(x_{M1}, t_1; \ldots; x_{MN}, t_N \mid x_0) \rho(x_0, 0^+) \,, \end{aligned} \tag{6.23}$$

where

$$\begin{aligned} & p_{S \mid x(0^+)}(x_{S1}, t_1; \ldots; x_{SN}, t_N \mid x0) \\ & \quad = c_S(x_0) f_S(x_0, 0^+; y_{SN}, t_N) \prod_{k=0}^{N-1} G_S(x_{Sk}, t_k; x_{Sk+1}, t_{k+1}) \end{aligned} \tag{6.24a}$$

$$\begin{aligned} & p_{M \mid x(0^+)}(x_{M1}, t_1; \ldots; x_{MN}, t_N \mid x0) \\ & \quad = c_M(x_0) f_M(x_0, 0^+; y_{MN}, t_N) \prod_{k=0}^{N-1} G_M(x_{Mk}, t_k; x_{Mk+1}, t_{k+1}) \end{aligned} \tag{6.24b}$$

are the finite joint densities of $x_S(\cdot)$ and $x_M(\cdot)$ conditioned on $x(0^+)$. In (6.24a)–(6.24b), c_S and c_M are functions of x_0 only, which satisfy $(c_S c_M)(x)\rho(x, 0^+) = 1$. The factored form (6.23) for the joint densities of $x_S(\cdot)$ and $x_M(\cdot)$ implies they are conditionally independent given the initial value $x(0^+)$ of the joint process.

As a consequence, the conditional density of $x_S(\cdot)$ given $x_M(\cdot)$ is identical to the conditional density of $x_S(\cdot)$ given $x_M(0^+)$, so that we only need the measurement $x_M(0^+)$ to analyze S for $t \geq 0^+$. From (6.7), we see that the conditional density of $x_S(0^+)$ given $x_M(0^+) = x_M$ can be expressed as

$$\rho_{S|M}(x_S, 0^+ \mid x_M) = \rho_S(x_S) \exp(-R_M(x_S - x_M - \alpha_M)^2) / \rho_M(x_M, 0^+) \,, \tag{6.25}$$

where $\rho_M(x_M, 0^+)$ denotes the marginal density of $x_M(0^+)$. Conditioned on $x_M(0^+) = x_M$, $x_S(\cdot)$ forms a reciprocal diffusion with Hamiltonian H_S and end-point density

$$q_{S|M}(x_S, 0^+; y_S, 0^+ \mid x_M) = q(x_S, x_M, 0^+; y_S, x_M, 0^+) / \rho_M(x_M, 0^+) \,. \tag{6.26}$$

Taking into account the form (6.6) for the joint end-point density $q(x, 0^+; y, 0^+)$ at the end of the measurement, we find that the function $M_{S|M}$ corresponding to the conditional end-point density $q_{S|M}$ can be expressed as

$$\begin{aligned} M_{S|M}(a_S, b_S, 0^+ \mid x_M) &= M_S(a_S, b_S) \\ & \quad -R_M[(a_S - x_M + \alpha_M)^2 - b_S^2] - \gamma_M(a_S + b_S) + d(x_M) \,, \end{aligned} \tag{6.27}$$

where $d(x_M)$ depends on x_M only. According to (2.22a)–(2.22b), the conditional mean momentum and stress tensor are given by

$$w_{S|M}(x_S, 0^+ \mid x_M) = w_S(x_S) - \gamma_M \tag{6.28a}$$

$$\pi_{S|M}(x_S, 0^+ \mid x_M) = \pi_S(x_S) + \frac{R_M}{2} \,. \tag{6.28b}$$

Using the expression (6.25) for the conditional density $\rho_{S|M}$, it is then easy to verify that, provided $x_S(\cdot)$ is a quantum diffusion before the measurement, for $t \geq 0^+$, conditioned on the measurement $x_M(0^+)$, it is also a quantum diffusion, i.e., it obeys the closure rules (2.30a)–(2.30b). Specifically, with

$$S_{S|M}(x_S, 0^+ \mid x_M) = S_S(x_S) - \gamma_M x_S \,, \tag{6.29}$$

we have

$$w_{S|M}(x_S, 0^+ \mid x_M) = \nabla_S S_{S|M}(x_S, 0^+ \mid x_M) \tag{6.30a}$$

$$\pi_{S|M}(x_S, 0^+ \mid x_M) = -\frac{1}{4}\nabla_S^2 \ln \rho_{S|M}(x_S, 0^+ \mid x_M) \,. \tag{6.30b}$$

