EEC 250 Linear Systems and Signals

Lecture 8

Topics: a) Natural modes

b) Response to real and complex exponentials

c) System poles and zeros

Natural modes: Consider a CT system

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \quad , \quad \underline{x}(0) = \underline{x}_0 \tag{1}$$

$$\underline{y}(t) = C\,\underline{x}(t) + D\,\underline{u}(t) \tag{2}$$

with $\underline{x} \in \mathbb{R}^n$, $\underline{u} \in \mathbb{R}^m$ and $\underline{y} \in \mathbb{R}^p$. We assume that A is diagonalizable so that it admits n independent right eigenvectors $\underline{p}_i$, $1 \leq i \leq n$. Let λ_i be the eigenvalue corresponding to $\underline{p}_i$, so that $A\underline{p}_i = \lambda_i \underline{p}_i$. If $P = [\underline{p}_1 \cdots \underline{p}_n]$ is the matrix whose columns are the eigenvectors of A, and

$$P^{-1} = Q = \begin{bmatrix} \underline{q}_1^T \\ \underline{q}_i^T \\ \underline{q}_n^T \end{bmatrix}$$

the rows $\underline{q}_i^T$ $1 \leq i \leq n$ are the left eigenvectors of A, i.e. $\underline{q}_i^T A = \lambda_i \underline{q}_i^T$.

The matrix exponential e^{At} can then be expressed as

$$e^{At} = P\, e^{\Lambda t}\, Q \qquad (3a)$$

with

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & & 0 \\ & \ddots & & & \\ & & e^{\lambda_i t} & & \\ & & & \ddots & \\ 0 & & & & e^{\lambda_n t} \end{bmatrix} \qquad (3b)$$

The zero input response (ZIR) of the system is obtained by setting the input $u(t) \equiv 0$ in (1). We obtain

$$\begin{aligned} \underline{x}(t) &= e^{At}\, \underline{x}_0 \\ &= P\, e^{\Lambda t}\, \underline{z} = \sum_{i=1}^{n} z_i\, \underline{p}_i\, e^{\lambda_i t} \end{aligned} \qquad (4)$$

with

$$Q\, \underline{x}_0 = \underline{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_i \\ \vdots \\ z_n \end{bmatrix} \qquad (5)$$

We see from (4) that the ZIR is the linear combination of vector functions $\{\underline{p}_i\, e^{\lambda_i t},\ 1 \leq i \leq n\}$ which are called the natural modes of the system. Specifically, if the initial state vector $\underline{x}_0$ is expressed as a linear combination

$$\underline{x}_0 = P\, \underline{z} = \sum_{i=1}^{n} z_i\, \underline{p}_i \qquad (6)$$

of the eigenvectors $\underline{p}_i$ of A, with weights z_i, then the state vector $\underline{x}(t)$ for all times $t \geq 0$ is precisely the same linear combination of the eigenmodes

$\underline{p}_i \, e^{\lambda_i t}$.

Another way of interpreting physically the above result is as follows. Suppose $(\lambda, \underline{p})$ is an eigenvalue / eigenvector pair of A such that λ is real. Then if $\underline{x}_0 = \underline{p}$, we have

$$\underline{x}(t) = e^{\lambda t} \underline{p} \quad \text{for} \quad t \geqslant 0 . \qquad (7)$$

In other words if the initial state vector $\underline{x}_0$ is colinear with $\underline{p}$, the state $\underline{x}(t)$ remains colinear with $\underline{p}$, but its length is scaled by the factor $e^{\lambda t}$. If $\lambda > 0$, this factor grows to infinity, whereas if $\lambda < 0$ it decays to zero.

For the case when λ is complex, since A is real, $(\lambda, \underline{p})$ and $(\lambda^*, \underline{p}^*)$ are two complex eigenvalue / eigenvector pairs. Let $\lambda = \sigma + j\omega$ and $\underline{p} = \underline{p}_R + j\,\underline{p}_I$, and let $z = a + jb$ be an arbitrary complex number. Then for

$$\underline{x}_0 = z\underline{p} + z^* \underline{p}^* = 2\,\mathrm{Re}\,(z\underline{p}) = 2\,(a\underline{p}_R - b\,\underline{p}_I)$$

we have

$$\underline{x}(t) = z\underline{p}\,e^{\lambda t} + z^* \underline{p}^* e^{\lambda^* t} = 2\,\mathrm{Re}\,(z\underline{p}\,e^{\lambda t})$$

