EEC 250 Linear Systems and Signals

Lecture 14

Topics: a) Properties of linear state feedback

b) Pole assignment: single input case

c) Pole assignment: multiple input case

Linear state feedback: Consider the state-space model

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \qquad (1a)$$

$$\underline{y}(t) = C\,\underline{x}(t) + D\,\underline{u}(t) \qquad (1b)$$

where $\underline{u} \in \mathbb{R}^m$, $\underline{x} \in \mathbb{R}^n$ and $\underline{y} \in \mathbb{R}^p$. If we apply the state feedback law

$$\underline{u}(t) = \underline{v}(t) - K\,\underline{x}(t) \qquad (2)$$

to the system (1a)-(1b), where $\underline{v} \in \mathbb{R}^m$ and K is a $m \times n$ matrix, the closed-loop system takes the form

$$\dot{\underline{x}}(t) = (A - BK)\,\underline{x}(t) + B\,\underline{v}(t) \qquad (3a)$$

$$\underline{y}(t) = (C - DK)\,\underline{x}(t) + D\,\underline{v}(t)\,. \qquad (3b)$$

The state feedback transformation (2) has the following properties.

Property 1: If the pair (A, B) is reachable, so is $(A-BK, B)$.

Proof: If (A, B) is reachable over $[0,T]$, starting from a zero initial state and for an arbitrary vector $\underline{z} \in \mathbb{R}^n$, we can find an input function $\underline{u}^*(t)$, $0 \leq t \leq T$ such that if $\underline{x}^*(t)$, $0 \leq t \leq T$ is the corresponding state trajectory, we have $\underline{x}^*(0) = \underline{0}$ and $\underline{x}^*(T) = \underline{z}$. Then, if we apply the input function

$$\underline{v}^*(t) = \underline{u}^*(t) + K\,\underline{x}^*(t) \tag{4}$$

to the closed-loop system (3a)-(3b), the state trajectory $\underline{x}^*(t)$ remains the same, so that starting from $\underline{x}^*(0) = \underline{0}$, we reach $\underline{x}^*(T) = \underline{z}$ at time $t = T$. Thus, the pair $(A-BK, B)$ is reachable.

Another way of proving this result consists in noting that

$$[sI - A + BK \quad B] = [sI - A \quad B] \begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \tag{5}$$

where the matrix $\begin{bmatrix} I & 0 \\ K & I \end{bmatrix}$ is invertible, so that the matrix $[sI-A+BK \quad B]$ has full rank for all s if and only if $[sI-A \quad B]$ has the same property.

<u>Property 2</u>: The zeros of the system are not affected by state feedback.

<u>Proof</u>: For a system with as many inputs as outputs, i.e. with $m = p$, the zeros of the transfer matrix

$$H(s) = C(sI-A)^{-1}B + D$$

are the zeros of the determinant of the system matrix

$$\begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix}$$

Since the closed-loop system matrix can be expressed as

$$\begin{bmatrix} sI-A+BK & B \\ -C+DK & D \end{bmatrix} = \begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix}\begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \tag{6}$$

we have

$$\det\begin{bmatrix} sI-A+BK & B \\ -C+DK & D \end{bmatrix} = \det\begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix} \tag{7}$$

so that the zeros of the system are preserved under state feedback.

<u>Property 3</u>: The eigenvalues of the unreachable part of the system are not affected by state feedback.

<u>Proof</u>: Since the characteristic polynomial of a system is not affected by a similarity transformation, without loss of generality we can assume that the system (1a) has been decomposed into its reachable and unreachable components, so that it takes the form

$$\begin{bmatrix} \dot{\underline{x}}_r \\ \dot{\underline{x}}_{\bar{r}} \end{bmatrix} = \begin{bmatrix} A_r & A_{r\bar{r}} \\ 0 & A_{\bar{r}} \end{bmatrix}\begin{bmatrix} \underline{x}_r \\ \underline{x}_{\bar{r}} \end{bmatrix} + \begin{bmatrix} B_r \\ 0 \end{bmatrix}\underline{u} \; . \tag{8}$$

