Old and New Results in Robust Hypothesis Testing

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Outline

- **Binary Hypothesis Testing**
- Robust Hypothesis Testing
- Huber's Clipped LR Test
- Robustness with a KL Divergence Tolerance
- Simulations

Binary Hypothesis Testing

- Consider observation $Y \in \mathbb{R}$ where under hypothesis H_0, Y has probability density $f_0(y)$ and under H_1, it has density $f_1(y)$.
- Given Y, we need to decide between H_1 or H_0. We use a randomized decision rule $\delta \in \mathcal{D}$, where given $Y = y$, we select H_1 with probability $\delta(y)$ and H_0 with probability $1 - \delta(y)$, where $0 \leq \delta(y) \leq 1$ for all $y \in \mathbb{R}$. Note that set $\mathcal{D}$ is convex.
- Bayesian hypothesis testing assumes a priori probabilities

$$\pi_0 = P[H_0] \quad , \quad \pi_1 = 1 - \pi_0 = P[H_1]$$

and costs C_M and C_F for a miss (deciding H_0 when H_1 holds) and a false alarm (deciding H_1 when H_0 holds), respectively.

Binary hypothesis testing(cont'd)

Let

$$
\begin{aligned}
P_F(\delta, f_0) &= \int_{-\infty}^{\infty} \delta(y) f_0(y) dy \\
P_M(\delta, f_1) &= \int_{-\infty}^{\infty} (1 - \delta(y)) f_1(y) dy
\end{aligned}
$$

denote the probability of false alarm and of a miss under H_0 and H_1, respectively. The optimal Bayesian test minimizes the risk

$$
\begin{aligned}
R(\delta, f_0, f_1) &= C_F P_F(\delta, f_0)\pi_0 + C_M P_M(\delta, f_1)\pi_1 \\
&= C_M \pi_1 + \int_{-\infty}^{\infty} \delta(y)[C_F \pi_0 f_0(y) - C_M \pi_1 f_1(y)] dy \, .
\end{aligned}
$$

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Binary hypothesis testing(cont'd)

Optimal Bayesian test: Let $L(y) = f_1(y)/f_0(y)$ = likelihood ratio (LR) and $\tau_B = C_F\pi_0/(C_M\pi_1)$. The test minimizing the Bayesian risk is given by

$$\delta(y) = \begin{cases} 1 & L(y) > \tau_B \\ 0 & L(y) < \tau_B \\ \text{arbitrary} & L(y) = \tau_B \ , \end{cases}$$

and randomization is not needed.

Neyman-Pearson test (of type I): Minimizes $P_M(\delta, f_1)$ under the constraint $P_F(\delta, f_0) \leq \alpha$. Solution:

$$\delta(y) = \begin{cases} 1 & L(y) > \tau \\ 0 & L(y) < \tau \\ p & L(y) = \tau \ . \end{cases}$$

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Binary hypothesis testing(cont'd)

- The threshold τ and randomization probability p are selected as follows. Let $F_L(\ell|H_0) = P[L \leq \ell|H_0]$ denote the cumulative probability distribution of likelihood ratio L under H_0. Then $F_L(\tau|H_0) = 1 - \alpha$ and $p = 0$ if $1 - \alpha$ is in the range of $F_L(\ell|H_0)$, and if

$$F_L(\tau_-|H_0) < 1 - \alpha < F_L(\tau|H_0)$$

then

$$p = \frac{F_L(\tau|H_0) - (1 - \alpha)}{F_L(\tau|H_0) - F_L(\tau_-|H_0)}$$

- Both the Bayesian and NP tests rely on the LR function $L(y)$. Only the threshold selection changes.

