UNIVERSITY OF CALIFORNIA, DAVIS
Department of Electrical and Computer Engineering

EEC250 — Linear Systems and Signals — Fall 2009

Lecture 7

Topics:

a) Properties of the matrix exponential

b) Eigenvalue/eigenvector expressions for e^{At}

Matrix exponential: Let A be an $n \times n$ real matrix. Then the matrix exponential $\Phi(t) = \exp(At)$ is defined as the solution of the differential equation

$$\Phi(t) = A\Phi(t) \tag{1}$$

for $t \geq 0$, with initial condition $\Phi(0) = I_n$. If $F(s)$ denotes the Laplace transform of $\Phi(t)$, by Laplace transforming equation (1), we obtain

$$sF(s) - \Phi(0) = AF(s) ,$$

so that

$$(sI - A)F(s) = \Phi(0) = I_n .$$

This implies $F(s) = (sI - A)^{-1}$, so that $\exp(At)$ can also be obtained through the Laplace transform relation

$$\exp(At) \overset{LT}{\longleftrightarrow} (sI - A)^{-1} . \tag{2}$$

Property 1: $\exp(At)$ admits the power series expansion

$$\exp(At) = I_n + At + \frac{(At)^2}{2!} + \ldots + \frac{(At)^k}{k!} + \ldots , \tag{3}$$

for $t \geq 0$, which can be viewed as a matrix version of the power series expansion

$$\exp(at) = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}$$

of the scalar exponential function. To verify (3), denote the power series on the right hand side of (3) by $\Phi(t)$ and assume that it converges and can be differentiated term by term. Differentiating each term gives

$$\frac{d}{dt}\Phi(t) = A\left(I_n + At + \ldots + \frac{(At)^{k-1}}{(k-1)!} + \ldots\right) = A\Phi(t)$$

for $t \geq 0$, with $\Phi(0) = I_n$, so that $\Phi(t)$ obeys the differential equation (1) defining the matrix exponential.

Another way of deriving (3) relies on the observation that for s sufficiently large, $(sI - A)^{-1}$ admits the power series expansion

$$(sI-A)^{-1} = \frac{1}{s}\Big(I - \frac{A}{s}\Big)^{-1} = \frac{1}{s}\Big[I + \frac{A}{s} + \Big(\frac{A}{s}\Big)^2 + \ldots + \Big(\frac{A}{s}\Big)^k + \ldots\Big]. \tag{4}$$

Then, using the fact that

$$\frac{t^k}{k!} \overset{\mathrm{LT}}{\longleftrightarrow} \frac{1}{s^{k+1}},$$

and taking the inverse Laplace transform of (4) gives (3).

Examples: (i) Consider the $r \times r$ nilpotent matrix

$$N = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ & 0 & 1 & \ddots & 0 \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{bmatrix}$$

which has ones on its first superdiagonal, and zeros everywhere else. It has property that for $\ell < r$ the matrix

$$N^\ell = \begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ & 0 & \ldots & 0 & 1 & 0 & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & & & 0 & 1 \\ & & & & \ddots & \ddots & 0 \\ & & & & & 0 & \\ 0 & & & & & & 0 \end{bmatrix}$$

has ones on its ℓ-th superdiagonal, and zeros everywhere else, and $N^\ell = 0$ for $\ell \geq r$. Then in the power series expansion (3) for $\exp(Nt)$, only the first r terms are nonzero, so that

$$\begin{aligned} \exp(Nt) &= I_r + Nt \ldots + \frac{(Nt)^{r-1}}{(r-1)!} \\ &= \begin{bmatrix} 1 & t & t^2/2 & & & t^{r-1}/(r-1)! \\ 0 & 1 & t & t^2/2 & & \\ & 0 & 1 & t & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & 0 & 1 & t \\ 0 & & & & 0 & 1 \end{bmatrix}. \end{aligned}$$