Under the state feedback law

$$\underline{u} = \underline{v} - [\, K_r \;\; K_{\bar{r}} \,] \begin{bmatrix} \underline{x}_r \\ \underline{x}_{\bar{r}} \end{bmatrix} \tag{9a}$$

the closed-loop system takes the form

$$\begin{bmatrix} \dot{\underline{x}}_r \\ \dot{\underline{x}}_{\bar{r}} \end{bmatrix} = \begin{bmatrix} A_r - B_r K_r & A_{r\bar{r}} - B_r K_{\bar{r}} \\ 0 & A_{\bar{r}} \end{bmatrix} \begin{bmatrix} \underline{x}_r \\ \underline{x}_{\bar{r}} \end{bmatrix} + \begin{bmatrix} B_r \\ 0 \end{bmatrix} v \, , \tag{9b}$$

so that the closed-loop characteristic polynomial is given by

$$a_{CL}(s) = \det \begin{bmatrix} sI - A_r + B_r K_r & -A_{r\bar{r}} + B_r K_{\bar{r}} \\ 0 & sI - A_{\bar{r}} \end{bmatrix}$$

$$= \det(sI - A_r + B_r K_r)\, \det(sI - A_{\bar{r}}) \, . \tag{10}$$

This expression shows that the characteristic polynomial $a_{\bar{r}}(s) = \det(sI - A_{\bar{r}})$ of the unreachable part of the system is unaffected by state feedback. Thus, no matter how we select the feedback gain matrix K, the eigenvalues of $A_{\bar{r}}$, i.e. the unreachable modes of the system, form a subset of the closed-loop eigenvalues.

This last property can be discussed more clearly in light of the following theorem, which constitutes the main result of this lecture.

Pole assignment theorem: The pair (A, B) is reachable if and only if we can move arbitrarily the eigenvalues of A by state feedback, which means that given an arbitrary characteristic polynomial

$$\alpha(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \dots + \alpha_n$$

we can always find a feedback gain matrix K such that

$$\det(sI - A + BK) = \alpha(s) \quad . \tag{11}$$

Clearly, the Property 3 of linear state feedback implies that if a system is not reachable, we cannot move arbitrarily its eigenvalues by state feedback, since the eigenvalues of its unreachable part are fixed. In the remainder of this lecture, we shall prove the converse, namely that if the pair (A, B) is reachable its eigenvalues can be assigned arbitrarily by state feedback.

This result provides an engineering motivation for the concept of reachability, which until now was rather abstract. It indicates that if the pair (A, B) is reachable, and if all the entries x_i of the state vector $\underline{x}$ can be measured directly, the dynamics of the system are completely under our control, if we apply state feedback.

Specifically, let μ_i $1 \leq i \leq n$ be the desired closed-loop eigenvalues, which for simplicity are assumed distinct, and let the feedback matrix K be selected such that (11) is satisfied, with

$$\alpha(s) = \prod_{i=1}^{n} (s - \mu_i) \quad . \tag{12}$$

Then, with $v(t) \equiv 0$, the solution of the closed-loop system

$$\dot{\underline{x}}(t) = (A - BK)\, \underline{x}(t) \qquad \underline{x}(0) = \underline{x}_0 \tag{13}$$

is given by

$$x(t) = e^{(A-BK)t}\, \underline{x}_0 \quad , \tag{14}$$

where the eigenvalues of $e^{(A-BK)t}$ are $e^{\mu_i t}$ $1 \leq i \leq n$, so that by appropriately selecting the μ_is, we can ensure that $\underline{x}(t)$ decays to zero as rapidly as desired. It is the ability to change arbitrarily the dynamics of a reachable system by state feedback that makes the reachability property useful from an engineering viewpoint.

Single-input case: We first prove the pole assignment theorem for single-input systems. To do so, we shall rely on the following lemma.

Lemma 1: If the pair (A, b) is reachable with

$$\det(sI - A) = a(s) = s^n + a_1 s^{n-1} + \ldots + a_n \; ,$$

it is linked by a similarity transformation to the controller form pair

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ & 1 & \ddots & \vdots \\ 0 & & \ddots\;\; 1 & 0 \end{bmatrix} \qquad b_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{15}$$

<u>Proof</u>: If (A, b) is reachable, the $n \times n$ reachability matrix

$$R = [\, b \;\; Ab \; \ldots \; A^{n-1} b \,]$$

is invertible. Let $\underline{f}^T$ be the last row of the inverse of R, so that

$$\underline{f}^T R = [\, 0 \;\; \ldots \; 0 \;\; 1 \,] \tag{16}$$

Then, consider the matrix

$$S = \begin{bmatrix} \underline{f}^T A^{n-1} \\ \vdots \\ \underline{f}^T A \\ \underline{f}^T \end{bmatrix} \tag{17}$$

The matrix S is invertible, since by taking into account the definition (16) of $\underline{f}^T$ and the structure of the reachability matrix R, we find

$$SR = U \quad , \tag{18}$$

where U is an upper triangular matrix with ones along its diagonal, i.e. $u_{ij} = 0$ for $i > j$ and $u_{ii} = 1$. This implies

$$\det S \;\det R = 1 \tag{19}$$

so that S is invertible.