□

The $R_M/2$ term which is added to the a priori stress tensor π_S to obtain the a posteriori stress tensor $\pi_{S|M}$ in (6.28b) admits a simple interpretation. Specifically, the Gaussian wavepacket (6.18a)–(6.18b) which is employed to measure the position of S has for variance $(2R_M)^{-1}$. Because of Heisenberg's uncertainty principle, our improved knowledge of the position of S comes at the price of an increase of $R_M/2$ in the variance of its momentum.

Example 6.1 To illustrate the above results, consider the case where S is a free particle, whose initial wavefunction is itself a minimum uncertainty wavepacket, so that in (6.12b) we have

$$R_S(x_S) = -\frac{R_S}{2}(x_S - \alpha_S)^2 + \beta_S \tag{6.31b}$$

$$S_S(x_S) = \gamma_S x_S \,. \tag{6.31b}$$

In (6.31b)–(6.31b), the parameters α_S and γ_S specify the a priori classical trajectory

$$x_{CS}(t) = \alpha_S + \gamma_S t \tag{6.32}$$

about which S would evolve in the absence of measurement. By substituting the Gaussian densities for the S and M wavepackets inside the expression (6.7), we find that after the measurement, $x(0^+)$ is Gaussian with mean

$$m = \begin{bmatrix} \alpha_S \\ \alpha_S + \alpha_M \end{bmatrix} \tag{6.33a}$$

and variance

$$K = \frac{1}{2}\begin{bmatrix} r_S^2 & r_S^2 \\ r_S^2 & r_S^2 + r_M^2 \end{bmatrix} \,, \tag{6.33b}$$

where

$$r_S^2 = R_S^{-1} \qquad r_M^2 = R_M^{-1} \tag{6.34}$$

are the a priori r^2 minimum uncertainty wavepacket parameters for S and M, respectively. Then, from (6.6) and (6.26), the conditional end-point density of $x_S(\cdot)$ given $x_M(0^+) = x_M$ can be expressed as

$$q_{S|M}(x_S, 0^+; y_S, 0^+ \mid x_M) = \exp\Big(R_{S|M}(x_S - \alpha_{S|M})(y_S - \alpha_{S|M}) - \gamma_{S|M}(y_S - x_S) + \beta_{S|M}\Big) \,, \tag{6.35}$$

with

$$R_{S|M} = R_S + R_M \tag{6.36a}$$
$$\gamma_{S|M} = \gamma_S - \gamma_M \tag{6.36b}$$
$$R_{S|M}\alpha_{S|M} = R_S\alpha_S + R_M(x_M - \alpha_M)\,. \tag{6.36c}$$

In this expression, $\alpha_{S|M}$ is the conditional mean of $x_S(0^+)$ given the measurement $x_M(0^+) = x_M$ and $\gamma_{S|M}$ is the new classical momentum for S, which includes a component $-\gamma_M$ due to the interaction between S and M. But in (6.35) we immediatedly recogize the end-point density for a minimum uncertainty wavepacket centered about the a posteriori classical trajectory

$$x_{CS|M}(t) = \alpha_{S|M} + \gamma_{S|M}t\,, \tag{6.37}$$

and with parameter

$$r^2_{S|M} = [r_S^{-2} + r_M^{-2}]^{-1} \tag{6.38}$$

so that conditioned on $x_M(0^+) = x_M$, $x_S(t)$ is a Gaussian process with mean $x_{CS|M}(t)$ and covariance (3.4) with $r^2 = r^2_{S|M}$. Note that (6.36c) and (6.38) are the standard expressions for the conditional mean and variance of a random variable $x_S(0) \sim N(\alpha_S, r_S^2/2)$ given a measurement

$$x_M(0) = \alpha_M + x_S(0) + n_M\,, \tag{6.39}$$

where the error $n_M \sim N(0, r_M^2/2)$ is independent of $x_S(0)$. □

An interesting aspect of the above example is that the measurement process affects the classical trajectory about which the wavepacket evolves: the a priori classical trajectory $x_{CS}(t)$ given by (6.32) is replaced by the a posteriori trajectory $x_{CS|M}(t)$ specified by (6.37). This may explains why in the two-slit diffraction experiment, the interference trajectories vanish when a measurement device is employed to determine the path followed by particles. Specifically, for a particle following an interference trajectory, the position estimate generated by the measuring device is necessarily $\pm\ell$ (the interference trajectories are not detectable), so that the a-posteriori classical trajectory is required to go through either the top or bottom slit, thus eliminating all interference trajectories.