(ii) Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then $A^{2p} = (-1)^p I_2$ and $A^{2p+1} = (-1)^p A$, where I_2 denotes the 2×2 identity matrix, so that

$$\begin{aligned} \exp(At) &= \Big(1 - \frac{t^2}{2!} + \frac{t^4}{4!} \dots\Big) I_2 + \Big(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\Big) A \\ &= \cos t I_2 + \sin t A = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} . \end{aligned}$$

Property 2: The matrix exponential has the *transition property*

$$\exp(A(t_1 + t_2)) = \exp(At_1)\exp(At_2) \tag{5}$$

for t_1, $t_2 \geq 0$. To see how this propoerty arises, consider solving the differential equation (1) for $0 \leq t \leq t_1 + t_2$. We can solve the equation in one step over the whole interval, in which case the solution at $t = t_1 + t_2$ is $\exp(A(t_1 + t_2))$. Alternatively, we can first solve the equation over $[0, t_2]$ and then over $[t_2, t_1 + t_2]$. In this case the solution at $t = t_2$ is $\exp(At_2)$, and the solution for $t_2 \leq t \leq t_1 + t_2$ is obtained by solving

$$\frac{d}{dt}\Phi(t) = A\Phi(t)$$

over $[t_2, t_1 + t_2]$ with initial condition $\Phi(t_2) = \exp(At_2)$. But the system is LTI, so that the solution at $t = t' + t_2$ with $t' \geq 0$ is given by

$$\Phi(t) = \exp(At')\Phi(t_2) .$$

Setting $t = t_1 + t_2$ and $t' = t_1$ in this identity gives (5).

Property 3: In general if A and B are two arbitrary matrices

$$\exp((A + B)t) \neq \exp(At)\exp(Bt) . \tag{6}$$

To see this, let

$$A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} .$$

Both A and B are nilpotent with

$$\exp(At) = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \exp(Bt) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} ,$$

so

$$\exp(At)\exp(Bt) = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} 1 - t^2 & -t \\ t & 1 \end{bmatrix} .$$

On the other hand

$$A + B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} ,$$

and it was shown earlier that

$$\exp((A + B)t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

which is clearly different from $\exp(At)\exp(Bt)$.

However *if A and B commute*, i.e., $AB = BA$, we have

$$\exp((A+B)t) = \exp(At)\exp(Bt)\ . \tag{7}$$

To prove this, note that when A and B commute

$$(A+B)^k = A^k + \binom{k}{1} A^{k-1}B + \ldots + \binom{k}{1} AB^{k-1} + B^k\ .$$

Taking this identity into account, and multiplying power series term by term gives

$$\begin{aligned}\exp(At)\exp(Bt) &= \left[I + At + \frac{(At)^2}{2!} + \ldots\right]\left[I + Bt + \frac{(Bt)^2}{2!} + \ldots\right] \\ &= I + (A+B)t + \frac{((A+B)t)^2}{2!} + \ldots = \exp((A+B)t)\ .\end{aligned}$$

Examples: (i) Consider the $r \times r$ Jordan block

$$J = \begin{bmatrix} \lambda & 1 & & & 0 \\ & \lambda & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{bmatrix} = \lambda I_r + N$$

where λI_r and N commute. We have

$$e^{\lambda I_r t} = \left(1 + \lambda t + \frac{(\lambda t)^2}{2!} + \ldots\right) I_r = e^{\lambda t} I_r$$

and

$$\begin{aligned}\exp(Jt) &= \exp((\lambda I_r + N)t) = \exp(\lambda I_r t)\exp(Nt) \\ &= \exp(\lambda t)\begin{bmatrix} 1 & t & & & t^{r-1}/(r-1)! \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & t \\ 0 & & & & 1 \end{bmatrix}.\end{aligned}$$

(ii) Let

$$A = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} = \sigma_2 + \omega Q$$

with

$$Q \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where I_2 and Q commute. Then

$$\exp(At) = \exp(\sigma t)\exp(\omega Q t) = \exp(\sigma t)\begin{bmatrix} \cos\omega t & -sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix} .$$

Property 4: $\exp(At)$ is an *invertible matrix* for all $t \geq 0$ and

$$\Big(\exp(At)\Big)^{-1} = \exp((-A)t) . \tag{8}$$

To see this, note that A and $-A$ commute and $A + (-A) = 0$, so that

$$e^{0t} = I_n = \exp(At)\exp((-A)t) .$$

Eigenvalue/eigenvector expressions: The matrix exponential $\exp(At)$ can also be expressed in terms of the eigenvalues and eigenvectors (or generalized eigenvectors) of A.