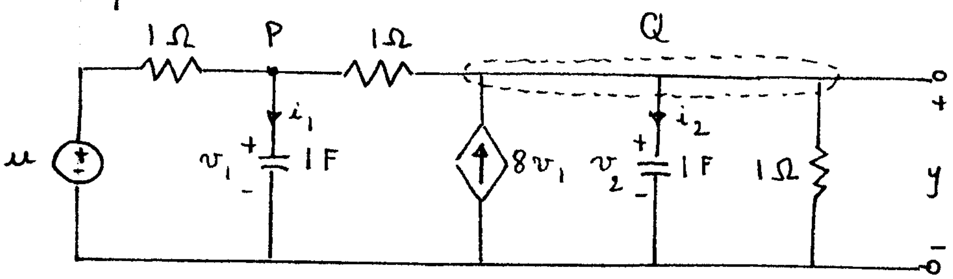
$$= 2\,\mathrm{Re}\,(z(t)\,\underline{p}\,)\,e^{\sigma t} = 2\,(a(t)\,\underline{p}_R - b(t)\,\underline{p}_I)\,e^{\sigma t} \qquad (8a)$$

with $z(t) = z\,e^{j\omega t} = a(t) + j\,b(t)$, where

$$\begin{bmatrix} a(t) \\ b(t) \end{bmatrix} = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} . \qquad (8b)$$

The relations (7a) and (7b) show that if $\underline{x}_0$ is a linear combination of $\underline{p}_R$ and $\underline{p}_I$, the state $\underline{x}(t)$ remains a combination of $\underline{p}_R$ and $\underline{p}_I$, where $\underline{x}(t)$ is obtained from $\underline{x}_0$ by performing a rotation by an angle ωt in the (a,b) plane, followed by scaling by a factor $e^{\sigma t}$.

<u>Example 1</u>: Consider the circuit



with initial conditions $v_1(0) = 2\,V$, $v_2(0) = 1\,V$. Applying the KCL at points P and Q gives

at P: $(v_1 - u) + i_1 + (v_1 - v_2) = 0$

at Q: $(v_2 - v_1) - 8v_1 + i_2 + v_2 = 0$

This implies

$$\dot{v}_1 = i_1 = -2v_1 + v_2 + u$$

$$\dot{v}_2 = i_2 = 9v_1 - 2v_2$$

which yields the state-space model

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 9 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} .$$

We have

$$sI - A = \begin{bmatrix} s+2 & -1 \\ -9 & s+2 \end{bmatrix}$$

so that

$$a(s) = \det(sI - A) = (s+2)^2 - 9 = (s-1)(s+5) .$$

The eigenvector corresponding to the eigenvalue $\lambda_1 = 1$ is given by

$$\underbrace{\begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix}}_{\lambda_1 I - A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{p}_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{p}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} .$$

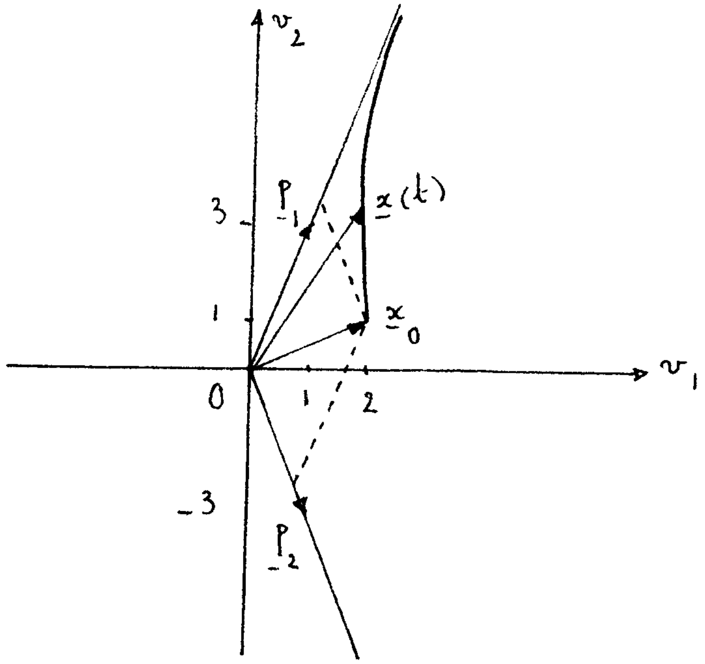
Similarly for the eigenvalue $\lambda_2 = -5$, one gets

$$\underbrace{\begin{bmatrix} -3 & -1 \\ -9 & -3 \end{bmatrix}}_{\lambda_2 I - A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{p}_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{p}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

The initial state vector $\underline{x}_0$ can be expressed in terms of eigenvectors $\underline{p}_1$ and $\underline{p}_2$ as

$$\underbrace{\begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix}}_{\underline{x}_0} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{7}{6} \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\underline{p}_1} + \frac{5}{6} \underbrace{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}_{\underline{p}_2} ,$$

as shown by the figure below.

