An Unbiased Explicit Adaptive Gain and Time-delay Estimation Algorithm

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Outline

- **Motivation**
- Adaptive Gain-Delay Estimation
- Unbiased Algorithm
- Simulations
- Conclusion

Motivation

- Some applications of gain and time-delay adaptive estimation, such as 2-channel time-interleaved ADC calibration require low complexity and high precision estimates.
- Low complexity $\Rightarrow$ need to use an **explicit algorithm**.
- A common explict gain and time-delay adaptive estimation algorithm, due to Kong and Solo (1991), exhibits a gain bias inversely proportional to SNR.
- Source of the bias: the power of the noise component of the error is **gain dependent**.

Motivation (cont.)

- By **normalizing the error** an unbiased adaptive explicit gain and time-delay algorithm is proposed.
- Unbiasedness is demonstrated both theoretically by using the ODE method to analyze convergence of the algorithm, and through simulations.

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Adaptive Gain-Delay Estimation

Consider observations

$$\begin{aligned} y_1(n) &= s(n) + v_1(n) \\ y_2(n) &= h(n, \boldsymbol{\theta}) * s(n) + v_2(n) \;, \end{aligned}$$

where $s(n)$ = unknown signal, $v_i(n)$ = independent zero-mean WGNs with variance v for $i = 1,\ 2$,

$$\boldsymbol{\theta} = \begin{bmatrix} g \\ d \end{bmatrix} = \text{ unknown parameter vector}$$

with g = gain d = delay, and

$$h(n, \boldsymbol{\theta}) = g \frac{\sin(\pi(n-d))}{\pi(n-d)}$$

= impulse response of gain-delay filter $H(e^{j\omega}, \theta) = g e^{-j\omega d}$.

Adaptive Gain-Delay Estimation (cont'd)

Consider error

$$\begin{aligned} e_{\mathrm{B}}(n,\hat{\boldsymbol{\theta}}) &= y_2(n) - h(n,\hat{\boldsymbol{\theta}}) * y_1(n) \\ &= y_2(n) - \hat{g}y_1(n-\hat{d}) \,, \end{aligned}$$

where

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{g} \\ \hat{d} \end{bmatrix} = \text{ estimated parameter vector .}$$

The explicit adaptive gain-delay algorithm proposed by Kong and Solo (1991) uses a stochastic gradient algorithm to minimize

$$J_{\mathrm{B}}(\hat{\boldsymbol{\theta}}) = \frac{1}{2}E[e_{\mathrm{B}}^2(\hat{\boldsymbol{\theta}})] \,.$$

Adaptive Gain-Delay Estimation (cont'd)

This gives

$$\hat{\boldsymbol{\theta}}(n+1) = \hat{\boldsymbol{\theta}}(n) + \mu_{\mathrm{B}} e_{\mathrm{B}}(n, \hat{\boldsymbol{\theta}}) \mathbf{q}(n, \hat{\boldsymbol{\theta}}) \ ,$$

where $\mu_{\mathrm{B}} =$ step size, and

$$\begin{aligned} \mathbf{q}(n, \hat{\boldsymbol{\theta}}) = \begin{bmatrix} q_1(n, \hat{\boldsymbol{\theta}}) \\ q_2(n, \hat{\boldsymbol{\theta}}) \end{bmatrix} \quad &= \quad -\nabla_{\hat{\boldsymbol{\theta}}} e_{\mathrm{B}}(n, \hat{\boldsymbol{\theta}}) \\ &= \quad \nabla_{\hat{\boldsymbol{\theta}}} h(n, \hat{\boldsymbol{\theta}}) * y_1(n) \ , \end{aligned}$$