Case 1: A is diagonalizable: In this case A admits n independent eigenvectors $\mathbf{p}_i$ corresponding to eigenvalues λ_i, i.e.

$$A\mathbf{p}_i = \lambda_i \mathbf{p}_i \tag{9}$$

for $1 \leq i \leq n$. The relations (9) can be combined as a single matrix equation

$$AP = P\Lambda$$

with

$$P = \begin{bmatrix} \mathbf{p}_1 & \dots & \mathbf{p}_i & \dots & \mathbf{p}_n \end{bmatrix} \text{ and } \Lambda = \text{diag}\,\{\lambda_i,\, 1 \leq i \leq n\} ,$$

so that $A = P\Lambda P^{-1}$. The matrix

$$P^{-1} = Q = \begin{bmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_i^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix}$$

yields the left eigenvectors of A since $P^{-1}A = P^{-1}\Lambda$, or equivalently

$$\mathbf{q}_i^T A = \lambda_i \mathbf{q}_i^T ,$$

for $1 \leq i \leq n$. Then

$$A^2 = (P\Lambda P^{-1})(P\Lambda P^{-1}) = P\Lambda^2 P^{-1}$$

and $A^k = P\Lambda^k P^{-1}$, so that

$$\begin{aligned} \exp(At) &= \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = P\Big[\sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!}\Big]P^{-1} \\ &= P\exp(\Lambda t)P^{-1} . \end{aligned} \tag{10}$$

The main advantage of expression (10) is that, since Λ is diagonal, its exponential matrix is easy to compute and is also diagonal, i.e.

$$e^{\Lambda t} = \text{diag}\,\{\exp(\lambda_i t),\, 1 \le i \le n\}\,.$$

The identity (10) expresses $\exp(At)$ completely in terms of the eigenvalues and right and left eigenvectors of A. To see this, note that

$$\begin{aligned}
\exp(At) &= P\exp(\Lambda t)Q \\
&= \begin{bmatrix} \mathbf{p}_1 & \dots & \mathbf{p}_i & \dots & \mathbf{p}_n \end{bmatrix} \text{diag}\,(\exp(\lambda_i t),\, 1 \le i \le n) \begin{bmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_i^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix}
\end{aligned}$$

can be rewritten as

$$e^{At} = \sum_{i=1}^{n} \mathbf{p}_i \mathbf{q}_i^T \exp(\lambda_i t)\,. \tag{11}$$

Example: Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ 6 & 0 & -6 \end{bmatrix}.$$

Then

$$sI - A = \begin{bmatrix} s-1 & 0 & 2 \\ 0 & s+1 & 0 \\ -6 & 0 & s+6 \end{bmatrix}$$

and

$$a(s) = \det(sI - A) = (s+1)\big[(s-1)(s+6) + 12\big] = (s+1)(s+2)(s+3)\,.$$

The eigenvector $\mathbf{p}_1$ corresponding to $\lambda_1 = -1$ is obtained by solving

$$(\lambda_1 I - A)\mathbf{p}_1 = \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ -6 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives $x_1 = x_3 = 0$ with x_2 free, and since the scaling of $\mathbf{p}_1$ is arbitrary, we select