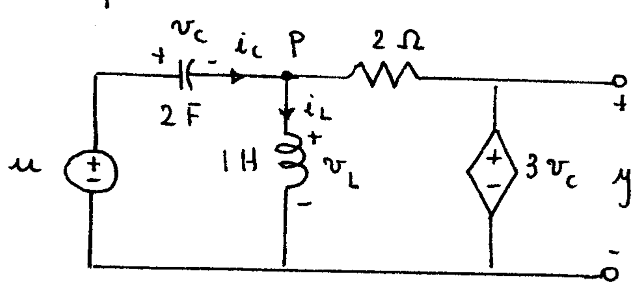


For $t \geq 0$, we have

$$\underline{x}(t) = \frac{7}{6}\,\underline{p}_1\, e^{t} + \frac{5}{6}\,\underline{p}_2\, e^{-5t}$$

where the first mode $\underline{p}_1 e^{t}$ is unstable, whereas the second mode $\underline{p}_2 e^{-5t}$ is stable. Thus as $t \to \infty$, $\underline{x}(t) \simeq \frac{7}{6}\,\underline{p}_1\, e^{t}$ grows to infinity. ■

<u>Example 2</u>: Consider the circuit



with initial conditions $v_c(0) = 1\,V$, $i_L(0) = 1\,A$. The KVL along the left loop gives

$$\dot{i}_L = v_L = u - v_c\,,$$

and KCL at node P yields

$$2\,\dot{v}_c = i_c = i_L + \frac{u - 4v_c}{2} .$$

The resulting state-space model is given by

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1/2 & -1 \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} 1 \\ 1/4 \end{bmatrix} u$$

$$y = [\,0 \quad 3\,] \begin{bmatrix} i_L \\ v_c \end{bmatrix}$$

We have

$$sI - A = \begin{bmatrix} s & 1 \\ -1/2 & s+1 \end{bmatrix}$$

and

$$a(s) = \det(sI - A) = s(s+1) + 1/2 = (s+1/2)^2 + 1/4$$

$$= \left(s + \tfrac{1}{2}(1+j)\right)\left(s + \tfrac{1}{2}(1-j)\right)$$

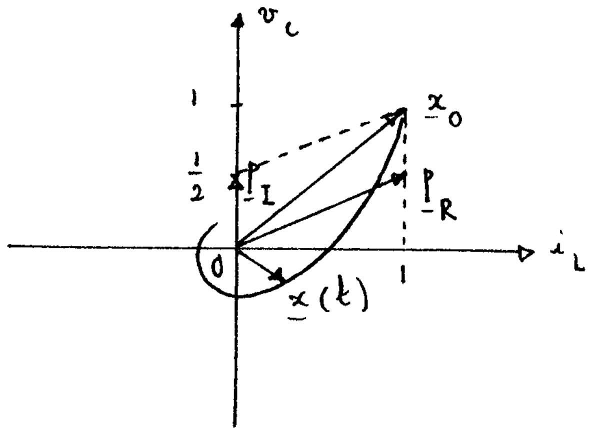
Thus A has two complex conjugate eigenvalues $\lambda_{\pm} = -\frac{1}{2}(1 \pm j)$. The eigenvector corresponding to $\lambda_+ = -\frac{1}{2}(1+j)$ is given by

$$\underbrace{\begin{bmatrix} -\frac{1}{2}(1+j) & 1 \\ -\frac{1}{2} & \frac{1}{2}(1-j) \end{bmatrix}}_{\lambda_+ I - A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{p}_+} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{p}_+ = \begin{bmatrix} 1 \\ \frac{1}{2}(1+j) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}}_{\underline{p}_R} + j \underbrace{\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}}_{\underline{p}_I}$$