with

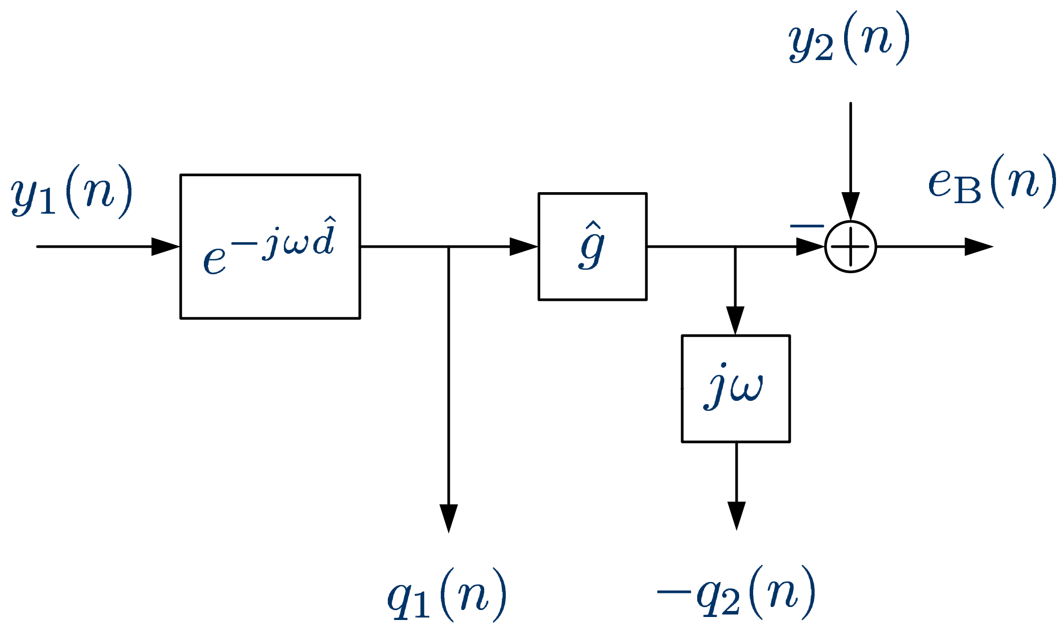
$$\begin{aligned} \nabla_{\hat{\boldsymbol{\theta}}} \quad &\triangleq \quad \left[\begin{array}{cc} \frac{\partial}{\partial \hat{g}} & \frac{\partial}{\partial \hat{d}} \end{array} \right]^T \\ &= \quad \text{gradient with respect to vector} \hat{\boldsymbol{\theta}} \ . \end{aligned}$$

Adaptive Gain-Delay Estimation (cont'd)

In the frequency domain

$$\nabla_{\hat{\boldsymbol{\theta}}} H(e^{j\omega}, \hat{\boldsymbol{\theta}}) = \begin{bmatrix} 1 \\ -j\omega\hat{g} \end{bmatrix} e^{-j\omega\hat{d}},$$

so that $e_B(n)$, $q_1(n)$ and $q_2(n)$ can be obtained as shown below.



Adaptive Gain-Delay Estimation (cont'd)

Bias evaluation: Assume unknown signal $s(n)$ is WSS with PSD $S(e^{j\omega})$ and total power P. The error

$$\begin{aligned} e_{\mathrm{B}}(n,\hat{\boldsymbol{\theta}}) &= gs(n-d) - \hat{g}s(n-\hat{d}) \\ &\quad + v_2(n) - \hat{g}v_1(n-\hat{d}) \ , \end{aligned}$$

so

$$\begin{aligned} J_{\mathrm{B}}(\hat{\boldsymbol{\theta}}) &= \frac{1}{4\pi}\int_{-\pi}^{\pi} |ge^{-j\omega d} - \hat{g}e^{-j\omega\hat{d}}|^2 S(e^{j\omega})d\omega \\ &\quad + (1+\hat{g}^2)\frac{v}{2} \ . \end{aligned}$$

Adaptive Gain-Delay Estimation (cont'd)

In the vicinity of g and d

$$J_{\mathrm{B}}(\hat{\boldsymbol{\theta}}) \approx \frac{A}{2}(\hat{g}-g)^2 + \frac{B}{2}(\hat{d}-d)^2 + (1+\hat{g}^2)\frac{v}{2}\,,$$

with

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi}^{\pi} S(e^{j\omega})d\omega = P \\ B &= \frac{1}{2}\int_{-\pi}^{\pi} \omega^2 S(e^{j\omega})d\omega\,. \end{aligned}$$