The first column of relation (18) reduces to

$$Sb = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = b_c \qquad (20)$$

Furthermore, by taking into account Cayley-Hamilton's identity

$$A^n = -a_1 A^{n-1} - a_2 A^{n-2} \cdots - a_n I_n$$

we obtain

$$SA = \begin{bmatrix} f^T A^n \\ \vdots \\ f^T A^2 \\ f^T A \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \begin{bmatrix} f^T A^{n-1} \\ \vdots \\ f^T A \\ f^T \end{bmatrix}$$

$$= A_c S \quad . \qquad (21)$$

Denoting $T \triangleq S^{-1}$, this shows

$$b_c = T^{-1} b \qquad A_c = T^{-1} A T \ ,$$

so that (A, b) and (A_c, b_c) are related by the similarity transformation T. ∎

<u>Remark</u>: From identities (20) and (21) we find

$$S R = R_c \qquad (22)$$

where $R_c = [\, b_c \ A_c b_c \ \cdots \ A_c^{n-1} b_c \,]$ is the reachability matrix of the pair (A_c, b_c), so that the upper triangular matrix U on the right hand side of (18) is R_c, and

$$T = R R_c^{-1}. \tag{23}$$

By induction, it is also easy to verify that

$$A_c b_c + a_1 b_c = \underline{e}_2 \tag{24a}$$

$$A_c^2 b_c + a_1 A_c b_c + a_2 b_c = \underline{e}_3 \tag{24b}$$

$$\cdots$$

$$A_c^{n-1} b_c + a_1 A_c^{n-2} b_c + \ldots + a_{n-1} b_c = \underline{e}_n \tag{24c}$$

where

$$\underline{e}_k = \begin{bmatrix} 0 \\ \\ 0 \\ 1 \\ 0 \\ \\ 0 \end{bmatrix} \leftarrow \text{kth entry}$$

is the kth unit basis vector of $\mathbb{R}^n$. This implies

$$R_c^{-1} = \begin{bmatrix} 1 & a_1 & a_2 & & a_{n-1} \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_2 \\ & 0 & & \ddots & a_1 \\ & & & & 1 \end{bmatrix}, \tag{25}$$

so that R_c^{-1} is upper triangular, and its entries are constant along diagonals. A matrix with this structure is said to be a Toeplitz matrix. The identities (23) and (25) will be useful in the following discussion.

<u>Pole assignment formula</u>: Given a reachable pair (A, b), we can now

compute a feedback gain matrix $k = [k_1 \ldots k_n]$ which satisfies

$$\det(sI - A + bk) = \alpha(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n \qquad (26)$$

where the characteristic polynomial $\alpha(s)$ is arbitrary.

The derivation of an expression for k is performed in two steps. These two steps are only required to obtain a formula for k. Once this formula is available, it should be used directly.

Step 1: We convert the open-loop system

$$\dot{\underline{x}} = A\underline{x} + bu \qquad (27)$$

to the controller form

$$\dot{\underline{x}}_c = A_c \underline{x}_c + b_c u \qquad (28)$$

by applying the similarity transformation $\underline{x} = T\underline{x}_c$ with T given by (23). Then, if we apply the feedback law

$$u = v - \kappa x_c \qquad (29)$$

with $\kappa = [\kappa_1 \; \kappa_2 \ldots \kappa_n]$, the closed-loop dynamics take the form

$$\dot{\underline{x}}_c = (A_c - b_c\kappa)\underline{x}_c + b_c v \qquad (30a)$$

with

$$A_c - b_c\kappa = \begin{bmatrix} -(a_1+\kappa_1) & -(a_2+\kappa_2) & \cdots & -(a_n+\kappa_n) \\ 1 & 0 & \cdots & 0 \\ & 1 & \ddots & \vdots \\ 0 & & \ddots & \\ & & 1 & 0 \end{bmatrix}. \qquad (30b)$$

Since the matrix $A_c - b_c K$ is in companion form, its characteristic polynomial can be obtained by inspection. It is given by

$$\det(sI - A_c + b_c K) = s^n + (a_1 + K_1)s^{n-1} + \ldots + (a_n + K_n) \quad . \qquad (31)$$