and the eigenvector corresponding to $\lambda_- = \lambda_+^*$ is $\underline{p}_- = \underline{p}_+^*$. The initial state vector $\underline{x}_0$ can be expressed in terms of $\underline{p}_R$ and $\underline{p}_I$ as

$$\underbrace{\begin{bmatrix} i_L(0) \\ v_c(0) \end{bmatrix}}_{\underline{x}_0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \underbrace{\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}}_{\underline{p}_R} + 1 \underbrace{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}_{\underline{p}_I}$$

as shown below



Thus $a = 1/2$, $b = -1/2$, and for $t \geq 0$

$$\underline{x}(t) = [(\cos\omega t + \sin\omega t)\,\underline{p}_R + (-\sin\omega t + \cos\omega t)\,\underline{p}_I]\, e^{-t/2}$$
$$= \sqrt{2}\, [\cos(\omega t - \tfrac{\pi}{4})\,\underline{p}_R - \sin(\omega t - \tfrac{\pi}{4})\,\underline{p}_I]\, e^{-t/2}$$

follows a spiraling motion towards zero.

<u>Response to real exponentials</u>: To obtain a solution of the state equation (1) corresponding to an exponential input $\underline{u}(t) = \underline{u}\, e^{vt}$, where $\underline{u} \in \mathbb{R}^m$ and v is not an eigenvalue of A, we decompose the solution $\underline{x}(t)$ as

$$\underline{x}(t) = \underline{x}_h(t) + \underline{x}_p(t)\,, \qquad (9)$$

where $\underline{x}_h(t)$ and $\underline{x}_p(t)$ are homogeneous and particular solutions.

Homogeneous solution: It is an arbitrary solution of the homogeneous equation

$$\dot{\underline{x}}(t) = A\,\underline{x}\,(t)$$

obtained by setting $\underline{u}(t) \equiv 0$ in (1). Thus, it is a superposition of natural modes, i.e.

$$\underline{x}_h(t) = \sum_{i=1}^{n} z_i\,\underline{p}_i\,e^{\lambda_i t} = P\,e^{\Lambda t}\,\underline{z} \tag{10}$$

where the coefficients z_i are arbitrary and will be selected to match the initial state vector $\underline{x}_0$.

Particular solution: It is assumed to have the form $\underline{x}_p(t) = \underline{x}\,e^{vt}$, where $\underline{x} \in \mathbb{R}^n$ is a constant vector to be determined. Substituting $\underline{u}(t) = \underline{u}\,e^{vt}$ and $\underline{x}_p(t)$ inside (1) yields

$$v\,\underline{x}\,e^{vt} = (A\,\underline{x} + B\,\underline{u})\,e^{vt}$$

so that $\underline{x} = (vI-A)^{-1}B\,\underline{u}$, where $vI-A$ is invertible since it was assumed that v is not an eigenvalue of A. Thus

$$\underline{x}_p(t) = (vI-A)^{-1}B\,\underline{u}\,e^{vt}. \tag{11}$$

Complete solution: We have

$$\underline{x}(t) = \underline{x}_h(t) + \underline{x}_p(t) = P\,e^{\Lambda t}\,\underline{z} + (vI-A)^{-1}B\,\underline{u}\,e^{vt}.$$

The vector $\underline{z}$ is selected to ensure that the initial condition $\underline{x}(0) = \underline{x}_0$ is satisfied. This gives

$$P\underline{z} + (\nu I - A)^{-1} B \underline{u} = \underline{x}_0$$

so that

$$\underline{z} = Q \left[\underline{x}_0 - (\nu I - A)^{-1} B \underline{u} \right]$$

Noting that $e^{At} = P e^{\Lambda t} Q$, we get

$$\underline{x}(t) = e^{At} \left[\underline{x}_0 - (\nu I - A)^{-1} B \underline{u} \right] + (\nu I - A)^{-1} B \underline{u} \, e^{\nu t} . \qquad (12)$$

In (12), we note that the homogeneous solution

$$\underline{x}_h(t) = e^{At} \left[\underline{x}_0 - (\nu I - A)^{-1} B \underline{u} \right] \neq ZIR = e^{At} \underline{x}_0 .$$