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Robust Hypothesis Testing

- The actual probability densities g_0 and g_1 of observation Y under H_0 and H_1 may differ slightly from the nominal densities f_0 and f_1. Assume $g_j \in \mathcal{F}_j$, where $\mathcal{F}_j$ denotes a convex neighborhood of f_j for $j = 0,\ 1$.
- Let $\mathcal{F} = \mathcal{F}_0 \times \mathcal{F}_1$. The **robust Bayesian hypothesis problem** can be expressed as

$$\min_{\delta \in \mathcal{D}} \max_{(g_0, g_1) \in \mathcal{F}} R(\delta, g_0, g_1)\ .$$

Since $R(\delta, g_0, g_1)$ is separately linear with respect to δ, and (g_0, g_1), the min-max problem has a convex-concave structure. For appropriate choices of metrics, $\mathcal{D}$ and $\mathcal{F}$ are compact, so by Von-Neumann's minimax theorem, there exists a saddle point (δ_R, g_0^L, g_1^L) satisfying

$$R(\delta_R, g_0, g_1) \leq R(\delta_R, g_0^L, g_1^L) \leq R(\delta, g_0^L, g_1^L)\ . \qquad (1)$$

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Robust Hypothesis Testing (cont'd)

- Here δ_R = robust test, and (g_0^L, g_1^L) = least-favorable densities. The second inequality in (1) implies δ_R is the optimum Bayesian test for the pair (g_0^L, g_1^L), so δ_R can be expressed as the LR test

$$L_L(y) = \frac{g_1^L(y)}{g_0^L(y)} \underset{H_0}{\overset{H_1}{\gtrless}} \tau_B \ .$$

- Since $R(\delta, g_0, g_1)$ is a fixed linear combination of $P_M(\delta, g_1)$ and $P_F(\delta, g_0)$, the first inequality in (1) is equivalent to

$$P_F(\delta_R, g_0) \leq P_F(\delta_R, g_0^L) \quad , P_M(\delta_R, g_1) \leq P_M(\delta_R, g_1^L) \tag{2}$$

for all $g_0 \in \mathcal{F}_0$ and $g_1 \in \mathcal{F}_1$.

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Robust Hypothesis Testing (cont'd)

- The **robust NP test** solves

$$\min_{\delta \in \mathcal{D}_\alpha} \max_{g_1 \in \mathcal{F}_1} P_M(\delta, g_1) \ , \tag{3}$$

where

$$\mathcal{D}_\alpha = \{\delta \in \mathcal{D} : \max_{g_0 \in \mathcal{F}_0} P_F(\delta, g_0)\}$$

is the set of decision rules of size less than α. Since $P_F(\delta, g_0)$ is a convex function of δ for each $g_0 \in \mathcal{F}_0$, so is

$$\max_{g_0 \in \mathcal{F}_0} P_F(\delta, g_0) \ ,$$

hence $\mathcal{D}_\alpha$ is convex.

- The cost function $P_F(\delta, g_1)$ has a convex concave structure, so a saddle point exist, and δ_R is the optimal NP test for least favorable observation densities (g_0^L, g_1^L).

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Huber's Clipped LR Test

- Different choices of neighborhoods $\mathcal{F}_j$ yield different robust tests. Let $G_j(y)$ and $F_j(y)$ denote the cumulative probability distribution functions corresponding to the actual and nominal densities $g_j(y)$ and $f_j(y)$ for $j = 0,\ 1$. For some numbers $0 \leq \epsilon_0, \epsilon_1, \nu_0, \nu_1 < 1$, Huber considered neighborhoods

$$\begin{aligned} \mathcal{F}_0 &= \{g_0 \, : \, G_0(y) \geq (1-\epsilon_0)F_0(y) - \nu_0 \text{ for all } y \in \mathbb{R}\} \\ \mathcal{F}_1 &= \{g_1 \, : \, 1 - G_1(y) \geq (1-\epsilon_1)(1-F_1(y)) - \nu_1 \text{ for all } y \in \mathbb{R}\} \,. \end{aligned}$$

- The constraints specifying $\mathcal{F}_j$ are linear in g_j, so the neighborhoods are *convex*.

- Since functions $P_M(\delta_R, g_1)$ and $P_F(\delta_R, g_0)$ are linear in g_1 and g_0, the maximization (2) for the least-favorable densities is a linear programming problem, so solutions will be located on the boundary of $\mathcal{F}_j$.