Matching its coefficients with those of the desired characteristic polynomial $\alpha(s)$ yields

$$K = [K_1 \; K_2 \ldots K_n] = [\alpha_1 - a_1 \;\; \alpha_2 - a_2 \; \ldots \; \alpha_n - a_n]$$
$$= (\underline{\alpha} - \underline{a})^T \quad , \qquad (32)$$

where

$$\underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} \qquad \underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{bmatrix}$$

are the vectors of coefficients of the closed-loop and open-loop characte--ristic polynomials, respectively. The above discussion shows that when a reachable system has been placed in controller form, the pole assignment problem can be solved by inspection. This explains why the ~~name~~ "controller" name is attached to this form. The expression (32) shows also that the gain K_i that needs to be applied to the state x_{c_i} in the feedback law (29) equals the difference $\alpha_i - a_i$ between the closed-loop and open-loop characteristic polynomial coefficients. This implies that the more we want to change the poles of the system, the larger the feedback gains, and thus the control effort, that must be applied.

<u>Step 2</u>: The feedback law (29) can be converted back to the original coordinate system, which yields

$$u = v - \kappa \underline{x}_c = v - k \underline{x} \qquad (33)$$

where

$$k = \kappa T^{-1} = (\underline{\alpha} - \underline{a})^T R_c R^{-1} . \qquad (34)$$

The expression (34) is the desired formula for the feedback gain matrix. Note that although it was derived by employing the controller form, it holds in arbitrary coordinates, and can be used without invoking this form.

<u>Direct method</u>: For systems of small dimension, it is sometimes easier to match the coefficients of s^i with $0 \le i \le n-1$ on both sides of the identity

$$\det(sI - A + bk) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n , \qquad (35)$$

thus yielding n equations in terms of the n unknowns $k_1, k_2 \dots k_n$. To see that these equations are linear, note that

$$\det \left[\begin{array}{c|c} sI - A & b \\ \hline -k & 1 \end{array} \right] = \det(sI - A + bk) , \qquad (36)$$

and since the determinant of a matrix is linear with respect to ea
row separately, we find that $\det(sI - A + bk)$ is a linear function of the entries k_i of k.

Example 1: Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{b} u$$

The open-loop characteristic polynomial is given by

$$a(s) = \det \begin{bmatrix} s & -1 \\ -9 & s \end{bmatrix} = s^2 - 9 = (s+3)(s-3) ,$$

so that the eigenvalues of A are 3 and -3, and thus A is unstable. The pair (A, b) is reachable since the reachability matrix

$$R = [\, b \;\; Ab \,] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

is invertible. Assume that we want to move both poles to $s = -1$, so that

$$\alpha(s) = (s+1)^2 = s^2 + 2s + 1 .$$

By matching the coefficients of $\alpha(s)$ with

$$\det(sI - A + bk) = \det \begin{bmatrix} s & -1 \\ -9-k_1 & s-k_2 \end{bmatrix} = s^2 - k_2 s - (k_1 + 9) ,$$

we find $k_2 = -2$, $k_1 = -10$. Equivalently, if we apply the formula (34), we find $R_c = I_2$ and

$$k = (\underline{\alpha} - \underline{a})^T R^{-1}$$
$$= \left([2 \quad 1] - [0 \quad -9] \right) \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = [-10 \quad -2] .$$

■

<u>Eigenvalue/eigenvector expression</u>: One disadvantage of the pole assignment formula (34) is that it expresses the gain matrix k as a function of the coefficients of the closed-loop and open-loop characteristic polynomials $\alpha(s)$ and $a(s)$. These coefficients are hard to interpret physically. To gain a better intuitive understanding of the pole assignment problem, it is of interest to express k in terms of the open-loop eigenvalues and eigenvectors $(\lambda_i, \underline{p}_i)$, and the closed-loop eigenvalues μ_i, $1 \leq i \leq n$. We assume that the open-loop eigenvalues are distinct, i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$.

According to (36), we have

$$\alpha(s) = \det(sI - A + bk) = \det \begin{bmatrix} sI-A & b \\ -k & 1 \end{bmatrix}$$
$$= a(s)\,[1 + k(sI-A)^{-1}b\,] . \qquad (37)$$