This is due to the fact that the ZIR represents the contribution of the initial state vector $\underline{x}_0$ to the solution $\underline{x}(t)$, whereas the homogeneous solution $\underline{x}_h(t)$ represents the effect of the deviation of $\underline{x}_0$ from the steady-stat vector $(\nu I - A)^{-1} B \underline{u}$. Similarly, the particular solution $\underline{x}_p(t)$ given by (11) differs from

$$ZSR = \int_0^t e^{As} B \underline{u} \, e^{\nu(t-s)} ds$$

$$= \int_0^t e^{-(\nu I - A) s} ds \, B \underline{u} \, e^{\nu t} = \left[I - e^{-(\nu I - A) t} \right] (\nu I - A)^{-1} B \underline{u} \, e^{\nu t}$$

$$= (\nu I - A)^{-1} B \underline{u} \, e^{\nu t} - e^{At} (\nu I - A)^{-1} B \underline{u} . \qquad (13)$$

The output $\underline{y}(t)$ corresponding to (12) is given by

$$\underline{y}(t) = C\,\underline{x}(t) + D\,\underline{u}(t)$$
$$= C\,e^{At}\left[\underline{x}_0 - (vI-A)^{-1}B\,\underline{u}\right] + H(v)\,\underline{u}\,e^{vt} \qquad (14)$$

where

$$H(v) = C\,(vI-A)^{-1}B + D$$

is the system transfer function. Thus $\underline{y}(t)$ can be decomposed as

$$\underline{y}(t) = \underline{y}_T(t) + \underline{y}_{ss}(t) \qquad (15)$$

where

$$\underline{y}_{ss}(t) = H(v)\,\underline{u}\,e^{vt} \qquad (16a)$$
$$\underline{y}_T(t) = C\,e^{At}\left[\underline{x}_0 - (vI-A)^{-1}B\,\underline{u}\right] \qquad (16b)$$

can be viewed as the steady-state and transient responses of the system to the forcing exponential input $\underline{u}(t) = \underline{u}\,e^{vt}$.

In this context, two useful observations are as follows:

(i) If $\underline{x}_0 = (vI-A)^{-1}B\,\underline{u}$, then for v arbitrary we have

$$\underline{y}(t) = H(v)\,\underline{u}\,e^{vt} \qquad \text{for } t \geq 0.$$

(ii) If $\underline{x}_0$ is arbitrary, i.e. $\underline{x}_0 \neq (vI-A)^{-1}B\,\underline{u}$, and $v > \mathrm{Re}\,\lambda_i$ for all i, then

$$\underline{y}(t) \simeq \underline{y}_{ss}(t) = H(v)\,\underline{u}\,e^{vt}$$

as $t \to \infty$. In other words, if the forcing exponential e^{vt} decays more slowly, or grows more rapidly, than the natural modes $e^{\lambda_i t}$, then the steady-state response $\underline{y}_{ss}(t)$ approximates $\underline{y}(t)$ as $t \to \infty$.

<u>Response to complex exponentials</u> : Assume now that the forcing input is

$$\underline{u}(t) = \mathrm{Re}[\underline{u}\, e^{vt}] = \frac{1}{2}(\underline{u}\, e^{vt} + \underline{u}^* e^{v^* t}) \qquad (17)$$

where $\underline{u} \in \mathbb{C}^m$ is a complex vector and $v = \sigma + j\omega$ is complex. It is also assumed that v is not an eigenvalue of A. Since $\underline{u}\, e^{vt}$ is a complex vector, its real part is taken componentwise in (17). Observing that the particular solution corresponding to $\underline{u}_c(t) = \underline{u}\, e^{vt}$ is $\underline{x}_{pc}(t) = (vI-A)^{-1} B\, \underline{u}\, e^{vt}$, by superposition the particular solution corresponding to (17) is

$$\underline{x}_p(t) = \frac{1}{2}(\underline{x}_{pc}(t) + \underline{x}^*_{pc}(t)) = \mathrm{Re}[(vI-A)^{-1} B\, \underline{u}\, e^{vt}] . \qquad (18)$$

Substituting (18) inside expression (9) for the complete solution, and selecting the vector $\underline{z}$ parametrizing the homogeneous solution so that the initial condition $\underline{x}(0) = \underline{x}_0$ is satisfied, we find

$$\underline{x}_h(t) = e^{At}(\underline{x}_0 - \mathrm{Re}[(vI-A)^{-1} B\, \underline{u}]) . \qquad (19)$$

The system output can again be decomposed as indicated in (15) with

$$\underline{y}_{ss}(t) = \mathrm{Re}[H(v)\underline{u}\, e^{vt}] \qquad (20a)$$

$$\underline{y}_T(t) = C e^{At}(\underline{x}_0 - \mathrm{Re}[(vI-A)^{-1} B\, \underline{u}]) . \qquad (20b)$$

Furthermore, if the initial state vector $\underline{x}_0$ is arbitrary, and if the complex frequency v is located to the right of the rightmost eigenvalue of A in the complex plane, i.e. $\mathrm{Re}\, v > \mathrm{Re}\, \lambda_i$ for all i, then

$$\underline{y}(t) \simeq \mathrm{Re}\,[H(v)\,\underline{u}\,e^{vt}] \tag{21}$$

as $t \to \infty$.