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Huber's Clipped LR Test (cont'd)

Least-favorable densities: There exists $I = [y_L, y_U]$ such that over this interval

$$\begin{aligned} G_0^L(y) &= (1-\epsilon_0)F_0(y) - \nu_0 \\ G_1^L(y) &= (1-\epsilon_1)F_1(y) + \epsilon_1 + \nu_1 \ , \end{aligned}$$

so the least-favorable densities are on the boundary of sets $\mathcal{F}_0$ and $\mathcal{F}_1$. For $j = 0,\ 1$, this implies

$$g_j^L(y) = (1-\epsilon_j)f_j(y)$$

over I. Let

$$\begin{aligned} a(y) &= v' f_0(y) + w' f_1(y) \\ b(y) &= v'' f_0(y) + w'' f_1(y) \ , \end{aligned}$$

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Huber's Clipped LR Test (cont'd)

with

$$v' = \frac{\epsilon_1 + \nu_1}{1 - \epsilon_1} \quad , \quad v'' = \frac{\epsilon_0 + \nu_0}{1 - \epsilon_0}$$

$$w' = \frac{\nu_0}{1 - \epsilon_0} \quad , \quad w'' = \frac{\nu_1}{1 - \epsilon_1} \; .$$

Let $\ell_L = L(y_L)$, $\ell_U = L(y_U)$. Then

$$g_j^L(y) = c_j a(y) \quad , \quad y \leq y_L$$

$$g_j^L(y) = d_j b(y) \quad , \quad y \geq y_U \; ,$$

with

$$\frac{c_1}{c_0} = \frac{1 - \epsilon_1}{1 - \epsilon_0} \ell_L \quad , \quad \frac{d_1}{d_0} = \frac{1 - \epsilon_1}{1 - \epsilon_0} \ell_U \; .$$

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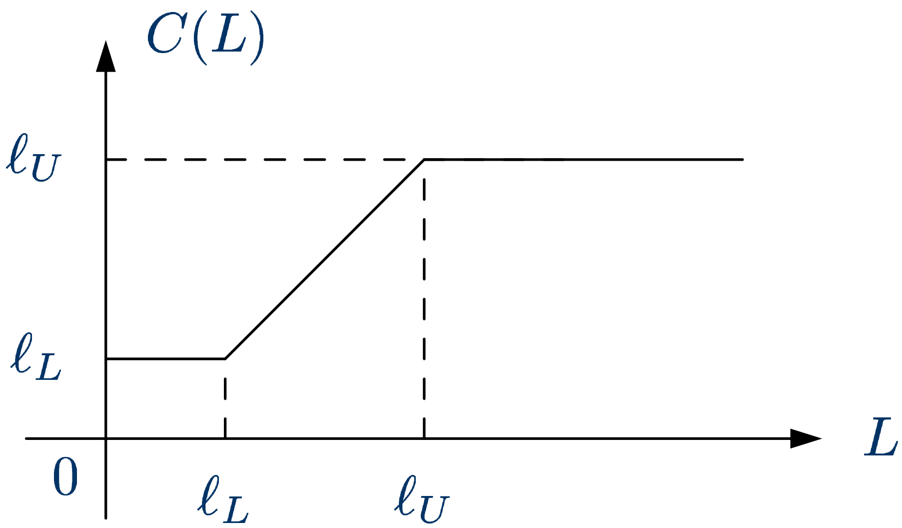
Huber's Clipped LR Test (cont'd)

Clipping transformation: The least-favorable LR can be expressed as

$$L_L(y) = \frac{g_1^L(y)}{g_0^L(y)} = \frac{1-\epsilon_1}{1-\epsilon_0} C(L(y))$$

where the clipping nonlinearity $C(\cdot)$ is shown below



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Huber's clipped LR test (cont'd)

Robust test: The decision rule

$$L_L(y) \underset{H_0}{\overset{H_1}{\gtrless}} \tau_B$$

can be rewritten as

$$C(L(y)) \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \frac{1-\epsilon_0}{1-\epsilon_1}\tau_B \ .$$

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Huber's clipped LR test (cont'd)

- For $\nu_0 = \nu_1 = 0$, the LF distributions belong to the *contamination class*

$$\mathcal{N}_j^C = \{g_j \,:\, G_j(y) = (1-\epsilon_j)F_j(y) + \epsilon_j H(y) \text{ for all } y\}$$

contained in $\mathcal{F}_j$, where $H(y)$ = arbitrary probability distribution.