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Similarly the eigenvector $\mathbf{p}_2$ corresponding to $\lambda_2 = -2$ is obtained by solving

$$(\lambda_2 I - A)\mathbf{p}_2 = \begin{bmatrix} -3 & 0 & 2 \\ 0 & -1 & 0 \\ -6 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which gives $x_2 = 0$ and $3x_1 = 2x_3$, so that we can select

$$\mathbf{p}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}.$$

Finally, to find eigenvector $\mathbf{p}_3$ corresponding to $\lambda_3 = -3$, we solve

$$(\lambda_3 I - A)\mathbf{p}_3 = \begin{bmatrix} -4 & 0 & 2 \\ 0 & -2 & 0 \\ -6 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives $x_2 = 0$ and $x_3 = 2x_1$, so that we can choose

$$\mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Inverting the matrix

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix}$$

gives

$$Q = P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \\ -3 & 0 & 2 \end{bmatrix},$$

so that

$$\begin{aligned} \exp(At) &= \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} \exp(-t) & 0 & 0 \\ 0 & \exp(-2t) & 0 \\ 0 & 0 & \exp(-3t) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \\ -3 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4\exp(-2t) - 3\exp(-3t) & 0 & 2[-\exp(-2t) + \exp(-3t)] \\ 0 & \exp(-t) & 0 \\ 6[\exp(-2t) - \exp(-3t)] & 0 & -3\exp(-2t) + 4\exp(-3t) \end{bmatrix}. \end{aligned}$$

In the expression (11) for $\exp(At)$, although $\exp(At)$ is real, the eigenvalues λ_i and eigenvectors $\mathbf{p}_i$ and $\mathbf{q}_i$ may be *complex*.

Example: Let

$$A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Then

$$sI - A = \begin{bmatrix} s+2 & 0 & 1 \\ 0 & s+2 & 0 \\ -2 & 0 & s \end{bmatrix},$$

and

$$a(s) = \det sI - A = (s^2 + 2s + 2)(s + 2) = (s + 1 - j)(s + 1 + j)(s + 2)i \, .$$

Thus A has two complex conjugate eigenvalues $\lambda_{\pm} = -1 \pm j$ and a real eigenvalue $\lambda_3 = -2$. The eigenvector $\mathbf{p}_+$coorresponding to λ_+ is obtained by solving

$$(\lambda_+ I - A)\mathbf{p}_+ = \begin{bmatrix} 1+j & 0 & 1 \\ 0 & 1+j & 0 \\ -2 & 0 & -1+j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which gives $x_2 = 0$, $x_3 = -(1+j)x_1$, so that we can select

$$\mathbf{P}_+ = \begin{bmatrix} 1 \\ 0 \\ -(1+j) \end{bmatrix}.$$

By observing that $\lambda_- = \lambda_+^*$ and A is real, and taking the complex conjugate of the eigenvector equation for λ_+ we find that

$$\mathbf{p}_- = \mathbf{p}_+^* = \begin{bmatrix} 1 \\ 0 \\ -(1-j) \end{bmatrix}$$

satisfies

$$(\lambda_- I - A)\mathbf{p}_- = \mathbf{0} \, ,$$

so it is the eigenvector corresponding to eigenvalue λ_-. Finally, the eigenvector corresponding to $\lambda_3 = -2$ is obtained by solving

$$(\lambda_3 I - A)\mathbf{p}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which yields $x_1 = x_3 = 0$, so that wecan select

$$\mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then if

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -(1+j) & -(1-j) & 0 \end{bmatrix}$$

we find

$$Q = P^{-1} = \begin{bmatrix} (1+j)/2 & 0 & j/2 \\ (1-j)/2 & 0 & -j/2 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$\exp(At) = P \exp(\Lambda t) Q$$

with

$$\exp(\Lambda t) = \begin{bmatrix} \exp(-t+jt) & 0 & 0 \\ 0 & \exp(-t-jt) & 0 \\ 0 & 0 & \exp(-2t) \end{bmatrix}.$$