<u>Frequency response</u>: A case which is of particular interest arises when A is asymptotically stable, so that all its eigenvalues are strictly in the left-half plane, and when $v = j\omega$. According to (21) the output corresponding to the input $u(t) = \mathrm{Re}\,[\underline{u}\,e^{j\omega t}]$ is given by

$$\underline{y}(t) \simeq \mathrm{Re}\,[H(j\omega)\,\underline{u}\,e^{j\omega t}] \tag{22}$$

as $t \to \infty$, where $H(j\omega)$ is the <u>matrix frequency response</u> of the system.

For the special case of a single input - single output (SISO) system, we can assume without loss of generality that $u = 1$, so that $u(t) = \mathrm{Re}\,[e^{j\omega t}] = \cos \omega t$, and if $H(j\omega)$ is expressed in terms of its amplitude and phase as $H(j\omega) = A(\omega)\,e^{j\varphi(\omega)}$, we have

$$y(t) \simeq A(\omega)\cos(\omega t + \varphi(\omega)) \quad \text{as } t \to \infty. \tag{23}$$

In other words, if the system is excited by a sinusoidal input with frequency ω, the output is a sinusoid with the same frequency, but scaled by the amplitude $A(\omega)$ and with the phase shift $\varphi(\omega)$.

<u>Example</u>: Consider the circuit of Example 2 with initial conditions $v_C(0) = 1\,V$, $i_L(0) = 1\,A$ and input $u(t) = \cos t = \mathrm{Re}\,[e^{jt}]$. We have

$$(sI-A)^{-1} = \frac{1}{(s+\frac{1}{2})^2+\frac{1}{4}} \begin{bmatrix} s+1 & -1 \\ 1/2 & s \end{bmatrix}$$

$$\leftrightarrow \quad e^{At} = e^{-t/2} \begin{bmatrix} \cos(t/2)+\sin(t/2) & -2\sin(t/2) \\ \sin(t/2) & \cos(t/2)-\sin(t/2) \end{bmatrix}$$

and

$$(sI-A)^{-1}b = \frac{1}{(s+\frac{1}{2})^2+\frac{1}{4}} \begin{bmatrix} s+3/4 \\ \frac{1}{4}(s+2) \end{bmatrix}$$

$$H(s) = c\,(sI-A)^{-1}b = \frac{\frac{3}{4}(s+2)}{(s+\frac{1}{2})^2+\frac{1}{4}} \ .$$

Setting $s=j$ gives

$$(jI-A)^{-1}b = \frac{1}{-\frac{1}{2}(1-2j)} \begin{bmatrix} j+3/4 \\ \frac{1}{4}(j+2) \end{bmatrix} = \begin{bmatrix} 1/2-j \\ -j/2 \end{bmatrix}$$

$$H(j) = -3j/2 \ .$$

The particular and homogeneous solutions are given respectively by

$$\underline{x}_p(t) = Re\left[(jI-A)^{-1}b\,e^{jt}\right] = Re\left[\begin{bmatrix} 1/2-j \\ -j/2 \end{bmatrix} e^{jt}\right]$$
$$= \begin{bmatrix} 1/2\cos t + \sin t \\ 1/2\sin t \end{bmatrix}$$

$$\underline{x}_h(t) = e^{At}\left(\underline{x}_0 - Re\left[(jI-A)^{-1}b\right]\right) = e^{At}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}\right)$$

$$= e^{-t/2} \begin{bmatrix} 1/2 \cos(t/2) - 3/2 \sin(t/2) \\ \cos(t/2) - 1/2 \sin(t/2) \end{bmatrix} .$$

Finally the system output is given by

$$y(t) = 3 v_c(t) = 3/2 \sin t + 3 e^{-t/2} (\cos(t/2) - 1/2 \sin(t/2))$$

so that as $t \to \infty$

$$y(t) \simeq y_{ss}(t) = 3/2 \sin t = \mathrm{Re}\, [H(j)\, e^{jt}] .$$

■

Poles and zeros: In our discussion of poles and zeros, we restrict our attention to SISO systems. The concepts of poles and zeros are not limited to SISO systems and have also been extended to multi-input multi-output (MIMO) systems, but these extensions are difficult to understand, unless one gains first a complete comprehension of the SISO case.