If $\Lambda = \mathrm{diag}\{\lambda_i\}$ and

$$P = [\,\underline{p}_1 \cdots \underline{p}_n\,] \qquad Q = P^{-1} = \begin{bmatrix} \underline{q}_1^T \\ \vdots \\ \underline{q}_n^T \end{bmatrix}$$

are the matrices formed by the right and left eigenvectors of A, it is easy to verify that the identity

$$(sI-A)^{-1} = P\,(sI-\Lambda)^{-1}\,Q = \sum_{i=1}^{n} \frac{\underline{p}_i\,\underline{q}_i^T}{s-\lambda_i} \tag{38}$$

holds. Substituting this expression inside (37) yields

$$\frac{\alpha(s)}{a(s)} = \frac{\prod_{j=1}^{n}(s-\mu_j)}{\prod_{j=1}^{n}(s-\lambda_j)} = 1 + \sum_{i=1}^{n} \frac{k\underline{p}_i\,\underline{q}_i^T b}{s-\lambda_i} \;, \tag{39}$$

and multiplying both sides of (39) by $s-\lambda_i$ and setting $s=\lambda_i$, we find

$$k\underline{p}_i\,\underline{q}_i^T b = \frac{\prod_{j=1}^{n}(\lambda_i-\mu_j)}{\prod_{j\neq i}(\lambda_i-\lambda_j)} \;. \tag{40}$$

To interpret the expression (40), it is convenient to assume that the left and right eigenvectors $\underline{q}_i$ and $\underline{p}_i$ have been selected in such a way that their norms are balanced, i.e. $\|\underline{q}_i\| = \|\underline{p}_i\|$. Then, from (40) we see that : (i) the magnitude of k increases as the separation $|\lambda_i-\mu_j|$ between the open- and closed-loop poles increases, (ii) the control effort $k\underline{p}_i$ exerted on the ith mode of the system increases as $\underline{q}_i^T b$ decreases, i.e. as the relative reachability of the ith mode decreases.

Denote

$$\Sigma = \mathrm{diag}\ \{ \underline{q}_i^T b \} \tag{41a}$$

and let

$$\pi = \left[\ \frac{\prod_{j=1}^{n} (\lambda_1 - \mu_j)}{\prod_{j \neq 1} (\lambda_1 - \lambda_j)} \quad \cdots \quad \frac{\prod_{j=1}^{n} (\lambda_n - \mu_j)}{\prod_{j \neq n} (\lambda_n - \lambda_j)} \ \right] \tag{41b}$$

Then (40) can be rewritten as

$$k = \pi\, \Sigma^{-1} Q . \tag{42}$$

<u>Example 2</u>: The system of Example 1 has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -3$ and the balanced right and left eigenvectors

$$\underbrace{\begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix}}_{\lambda_1 I - A} \underbrace{6^{-1/2} \begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\underline{p}_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \underbrace{6^{-1/2} [3 \quad 1]}_{\underline{q}_1^T} \underbrace{\begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix}}_{\lambda_1 I - A} = [0 \quad 0]$$

$$\underbrace{\begin{bmatrix} -3 & -1 \\ -9 & -3 \end{bmatrix}}_{\lambda_2 I - A} \underbrace{6^{-1/2} \begin{bmatrix} -1 \\ 3 \end{bmatrix}}_{\underline{p}_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \underbrace{6^{-1/2} [-3 \quad 1]}_{\underline{q}_2^T} \underbrace{\begin{bmatrix} -3 & -1 \\ -9 & -3 \end{bmatrix}}_{\lambda_2 I - A} = [0 \quad 0]$$

Then $\Sigma = -6^{-1/2} I_2$, and if the desired closed loop eigenvalues are $\mu_1 = \mu_2 = -$

we have

$$\pi = \left[\frac{(3+1)^2}{6} \quad \frac{(-3+1)^2}{-6} \right] = \left[\frac{8}{3} \ -\frac{2}{3} \right]$$

so that

$$k = \left[-\frac{8}{3} \quad \frac{2}{3} \right] \begin{bmatrix} 3 & 1 \\ -3 & 1 \end{bmatrix} = [-10 \ -2] \ ,$$

as expected. ■

Multiple-input case: As for the single-input case, our proof of the pole assignment theorem for multiple-input systems relies on the controller canonical form.

Controller form: Consider a reachable pair (A, B) where the input matrix

$$B = [b_1 \ \dots \ b_m]$$

has size $n \times m$. Without loss of generality, we assume the columns b_i $1 \leq i \leq m$ of B are linearly independent. Otherwise, the number of effective inputs to the system would be less than m. Specifically, if $\text{rank}(B) = \bar{m} < m$, we can write

$$B = \bar{B} \, G \tag{43}$$

where $\bar{B}$ and G have sizes $n \times \bar{m}$ and $\bar{m} \times m$, respectively. Then, if we set $\underline{\bar{u}}(t) = G \underline{u}(t)$, the dynamical system (1a) can be rewritten as

$$\underline{\dot{x}}(t) = A \underline{x}(t) + \bar{B} \underline{\bar{u}}(t) \tag{44}$$

which has only $\bar{m}$ inputs. In other words, a simple change of input coordinates can be used to reduce the number of system inputs.