- For $\epsilon_0 = \epsilon_1 = 0$, the LF densities belong to the *total variation class*

$$\mathcal{N}_j^{TV} = \{g_j \,:\, |g_j - f_j|_1 \leq 2\nu_j\}$$

contained in $\mathcal{F}_j$.

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Robustness with a KL Tolerance

- For $j = 0,\ 1$ consider neighborhoods

$$\mathcal{F}_j = \{g_j \ : \ D(g_j | f_j) \leq \epsilon\}$$

where

$$D(g|f) = \int_{-\infty}^{\infty} \ln(g(y)/f(y))g(y)dy$$

is the Kullback-Leibler divergence or relative entropy of density g with respect to f.

- $D(g|f)$ is convex in g, so $\mathcal{F}_j$ is convex. $D(g|f)$ is not a true distance, since it is not symmetric ($D(f|g) \neq D(g|f)$) and does not satisfy the triangle inequality. But $D(g|f) \geq 0$ with equality if and only if $g = f$.
- $D(g|f)$ and its dual $D^*(g|f) = D(f|g)$ admit a non-Riemannian differential geometric interpretation in terms of dual connections.

Robustness with a KL Tolerance

Assumptions:

i) The nominal LR $L(y) = f_1(y)/f_0(y)$ is monotone increasing in y.

ii) $f_1(y) = f_0(-y)$.

iii) $0 < \epsilon < D(f_{1/2}|f_0)$, where $f_{1/2}(y)$ is the mid-way density on the geodesic

$$f_u(y) = \frac{f_0^{1-u}(y) f_1^u(y)}{Z(u)}$$

linking f_0 and f_1. Here

$$Z(u) = \int_{-\infty}^{\infty} f_1^u(y) f_0^{1-u}(y) dy$$

= normalization constant.

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Robustness with a KL Tolerance (cont'd)

Robust test and LF densities: For a minimum probability of error criterion ($C_F = C_M = 1$) and equally likely hypotheses, there exists $y_U > 0$ such that

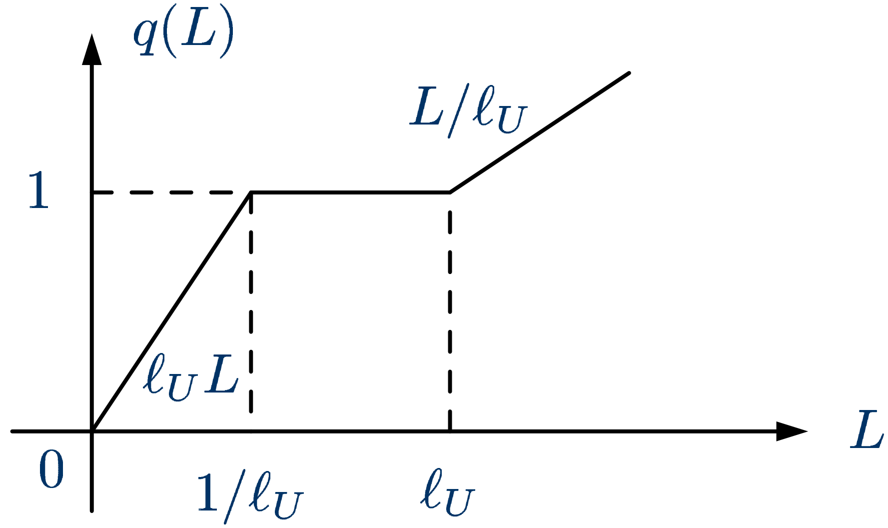
$$\delta_R(y) = \begin{cases} 1 & y > y_U \\ \frac{1}{2}\left[1 + \frac{\ln L(y)}{\ln \ell_U}\right] & -y_U \leq y \leq y_U \\ 0 & y < -y_U \,, \end{cases}$$

$$g_0^L(y) = \begin{cases} \ell_U f_0(y)/Z(y_U) & y > y_U \\ \ell_U^{1/2} f_1^{1/2} f_0^{1/2}(y)/Z(y_U) & -y_U \leq y \leq y_U \\ f_0(y)/Z(y_U) & y < -y_U \,, \end{cases}$$

$g_1^L(y) = g_0^L(-y)$, with $\ell_U = L(y_U)$.