Example 6.2 Assume now that S is a harmonic oscillator with potentials

$$A_S(x_S, t) = 0 \quad , \quad \phi_S(x_S, t) = (\omega x_S)^2/2\,, \tag{6.40}$$

which is initially in its coherent state, i.e.,

$$R_S(x_S) = -\frac{\omega}{2}(x - \alpha_S)^2 + \beta_S \quad , \quad S_x(x_S) = 0\,. \tag{6.41}$$

For simplicity we assume that $\alpha_M = \gamma_M = 0$ in the measuring Gaussian wavepacket. Because the initial wavepacket (6.41) has the same form as in Example 6.1, the conditional end-point density $q_{S|M}$ admits the form (6.35), with $\gamma_{S|M} = 0$, where $R_M = \omega$ in the expressions (6.36a) and (6.36c) for $R_{S|M}$ and $\alpha_{S|M}$. Let now

$$x_{CS|M}(t) = \alpha_{S|M}\cos(\omega t) \quad , \quad p_{CS|M}(t) = -\omega\alpha_{S|M}\sin(\omega t) \tag{6.42}$$

be the a posteriori classical trajectory and the corresponding momentum. Based on the results of [21, §4], the end-point density $q_{S|M}(x_S, 0^+; y_S, T \mid x_M)$ admits the structure

$$\begin{aligned}
&\ln q_{S|M}(x_S, 0^+; y_S, T) \\
&= -\frac{1}{2}\left[\begin{array}{cc} x_S - x_{CS|M}(0^+) & y_S - x_{CS|M}(T) \end{array}\right] Q_{S|M}(0,T) \left[\begin{array}{c} x_S - x_{CS|M}(0^+) \\ y_S - x_{CS|M}(T) \end{array}\right] \\
&\quad - \left[\begin{array}{cc} x_S - x_{CS|M}(0^+) & y_S - x_{CS|M}(T) \end{array}\right] \left[\begin{array}{c} p_{CS|M}(0^+) \\ -p_{CS|M}(T) \end{array}\right] + \beta_{S|M}(0,T)\,, \qquad (6.43)
\end{aligned}$$

where the 2×2 matrix $Q_{S|M}(0,t)$ satisfies the Riccati equation

$$\frac{dQ_{S|M}}{dt}(0,t) = Q_{S|M} \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right] Q_{S|M} + \left[\begin{array}{cc} 0 & 0 \\ 0 & \omega^2 \end{array}\right] \qquad (6.44a)$$

with initial condition

$$Q_{S|M}(0,0) = \left[\begin{array}{cc} 0 & R_{S|M} \\ R_{S|M} & 0 \end{array}\right]. \qquad (6.44b)$$

This gives

$$Q_{S|M}(0,T) = \frac{R_{S|M}}{\cos(\omega T)} \left[\begin{array}{cc} \kappa^{-1}\sin(\omega T) & 1 \\ 1 & \kappa \sin(\omega T) \end{array}\right], \qquad (6.45a)$$

where

$$\kappa = \omega / R_{S|M} \qquad (6.45b)$$

represents the percentage decrease in the variance of $x_S(0)$ brought about by the observation of $x_M(0^+)$.