If SNR $= P/v$, it is minimized by

$$\hat{g} = \frac{g}{1+(\mathrm{SNR})^{-1}} \approx [1-(\mathrm{SNR})^{-1}]g$$

and $\hat{d} = d$. **The gain estimate is biased!**

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Unbiased Algorithm

To remove the estimated gain dependence of the noise component of the error, use the scaled error

$$\begin{aligned} e(n,\hat{\boldsymbol{\theta}}) &= \frac{1}{(1+\hat{g}^2)^{1/2}} e_{\mathrm{B}}(n,\hat{\boldsymbol{\theta}}) \\ &= \frac{1}{(1+\hat{g}^2)^{1/2}} \left[y_2(n) - \hat{g} y_1(n-\hat{d})\right] , \end{aligned}$$

and use a stochastic gradient scheme to minimize

$$J(\hat{\boldsymbol{\theta}}) = \frac{1}{2} E[e^2(n,\hat{\boldsymbol{\theta}})] = \frac{1}{1+\hat{g}^2} J_{\mathrm{B}}(\hat{\boldsymbol{\theta}}) \,.$$

Unbiased Algorithm (cont.)

- The scale factor $(1+\hat{g}^2)^{-1}$ distorts the shape of objective function $J(\hat{\boldsymbol{\theta}})$ and may change the domain where it is convex.
- In the vicinity of $\boldsymbol{\theta}$, $J(\hat{\boldsymbol{\theta}})$ admits the approximation

$$J(\hat{\boldsymbol{\theta}}) \approx \frac{1}{2}\left(\frac{1}{(1+g^2)}[A(\hat{g}-g)^2 + B(\hat{d}-d)^2] + v\right),$$

whose minimum is $\boldsymbol{\theta} \Rightarrow$ **new algorithm will be unbiased.**

Unbiased Algorithm (cont.)

Unbiased adaptive algorithm:

$$\hat{\boldsymbol{\theta}}(n+1) = \hat{\boldsymbol{\theta}}(n) - \mu e(n, \hat{\boldsymbol{\theta}}) \nabla_{\hat{\boldsymbol{\theta}}} e(n, \hat{\boldsymbol{\theta}}) \ ,$$

where $\mu =$ step size, and

$$\frac{\partial e}{\partial \hat{g}} = -\frac{1}{(1+\hat{g}^2)^{3/2}} [\hat{g} y_2(n) + y_1(n-\hat{d})]$$

$$\frac{\partial e}{\partial \hat{d}} = -\frac{1}{(1+\hat{g}^2)^{1/2}} \frac{\partial h}{\partial \hat{d}}(n, \hat{\boldsymbol{\theta}}) * y_1(n) \ .$$

For small $\hat{d}$, the time-varying delay filter $e^{-j\omega\hat{d}}$ can be implemented with Farrow's expansion

$$e^{-j\omega\hat{d}} \approx 1 - j\omega\hat{d} + (-j\omega)^2 \frac{(\hat{d})^2}{2!} + \ldots + (-j\omega)^k \frac{(\hat{d})^k}{k!} \ ,$$

where k = approximation order.

Unbiased Algorithm (cont.)

Convergence analysis: By using the ODE/ stochastic averaging method, the sample paths $\hat{\boldsymbol{\theta}}(n)$ of the parameter estimates are approximated by the solutions of ODE

$$\frac{d\hat{\boldsymbol{\theta}}}{dt}(t) = -\nabla_{\hat{\boldsymbol{\theta}}} J(\hat{\boldsymbol{\theta}}(t)) \; .$$

Starting from an initial value $\hat{\boldsymbol{\theta}}(0)$ sufficiently close to $\boldsymbol{\theta}$, all trajectories converge to $\boldsymbol{\theta}$, and for small μ and large n, the estimated parameter vector $\hat{\boldsymbol{\theta}}(n)$ is $N(\boldsymbol{\theta}, \mu\mathbf{P})$ distributed.

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Simulations

- Assume $s(n)$ formed by the supperposition of 3 unit power sinusoids with frequencies 0.1π, 0.4π, and 0.8π and uniformly distributed random phases.
- Noise variance $v = 6 \times 10^{-2}$, so SNR $= 50$ (17dB).
- True gain $g = 0.9$ and delay $d = 0.1$ (10% mismatches between the two channels).
- Farrow expansion of order 2 used for delay filter. Differentiator $j\omega$ implemented by applying Kaiser window of length 81 and $\beta = 4$ to ideal impulse response

$$k(n) = \begin{cases} 0 & n = 0 \\ \frac{(-1)^n}{n} & n \neq 0 \,. \end{cases}$$