Consider the $n \times nm$ reachability matrix

$$R = [B \ AB \ \dots \ A^{n-1}B] .$$

Because the pair (A, B) is reachable, R admits n independent columns. There are many ways of generating n such columns, but we shall focus here on a specific method. We scan the columns of R <u>from left to right</u>, so that we consider successively the columns $b_1, b_2 \dots b_m$ $Ab_1, Ab_2 \dots Ab_m, A^{n-1}b_1, A^{n-1}b_2 \dots A^{n-1}b_m$. For each such column $\underline{r}$, we determine whether it is linearly independent of the columns located to its left. If yes, $\underline{r}$ is included in the basis of $\mathcal{R} = \mathbb{R}^n$ that we generate; otherwise it is discarded. An interesting observation in this context is that if a certain column, say $\underline{r}_k$, of R can be expressed in terms of the columns located to its left as

$$\underline{r}_k = \sum_{i<k} u_{ki} \underline{r}_i \qquad (45a)$$

then

$$A\underline{r}_k = \sum_{i<k} u_{ki} A\underline{r}_i \qquad (45b)$$

where the vectors $A\underline{r}_i$, $i<k$ are columns of R located to the left of $A\underline{r}_k$. In other words, $A\underline{r}_k, A^2\underline{r}_k \dots$ will also depend on previously

scanned columns of R. By using the above selection procedure, we generate a $n \times n$ matrix

$$\Gamma = [\Gamma_1 \ \Gamma_2 \ \dots \ \Gamma_m] \tag{46a}$$

$$\Gamma_i = [b_i \ Ab_i \ \dots \ A^{k_i - 1} b_i] \tag{46b}$$

whose columns form a basis of $\mathbb{R}^n$. Γ is constituted of chains $\{b_i \ Ab_i \dots A^{k_i-1} b_i\}$ obtained by successively applying A to each column b_i $1 \leq i \leq m$ of B. The lengths $k_1, k_2 \dots k_m$ of these chains are called the <u>controllability indices</u> of the pair (A, B). For convenience, we assume in the following that the inputs of the system are relabele in such a way that $k_1 \geq k_2 \dots \geq k_m$.

For $1 \leq i \leq m$, let $\underline{f}_i^T$ be the $(k_1 + k_2 + \dots + k_i)$-th row of the inverse of Γ, so that

$$f_i^T \ \Gamma = [0 \ \dots \ 0 \ \underset{\substack{\uparrow \\ (k_1 + k_2 + \dots + k_i)\text{-th position}}}{1} \ 0 \ \dots \ 0] \tag{47}$$

Then, consider the matrix

$$S = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_m \end{bmatrix} \qquad S_i = \begin{bmatrix} \underline{f}_i^T A^{k_i - 1} \\ \vdots \\ \underline{f}_i^T A \\ \underline{f}_i^T \end{bmatrix} \tag{48}$$

which is the generalization to multiple input systems of the matrix S considered in (17). As in the scalar case, S has the following property.

<u>Lemma 2</u> : S is invertible.

<u>Proof</u> : Let

$$\sum_{i=1}^{m} \sum_{j=1}^{k_i} u_{ij} \, \underline{f}_i^T A^j = \underline{0} \tag{49}$$

be a zero linear combination of the rows of S. Multiplying on the right by B and taking into account the property (47) of the row vectors $\underline{f}_i^T$, we find

$$[u_{1k_1} \; u_{2k_2} \cdots u_{mk_m}] \; V = \underline{0} \tag{50a}$$

where

$$V = \begin{bmatrix} \underline{f}_1^T A^{k_1-1} B \\ \underline{f}_2^T A^{k_2-1} B \\ \\ \underline{f}_m^T A^{k_m-1} B \end{bmatrix} = \begin{bmatrix} 1 & x & \cdots & x \\ & \ddots & \ddots & \vdots \\ & & \ddots & x \\ 0 & & & 1 \end{bmatrix} \tag{50b}$$

is upper triangular with ones along its diagonal and is thus invertible. This implies $u_{1k_1} = u_{2k_2} = \cdots = u_{mk_m} = 0$. Next, multiplying by AB, we find

$$[u_{1k_1-1} \; u_{2k_2-1} \cdots u_{mk_m-1}] \; V = \underline{0} \;, \tag{51}$$

so that by induction, all the coefficients u_{ij} of the linear combination (49) are zero. Consequently S is invertible, since its rows admit no nontrivial linear dependency relation. ∎