As shown in [21], the Green's function of the harmonic oscillator with potentials (6.40) is given by

$$G_S(x_S, s; y_S, t) = C(t-s) \exp\Big(-\frac{\omega}{2\sin(\omega(t-s))}[(x^2+y^2)\cos(\omega(t-s)) - 2xy]\Big) \qquad (6.46a)$$

with

$$C(t-s) = \left(\frac{\omega}{2\pi \sin(\omega(t-s))}\right)^{1/2} \qquad (6.46b)$$

for $t \geq s$. Note that the requirement that G_S should decay as $|y| \to \infty$ implies it is defined only for $t - s < \pi/2\omega$, which corresponds to one quarter of the period of the harmonic oscillator. Then, the a posteriori joint density of $x_S(0^+)$ and $x_S(T)$ given $x_M(0^+) = x_M$ can be expressed as

$$\begin{aligned}
p_{S|M}(x_S, 0^+; y_S, T \mid x_M) &= G_S(x_S, 0; y_S, T)\, q_{S|M}(x_S, 0^+; y_S, T \mid x_M) \\
&= N\left(\left[\begin{array}{c} x_{CS|M}(0^+) \\ x_{CS|M}(T) \end{array}\right], \mathbf{P}_{S|M}(0,T)\right), \qquad (6.47)
\end{aligned}$$

where

$$\mathbf{P}_{S|M}(0,T) = \frac{1}{2\omega}\left[\begin{array}{cc} \kappa & \kappa\cos(\omega T) - \sin(\omega T) \\ \kappa\cos(\omega T) - \sin(\omega T) & \kappa\cos^2(\omega T) + \kappa^{-1}\sin^2(\omega T) \end{array}\right] \qquad (6.48)$$

denotes the conditional covariance matrix of $x_S(0^+)$ and $x_S(T)$. In particular, when $T = \pi/2\omega$ is one quarter of a period of the harmonic oscillator, $\mathbf{P}_{S|M}(0,T)$ becomes singular, and if $z(t) = x_S(t) - x_{CS|M}(t)$ represents the deviation of $x_S(\cdot)$ from its a posteriori classical trajectory, we have

$$z(0) = -\kappa z(\pi/2\omega) \sim N(0, \frac{\kappa}{2\omega}) \,. \tag{6.49}$$

To obtain a complete statistical description of $z(t)$, note from the characterization of Gaussian reciprocal diffusions given in [19], that it satisfies the second-order stochastic differential equation

$$\begin{aligned} \mathcal{L}_H z(t) &= \xi(t) && \text{(6.50a)} \\ &\triangleq -\frac{d^2}{dt^2} - \omega^2 && \text{(6.50b)} \end{aligned}$$

with boundary conditions (6.49), where the noise $\xi(t)$ is a generalized Gaussian process independent of $z(0)$, with zero mean and covariance

$$E[\xi(t)\xi(s)] = \mathcal{L}_H \delta(t-s) \,. \tag{6.51}$$

The Green's function of the operator $\mathcal{L}_H$ with homogeneous Dirichlet conditions at $t = 0$ and $t = T = \pi/2\omega$ is given by

$$\Gamma_H(t,s) = \begin{cases} \frac{1}{\omega}\cos(\omega t)\sin(\omega s) & \text{for } \; t \geq s \\ \frac{1}{\omega}\sin(\omega t)\cos(\omega s) & \text{for } \; s \geq t \,. \end{cases} \tag{6.52}$$

As shown in [19], the solution of the stochastic boundary value problem (BVP) (6.49)–(6.51) is given by

$$z(t) = I(t) + \cos(\omega t) z(0) + \sin(\omega t) z(T) \,, \tag{6.53a}$$

where

$$I(t) = \int_0^T \Gamma_H(t,s)\xi(s)ds \tag{6.53b}$$

is a zero mean Gaussian process with covariance $\Gamma_H(t,s)$. Then, by using (6.49), and (6.52), we find that the conditional covariance of $z(t)$ is given by

$$K_{S|M}(t,s) = \frac{1}{2\omega}[\kappa\cos(\omega t)\cos(\omega s) + \kappa^{-1}\sin(\omega t)\sin(\omega s) - \sin(\omega|t-s|)] \,. \tag{6.54}$$