Robustness with a KL Tolerance (cont'd)

Nonlinear transformation: The least-favorable LR can be expressed as a nonlinear transformation $L_L = q(L)$ of the nominal LR.



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Robustness with a KL Tolerance (cont'd)

- $g_0^L(\cdot|y_U)$ is parametrized by y_U with $g_0^L = f_0$ for $y_U = 0$ and $\lim g_0^L = f_{1/2}$ as $y_U \to \infty$.
- y_U is selected such that $D(g_0^L(\cdot|y_U)|f_0) = \epsilon$. Relies on showing that

$$D(y_U) = D(g_0^L(\cdot|y_U)|f_0)$$

is a monotone increasing function of y_U.

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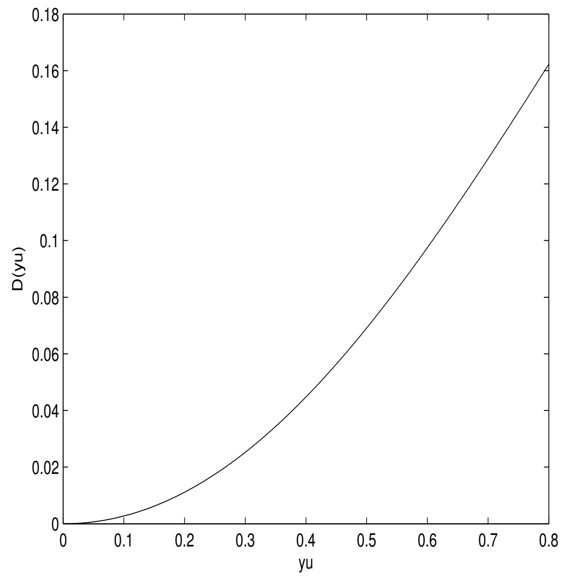
Simulations

Consider the nominal model

$$H_0 \,:\, Y = -1 + V \qquad H_1 \,:\, Y = 1 + V\ ,$$

with $V \sim N(0, \sigma^2)$, so $f_0 \sim N(-1, \sigma^2)$. $D(y_U)$ is plotted below for SNR =0dB ($\sigma = 1$).