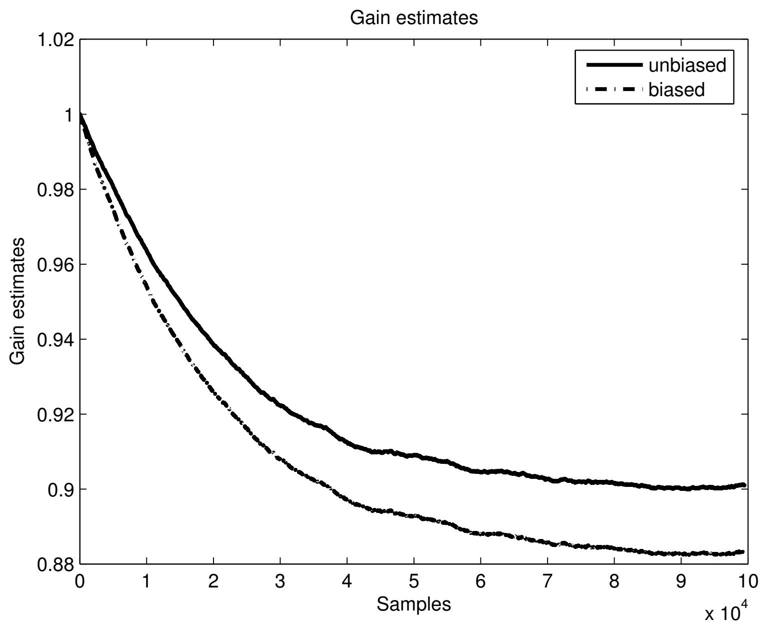
Simulations (cont.)

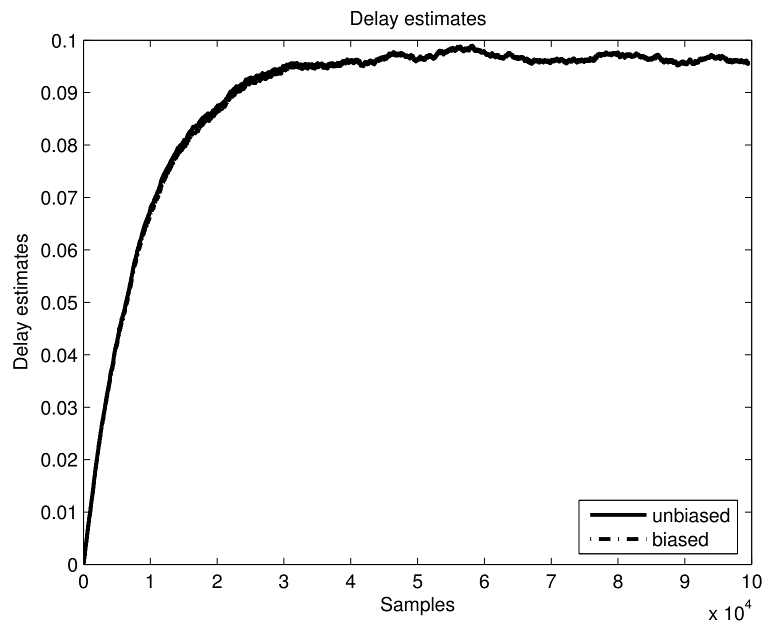
- Compare biased and unbiased algorithms with $\mu_{\mathrm{B}} = 0.5\mu$ and $\mu = 10^{-4.5}$ (factor $1/2$ Due to the relative scaling of J_{B} and J).
- Use $\hat{g}(0) = 1$ and $\hat{d}(0) = 0$ as initial estimates.
- Computed biased gain estimate: $\hat{g}_{\mathrm{B}} = 0.8851$, close to the value

$$\frac{1}{1 + (\mathrm{SNR})^{-1}} g = 0.8824$$

predicted by theoretical analysis.

Simulations (cont'd)





Estimated gains and delays for 3 sinusoids signal and 17dB SNR

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Conclusion

- By applying a $(1+\hat{g}^2)^{-1}$ scaling to the objective function used by the Kong-Solo explicit gain-delay estimation algorithm, an unbiased algorithm was derived.
- The scaling shifts the local minimum of objective function $J(\hat{\boldsymbol{\theta}})$ to the true value $\boldsymbol{\theta}$, but distorts the function far away from $\boldsymbol{\theta}$.
- The algorithm is best adapted to cases (like TIADC calibration) where an initial estimate $\hat{\boldsymbol{\theta}}(0)$ close to $\boldsymbol{\theta}$ is available.

Thank you!!