In the absence of measurement, i.e. when $\kappa = 1$, it reduces to the covariance

$$K_S(t,s) = \frac{1}{2\omega}(\cos(\omega|t-s|) - \sin(\omega|t-s|)) \tag{6.55}$$

of the *shifted cosine process* [7] which models the coherent state of the harmonic oscillator. This process is stationary, but the a posteriori covariance $K_{S|M}(t,s)$ is not stationary. In fact, the variance of $z(t)$ is given by

$$K_{S|M}(t,t) = \frac{1}{2\omega}(\kappa\cos^2(\omega t) + \kappa^{-1}\sin^2(\omega t)) \,, \tag{6.56}$$

which can be recognized as corresponding to a "squeezed state" of the harmonic oscillator (see [6, pp. 103–104]). At the times $t = n\pi/\omega$ with n integer when the classical trajectory $x_{CS|M}(t)$ reaches an extremum, the variance of the deviation process $z(t)$ equals $\kappa(2\omega)^{-1}$, and is thus "squeezed" below the variance $(2\omega)^{-1}$ of the coherent state. On the other hand, at the times $\pi/2\omega + n\pi/\omega$ when the classical trajectory passes through the origin, and the classical momentum reaches an extremum, the variance of the deviation process equals $\kappa^{-1}(2\omega)^{-1}$, which "stretches" the coherent state value.

In summary, after the coherent state of the harmonic oscillator is measured, the classical trajectory about which the wavepacket evolves is shifted from $x_{CS}(t)$ to $x_{CS|M}(t)$, to reflect the measured value of the initial position, and the state is "squeezed," to reflect the decrease in the position uncertainty due to the measurement. Unfortunately, because of the uncertainty principle, the improvement in the knowledge of the position comes at the price of a greater uncertainty in the momentum, which causes a stretching of the wavepacket when it passes though the origin.

Since the squeezed state process is Gaussian, its mean velocity and stress tensor take the form [19, 21]

$$v_{S|M}(x_S, t \mid x_M) = p_{CS|M}(t) + V_{S|M}(t)(x_S - x_{CS|M}(t)) \tag{6.57a}$$

$$\pi_{S|M}(x_S, t \mid x_M) = \pi_{S|M}(t)\,, \tag{6.57b}$$

where $V_{S|M}(t)$ and $\pi_{S|M}(t)$ are functions of t only, which can be expressed in terms of the covariance $K_{S|M}$ as

$$V_{S|M}(t) = \frac{1}{2}\left(\frac{\partial K_{S|M}}{\partial t}(t^+, t) + \frac{\partial K_{S|M}}{\partial t}(t^-, t)\right) K_{S|M}^{-1}(t, t) \tag{6.58a}$$

$$\pi_{S|M}(t) = \frac{1}{2}\left(\frac{\partial^2 K_{S|M}}{\partial t \partial s}(t^+, t) + \frac{\partial^2 K_{S|M}}{\partial t \partial s}(t^-, t)\right) - K_{S|M}(t, t) V_{S|M}^2(t) \tag{6.58b}$$

This gives

$$V_{S|M}(t) K_{S|M}(t, t) = \frac{(\kappa^{-1} - \kappa)}{2} \sin(\omega t)\cos(\omega t) \tag{6.59b}$$

$$\pi_{S|M}(t) = \frac{\omega}{2(\kappa \cos^2(\omega t) + \kappa^{-1}\sin^2(\omega t))} = \frac{K_{S|M}^{-1}(t, t)}{4}\,, \tag{6.59b}$$

where the last equality indicates that the closure rule (2.30b) holds, so that the squeezed state process is a quantum diffusion, as expected.

Finally, note that the BVP (6.49)–(6.51) defines the squeezed process $z(\cdot)$ for only one quarter of the period of the harmonic oscillator. To construct this process over longer periods of time, we may set $z(0) = z(\pi/\omega)$ in the boundary condition (6.49) and use it it to specify a stochastic BVP for the next quarter of a period. Thereafter, we may set $z(t) = z(t + \pi/\omega)$, so that the squeezed state process has half the oscillator's period. □

7 Conclusions

By employing a form of stochastic mechanics introduced in [21], reciprocal quantum diffusion models have been constructed for the two-slit particle diffraction experiment, and

for quantum measurements. The two-slit model was of compound type, in the sense it was formed by a pair $(x(t), J)$ representing a position coordinate and a discrete path index. This model was used to explain the disappearance of interference fringes when "which path" information becomes available. The key aspect of the measurement model was that, conditioned on the recorded measurement, the observed system still evolves according to a quantum diffusion. The disappearance of interference trajectories when the path taken by each diffracting particle is estimated, and the readjustment of the classical trajectories about which the wavepackets of Examples 6.1 and 6.2 evolve, after a measurement has been performed, strongly suggest that quantum mechanics is purely an information based theory, where physical reality does not exist independently of the information available to an observer.

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