We are now in position to construct the controller canonical form of (A, B). Let J_i and β_i be respectively $k_i \times k_i$ and $k_i \times 1$ matrices of the form

$$J_i = \begin{bmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} \qquad \beta_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{52}$$

and let $\bar{A}$ and $\bar{B}$ be the $n \times n$ and $n \times m$ matrices given by

$$\bar{A} = \text{diag}\{J_i, \ 1 \leq i \leq m\} \tag{53a}$$

$$\bar{B} = \text{diag}\{\beta_i, \ 1 \leq i \leq m\}. \tag{53b}$$

Then, we have the following result.

<u>Theorem</u> (controller canonical form). If (A, B) is reachable, by applying the similarity transformation $\underline{x} = T \underline{x}_c$ with $T = S^{-1}$, where S is given by (48), we obtain the controller form pair

$$A_c = T^{-1} A T = \bar{A} + \bar{B} L T \tag{54a}$$

$$B_c = T^{-1} B = \bar{B} V \tag{54b}$$

where V is given by (50b), and L is the $m \times n$ matrix

$$L = \begin{bmatrix} \underline{f}_1^T A^{k_1} \\ \underline{f}_2^T A^{k_2} \\ \vdots \\ \underline{f}_m^T A^{k_m} \end{bmatrix} . \tag{55}$$

<u>Proof</u>: Taking into account the structure (48) of S, we have

$$SA = \begin{bmatrix} S_1 A \\ S_2 A \\ \vdots \\ S_m A \end{bmatrix} \qquad SB = \begin{bmatrix} S_1 B \\ S_2 B \\ \vdots \\ S_m B \end{bmatrix} \tag{56a}$$

with

$$S_i A = \begin{bmatrix} \underline{f}_i^T A^{k_i} \\ \vdots \\ \underline{f}_i^T A^2 \\ \underline{f}_i^T A \end{bmatrix} = J_i \begin{bmatrix} \underline{f}_i^T A^{k_i - 1} \\ \vdots \\ \underline{f}_i^T A \\ \underline{f}_i^T \end{bmatrix} + \beta_i \underline{f}_i^T A^{k_i} \tag{56b}$$

$$S_i B = \begin{bmatrix} \underline{f}_i^T A^{k_i - 1} B \\ \vdots \\ \underline{f}_i^T A B \\ \underline{f}_i^T B \end{bmatrix} = \beta_i \underline{f}_i^T A^{k_i - 1} B \tag{56c}$$

so that

$$SA = \bar{A} S + B L \quad , \qquad S B = \bar{B} V \; , \tag{57}$$

which proves (54a)-(54b). ∎

A consequence of expressions (54a)-(54b) is that A_c and B_c have the structure

$$A_c = \left[\begin{array}{c|c|c|c}
\begin{matrix} \times & \cdots & \times \\ 1 & 0 \cdots & 0 \\ & \ddots & \vdots \\ 0 & 1 & 0 \end{matrix} & \begin{matrix} \times & \cdots & \times \\ & & \\ & 0 & \\ & & \end{matrix} & \cdots & \begin{matrix} \times & \cdots & \times \\ & & \\ & 0 & \\ & & \end{matrix} \\ \hline
\begin{matrix} \times & \cdots & \times \\ & & \\ & 0 & \\ & & \end{matrix} & \begin{matrix} \times & \cdots & \times \\ 1 & 0 & 0 \\ & \ddots & \\ 0 & 1 & 0 \end{matrix} & \cdots & \begin{matrix} \times & \cdots & \times \\ & & \\ & 0 & \\ & & \end{matrix} \\ \hline
& & \ddots & \\ \hline
\begin{matrix} \times & \cdots & \times \\ & & \\ & 0 & \\ & & \end{matrix} & \begin{matrix} \times & \cdots & \times \\ & & \\ & 0 & \\ & & \end{matrix} & \cdots & \begin{matrix} \times & \cdots & \times \\ 1 & 0 \cdots & 0 \\ & \ddots & \vdots \\ 0 & 1 & 0 \end{matrix}
\end{array}\right]
\begin{array}{l} \updownarrow k_1 \\ \\ \updownarrow k_2 \\ \\ \\ \updownarrow k_m \end{array}$$