Consider a SISO system (A, b, c, d) with transfer function

$$H(s) = c\,(sI-A)^{-1} b + d = \frac{b(s)}{a(s)}$$

where

$$a(s) = \det(sI-A) = \prod_{i=1}^{n} (s-\lambda_i)$$

$$b(s) = c\,(\widetilde{sI-A})\, b + d\, a(s) = \prod_{i=1}^{q} (s-z_i)$$

with $q \leqslant n$. Here $\widetilde{sI-A}$ denotes the adjugate matrix of $sI-A$. It is also assumed that the system (A, b, c, d) is irreducible, in the sense that the polynomials $a(s)$ and $b(s)$ do not have a common root. As will be shown in the following lectures, the irreducibility of a system is closely related to the notions of reachability and observability. Then from a purely mathematical point of view, the <u>poles</u> of a system are the roots λ_i of the denominator polynomial $a(s)$, whereas the <u>zeros</u> are the roots z_i of its numerator polynomial $b(s)$.

However, the above definition is not very illuminating, and we now provide a physical interpretation of poles and zeros.

<u>Poles</u>: The poles are the complex frequencies generated by the system in the absence of any input excitation, i.e. the poles and natural frequencies of a system coincide, provided the system is irreducible. The situation is as depicted below.

$$u(t) \equiv 0 \longrightarrow \boxed{H(s)} \xleftarrow{\ \underline{x}_0\ } \longrightarrow y(t) = \sum_{i=1}^{n} k_i \, e^{\lambda_i t}$$

To see this, note that if the input $u(t) \equiv 0$, the ZIR of the system is

$$\underline{x}(t) = \sum_{i=1}^{n} z_i \, \underline{p}_i \, e^{\lambda_i t} \qquad (24)$$

where the vector $\underline{z}$ with entries z_i is given by (5). The corresponding output is

$$y(t) = \sum_{i=1}^{n} k_i \, e^{\lambda_i t} \tag{25}$$

with $k_i = z_i \, c \underline{p}_i$. Thus provided that the weight z_i of the natural mode $\underline{p}_i e^{\lambda_i t}$ is nonzero, and provided that this natural mode is observable from the output $y(t)$, i.e. $c\underline{p}_i \neq 0$, the complex exponential $e^{\lambda_i t}$ appears in $y(t)$.

Furthermore if the initial state $\underline{x}_0 = \underline{p}_i$, we have

$$\underline{x}(t) = \underline{p}_i \, e^{\lambda_i t} \quad \text{and} \quad y(t) = c\underline{p}_i \, e^{\lambda_i t}$$

so that by properly selecting the initial state x_0, we can ensure that a single natural frequency appears in the output. The situation is therefore as follows: there is no input, but a pure complex exponential $e^{\lambda_i t}$ is generated by the system. The frequencies λ_i that are generated by a system are its poles. They can be viewed as describing how the system freely redistributes its internal energy, as represented by the initial state $\underline{x}_0$.

<u>Zeros</u>: The zeros are the complex frequencies absorbed by a system. They are the frequencies z_i such that for the input $u(t) = \mathrm{Re}\,[e^{z_i t}]$, we can find an initial state $\underline{x}_0$ such that $y(t) \equiv 0$ for all t. As

shown earlier the output corresponding to the input $u(t) = \mathrm{Re}\,[e^{z_i t}]$ is

$$y(t) = c\,e^{At}\left(\underline{x}_0 - \mathrm{Re}\,[(z_i I - A)^{-1} b]\right) + \mathrm{Re}\,[H(z_i)\,e^{z_i t}]\,. \qquad (26)$$

But if z_i is a system zero, we have $b(z_i) = H(z_i) = 0$, and selecting as initial state

$$\underline{x}_0 = \underline{m}_i = \mathrm{Re}\,[(z_i I - A)^{-1} b] \qquad (27)$$

we obtain $y(t) \equiv 0$ for $t \geqslant 0$. The situation is as shown below

$$u(t) = \mathrm{Re}\,[e^{z_i t}] \longrightarrow \boxed{H(s)}\overset{\downarrow\,\underline{m}_i}{} \longrightarrow y(t) \equiv 0$$

Thus, although we are exciting the system with the complex exponential input $u(t) = \mathrm{Re}\,[e^{z_i t}]$, nothing comes out! For this reason, the system zeros z_i are sometimes called transmission or blocking zeros.