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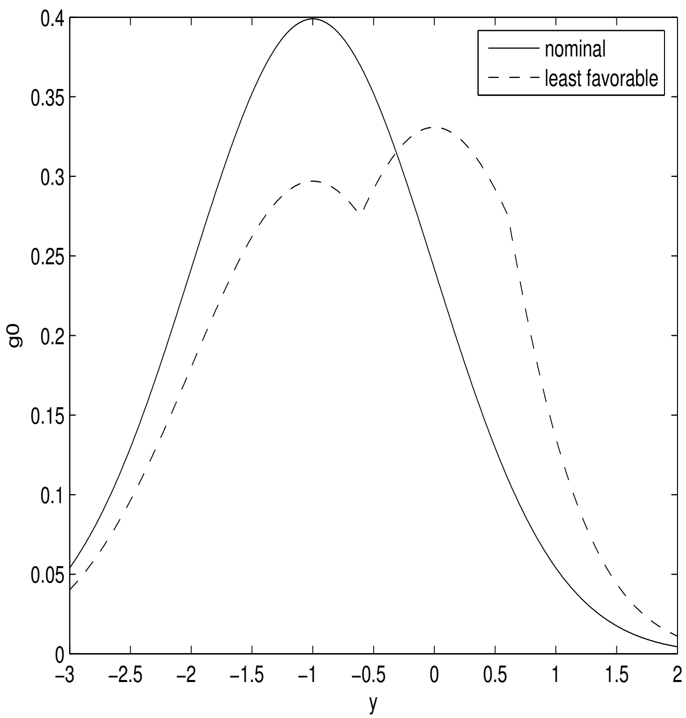
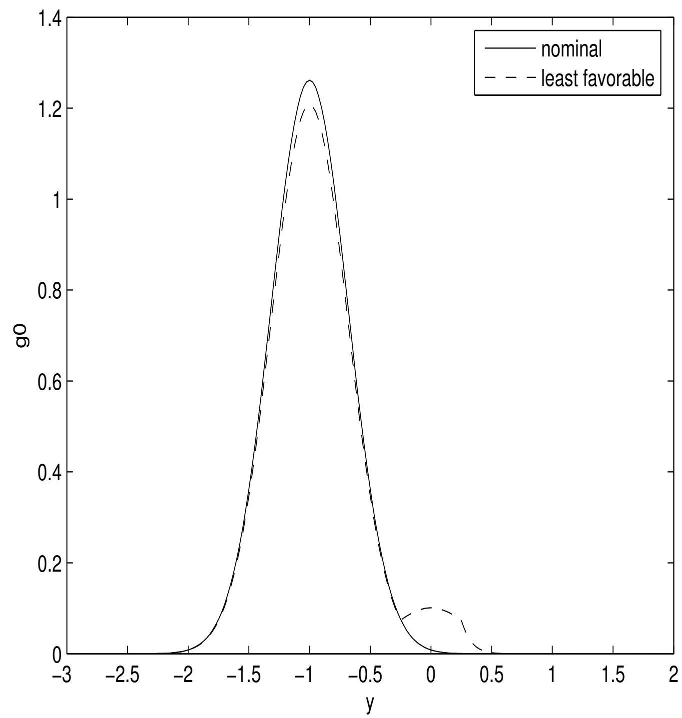
Simulations (cont'd)

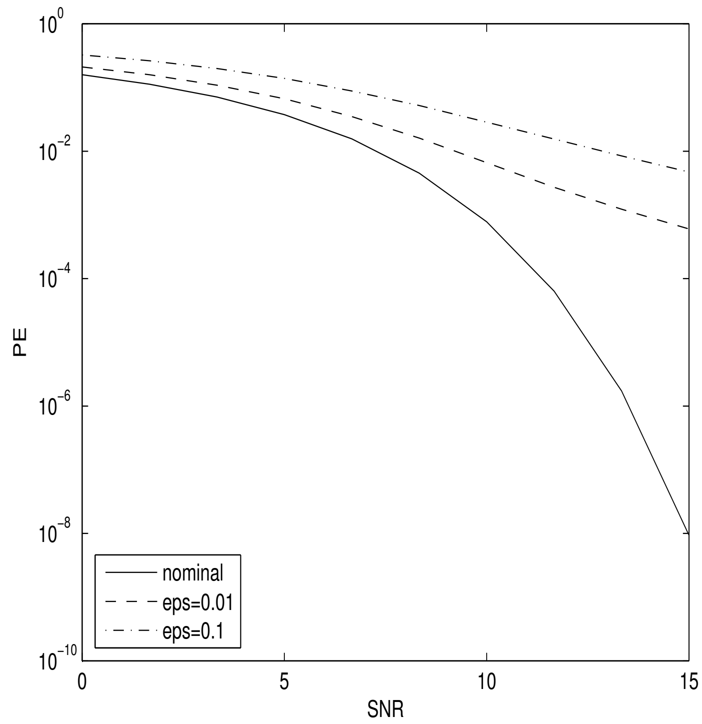
LF densities g_0^L for $\epsilon = 0.1$ and SNR = 0, 10dB.





Simulations (cont'd)

Comparison of worst-case $P[E]$ for test δ_R with $\epsilon = 0.01, 0.1$ against $P[E]$ for the Bayesian test on nominal model.



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Thank you!!