Real form of the eigenvalue/eigenvector expansion of $\exp(At)$: To obtain expressions involving only real quantities, one can proceed as follows. First observe that since A is a real matrix, if $(\lambda_i,\, \mathbf{p}_i)$ is an eigenvalue/eigenvector pair for A, so is $(\lambda_i^*,\, \mathbf{p}_i^*)$. Assume now that A has r pairs of complex conjugate eigenvalues and $q = n - 2r$ real eigenvalues. Then order the eigenvalues of A as $\{\lambda_1, \lambda_1^* \ldots, \lambda_k, \lambda_k^*, \ldots, \lambda_r, \lambda_r^*, \lambda_{2r+1}, \ldots, \lambda_n\}$. In the eigenvalue/eigenvector relation $A\mathbf{p}_k = \lambda_k \mathbf{p}_k$, the eigenvalue λ_k and eigenvector $\mathbf{p}_k$ can be decomposed into their real and imaginary parts as

$$\begin{aligned} \lambda_k &= a_k + jb_k \\ \mathbf{p}_k &= \mathbf{p}_k^R + j\mathbf{p}_k^I . \end{aligned}$$

This implies that

$$\begin{aligned} A\mathbf{p}_k^R &= a_k \mathbf{p}_k^R - b_k \mathbf{p}_k^I \\ A\mathbf{p}_k^I &= b_k \mathbf{p}_k^R + a_k \mathbf{p}_k^I . \end{aligned}$$

Then, if we consider the matrix

$$M = \begin{bmatrix} \mathbf{p}_1^R \mathbf{p}_1^I \ \ldots \ \mathbf{p}_k^R \mathbf{p}_k^I \ \ldots \ \mathbf{p}_r^R \mathbf{p}_r^I \ \mathbf{p}_{2r+1} \ \ldots \ \mathbf{p}_n \end{bmatrix},$$

M is real and

$$AM = MD$$

with

$$D = \begin{bmatrix} D_1 & & & & & & & \\ & \ddots & & & 0 & & & \\ & & D_k & & & & & \\ & & & \ddots & & & & \\ & & & & D_r & & & \\ & 0 & & & & \lambda_{2r+1} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_n \end{bmatrix}$$

where

$$D_k \triangleq \begin{bmatrix} a_k & b_k \\ -b_k & a_k \end{bmatrix}$$

If $N = M^{-1}$, this implies

$$A = MDN \tag{12}$$

where D is block diagonal. It is constituted of r 2×2 blocks corresponding to the r pairs $(\lambda_k,\, \lambda_k^*)$ of complex conjugate eigenvalues, as well as $n - 2r$ 1×1 blocks corresponding to the $n - 2r$ real eigenvalues $\lambda_{2r+1}, \ldots, \lambda_n$ of A.

The expression (12) for A implies

$$\exp(At) = M\exp(Dt)N\,, \tag{13}$$

with

$$\exp(Dt) = \begin{bmatrix} \exp(D_1t) & & & & & & \\ & \ddots & & & & 0 & \\ & & \exp(D_kt) & & & & \\ & & & \ddots & & & \\ & & & & \exp(D_rt) & & \\ & 0 & & & & \exp(\lambda_{2r+1}t) & \\ & & & & & & \ddots \\ & & & & & & & \exp(\lambda_nt) \end{bmatrix}$$

and

$$\exp(D_kt) = \exp(a_kt)\begin{bmatrix} \cos(b_kt) & \sin(b_kt) \\ -\sin(b_kt) & \cos(b_kt) \end{bmatrix},$$

where the expression (13) involves only real matrices.

Example: It was found earlier that the matrix

$$A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

has eigenvalues $\lambda_\pm = -1 \pm j$ and $\lambda_3 = -2$. Its eigenvectors are

$$\mathbf{p}_\pm = \begin{bmatrix} 1 \\ 0 \\ -(1 \pm j) \end{bmatrix} \quad \text{and} \quad \mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The vector $\mathbf{p}_+$ can be decomposed as $\mathbf{p}_+ = \mathbf{p}_R + j\mathbf{p}_I$ with

$$\mathbf{p}_R = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{p}_I = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Let

$$\begin{aligned} M &= \begin{bmatrix} \mathbf{p}_R & \mathbf{p}_I & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \\ N &= M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \\ D &= \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \end{aligned}$$