<u>Example</u>: For the circuit of Example 2, the transfer function

$$H(s) = \frac{\frac{3}{2}(s+2)}{(s+\frac{1}{2})^2 + \frac{1}{4}}$$

has a zero at $z = -2$. If we excite the system with the input $u(t) = e^{-2t}$ and select as initial state

$$\underline{x}_0 = (zI - A)^{-1} b = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

the output $y(t) \equiv 0$.

Zero characterization: Another characterization of the system zeros can be obtained by noting that if z_i is a zero of the transfer function $H(s)$ then

$$0 = H(z_i) = c\,(z_i I - A)^{-1} b + d$$

Denoting $\underline{l}_i = (z_i I - A)^{-1} b$ yields

$$c\underline{l}_i + d = 0$$

$$(z_i I - A)\,\underline{l}_i - b = \underline{0}$$

which can be rewritten in matrix form as

$$\begin{bmatrix} z_i I - A & -b \\ c & d \end{bmatrix} \begin{bmatrix} \underline{l}_i \\ 1 \end{bmatrix} = \underline{0} \,. \tag{28}$$

In other words, if z_i is a zero of $H(s)$ the $(n+1) \times (n+1)$ matrix

$$P(s) = \begin{bmatrix} sI - A & -b \\ c & d \end{bmatrix}$$

must be singular for $s = z_i$. Thus $s = z_i$ is a root of $p(s) = \det P(s)$. This can also be seen by noting that

$$p(s) = \det P(s) = \det(sI - A)\ \det\left(d + c(sI - A)^{-1} b\right)$$

$$= a(s)\,(\underbrace{d + c(sI - A)^{-1} b}_{H(s)}) = b(s) \tag{29}$$

where we have used the fact that, since $H(s) = d + c(sI - A)^{-1} b$ is scalar, $\det H(s) = H(s)$.

Thus, whereas the poles of $H(s)$ are the roots of $a(s) = \det(sI-A)$ its zeros are the roots of

$$b(s) = \det \begin{bmatrix} sI-A & -b \\ c & d \end{bmatrix} .$$

The matrix $P(s)$ provides more information than just the zeros of $H(s)$. Specifically if z_i is a zero of $H(s)$, and $\begin{bmatrix} \underline{\ell}_i \\ 1 \end{bmatrix}$ is a vector in the right null space of $P(z_i)$, by selecting

$$\underline{x}_0 = \mathrm{Re}[\underline{\ell}_i] = \mathrm{Re}[(z_i I - A)^{-1} b]$$

as initial state for the input $u(t) = \mathrm{Re}[e^{z_i t}]$, the output $y(t) \equiv 0$ for $t \geq 0$. In other words the vectors in the right null space of $P(z_i)$ provide the initial states $\underline{x}_0$ which ensure that the output is zero when the system is excited by the exponential input $\mathrm{Re}[e^{z_i t}]$.

Example: For the circuit of Example 2, we have

$$P(s) = \begin{bmatrix} sI-A & -b \\ c & d \end{bmatrix} = \begin{bmatrix} s & 1 & -1 \\ -1/2 & s+1 & -1/4 \\ 0 & 3 & 0 \end{bmatrix}$$

Laplace's expansion of $p(s) = \det P(s)$ with respect to the last row of $P(s)$ yields

$$p(s) = \frac{3}{4}(s+2)$$

so that $z = -2$ is a zero of $H(s)$. Furthermore

$$\underbrace{\left[\begin{array}{cc:c} -2 & 1 & -1 \\ -1/2 & -1 & -1/4 \\ \hdashline 0 & 3 & 0 \end{array}\right]}_{zI-A} \overbrace{\left[\begin{array}{c} -1/2 \\ 0 \\ \hdashline 1 \end{array}\right]}^{\underline{\ell}} = \underline{0}$$

so that $\underline{x}_0 = \underline{\ell} = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$ is the initial condition which, along with the input $u(t) = e^{-2t}$, gives $y(t) \equiv 0$. ∎