Then

$$\begin{aligned}
\exp(At) &= M\exp(Dt)N \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \exp(-t)\cos t & \exp(-t)\sin t & 0 \\ -\exp(-t)\sin t & \exp(-t)\cos t & 0 \\ 0 & 0 & \exp(-2t) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} .
\end{aligned}$$

Case 2: A is not diagonalizable. In this case A has a Jordan block of size 2 or larger, or equivalently it has a repeated eigenvalue which admits fewer independent eigenvectors than its multiplicity. In this case, we can find an invertible matrix T constituted of the eigenvectors and generalized eigenvectors of A such that

$$A = TJT^{-1} \tag{14}$$

with

$$J = \begin{bmatrix} J_1 & & & & 0 \\ & \ddots & & & \\ & & J_i & & \\ & & & \ddots & \\ 0 & & & & J_\ell \end{bmatrix}$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ 0 & & & \lambda_i \end{bmatrix} = n_i \times n_i \text{ Jordan block} ,$$

where $\sum_{i=1}^{\ell} n_i = n$. The matrix exponential $\exp(At)$ is then given by

$$\exp(At) = T\exp(Jt)T^{-1} , \tag{15}$$

with

$$\exp(Jt) = \begin{bmatrix} \exp(J_1 t) & & & & 0 \\ & \ddots & & & \\ & & \exp(J_i t) & & \\ & & & \ddots & \\ 0 & & & & \exp(J_\ell t) \end{bmatrix}$$

and

$$\exp(J_i t) = \exp(\lambda_i t) \begin{bmatrix} 1 & t & & \frac{t^{n_i-1}}{(n_i-1)!} \\ & \ddots & \ddots & \\ & & \ddots & t \\ 0 & & & 1 \end{bmatrix} .$$

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$sI_4 - A = \begin{bmatrix} s-1 & -2 & 0 & -1 \\ 0 & s-1 & 0 & 0 \\ 0 & 1 & s-1 & 0 \\ 0 & 0 & 0 & s-1 \end{bmatrix},$$

and using Laplace's expansion of $a(s) = \det sI - A$ with respect to the first column yields $a(s) = (s-1)^4$. The eigenvectors $\mathbf{p}$ of A corresponding to $\lambda_1 = 1$ satisfy

$$(\lambda_1 I - A)\mathbf{p} = \begin{bmatrix} 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives $x_2 = x_4 = 0$, so that A has the two eigenvectors

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since the number of eigenvectors (two) is less than the multiplicity (four) of λ_1, A is not diagonalizable. Observing that $(\lambda_1 I - A)^2 = 0$, we can conclude that A will have 2 Jordan blocks of size 2×2 corresponding to $\lambda_1 = 1$. The generalized eigenvector $\mathbf{g}_1$ corresponding to $\mathbf{p}_1$ is given by

$$(A - \lambda_1 I)\mathbf{g}_1 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which gives $x_2 = 0$ and $x_4 = 1$, so that we can select

$$\mathbf{g}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Similarly the generalized eigenvector $\mathbf{g}_2$ corresponding to $\mathbf{p}_2$ satisfies

$$(A - \lambda_1 I)\mathbf{g}_2 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We obtain $x_2 = -1$ and $x_4 = 2$, so that we can select

$$\mathbf{g}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}.$$

Then if

$$\begin{aligned}
T &= \begin{bmatrix} \mathbf{p}_1 & \mathbf{g}_1 & \mathbf{p}_2 & \mathbf{g}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \\
T^{-1} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \\
J &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned}$$

we have $A = TJT^{-1}$ and

$$\exp(At) = T\exp(Jt)T^{-1}$$

with

$$\exp(Jt) = \exp(t)\begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that the Jordan decomposition $A = TJT^{-1}$ may be complex. There exists a real form of this decomposition, but it will not be needed in the remainder of this course.