$$B_c = \left[\begin{array}{c}
\begin{matrix} 1 & \times & \cdots & \times \\ & & 0 & \end{matrix} \\ \hline
\begin{matrix} 0 & 1 & \times & \cdots & \times \\ & & 0 & & \end{matrix} \\ \hline
\vdots \\ \hline
\begin{matrix} 0 & \cdots & 0 & 1 \\ & & 0 & \end{matrix}
\end{array}\right]
\begin{array}{l} \updownarrow k_1 \\ \updownarrow k_2 \\ \\ \updownarrow k_m \end{array}$$

where the x s denote entries which may be nonzero. Clearly, A_c and B_c have a structure similar to the scalar controller form, except that we have now one controller block of size $k_i \times k_i$ with $1 \leq i \leq m$ for each system input. The first rows of each block of A_c and B_c correspond to the rows of LT and V. Note that the upper triangular structure of V, and thus of the nonzero rows of B_c, is a consequence of the ordering $k_1 \geqslant k_2 \dots \geqslant k_m$ of the controllability indices of B. The reverse ordering $k_1 \leqslant k_2 \dots \leqslant k_m$ would lead to a lower triangular structure.

<u>Example</u> 3 : Consider the pair

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -21 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} .$$

Its reachability matrix

$$R = \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 0 & 0 & 2 & 2 & 17 \\ 0 & 0 & 0 & 1 & 1 & 8 & 3 & 30 \\ 0 & 1 & 0 & 5 & 0 & 4 & -21 & -148 \end{array}\right]$$

has full rank, and the first 4 columns of R form a basis of $\mathbb{R}^4$, so that

$k_1 = k_2 = 2$ and

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

The 2nd and 4th row of Γ^{-1} are given by

$$\underline{f}_1^T = [0 \; 1 \; 0 \; 0] \qquad \underline{f}_2^T = [0 \; 0 \; 1 \; 0]$$

and

$$S = \begin{bmatrix} \underline{f}_1^T A \\ \underline{f}_1^T \\ \underline{f}_2^T A \\ \underline{f}_2^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The similarity transformation relating (A, B) to its controller form (A_c, B_c) is therefore given by

$$T = S^{-1} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{bmatrix},$$

and by observing that

$$V = \begin{bmatrix} \underline{f}_1^T A B \\ \underline{f}_2^T A B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} \underline{f}_1^T A^2 \\ \underline{f}_2^T A^2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 7 & 2 \\ 1 & 3 & -10 & 8 \end{bmatrix}$$

$$LT = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 1 & -5 & 8 & -36 \end{bmatrix}$$

we find

$$A_c = \left[\begin{array}{cc|cc} 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ \hline 1 & -5 & 8 & -36 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad B_c = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \end{array}\right] .$$

Pole assignment procedure: Once the system has been brought to the controller canonical form

$$\dot{\underline{x}}_c = A_c \, \underline{x}_c + B_c \, \underline{u} , \qquad (58)$$

it becomes easy to move the system poles.

Step 1: To simplify our discussion, we apply the transformation $\underline{u} = V^{-1} \bar{\underline{u}}$ to the input vector.

Step 2: Then, when we apply the feedback law

$$\bar{u} = \bar{v} - (LT + \bar{K}) \, \underline{x}_c \qquad (59)$$

to the system (58), the closed-loop dynamics take the form

$$\dot{\underline{x}}_c = (\bar{A} - \bar{B} \, \bar{K}) \, \underline{x}_c + \bar{B} \, \bar{v} \qquad (60)$$

where the rows of $\bar{K}$ can be used to assign arbitrarily the first row of each block of $\bar{A} - \bar{B}\,\bar{K}$. In particular, if we select

$$\bar{K} = \left[\begin{array}{ccc|ccc|c|ccc} \alpha_1 & \cdots & \alpha_{k_1} & \alpha_{k_1+1} & \cdots & \alpha_{k_2} & & \alpha_{\sum_{i=1}^{m-1} k_i+1} & & \alpha_n \\ 0 & \cdots\, 0 & -1 & 0 & \cdots & 0 & \cdots & & & \\ & & & 0 & \cdots\, 0 & -1 & & & 0 & \\ & & & & & & & & & \end{array}\right] \tag{61}$$
$$\underbrace{\qquad\qquad}_{k_1}\ \underbrace{\qquad\qquad}_{k_2}\ \cdots\ \underbrace{\qquad\qquad}_{k_m}$$

the closed-loop matrix

$$\bar{A} - \bar{B}\bar{K} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & \cdots & -\alpha_n \\ 1 & 0 & \cdots & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ 0 & & \ddots & \ddots & \vdots \\ & & & 1 & 0 \end{bmatrix} \tag{62}$$

is in companion form, so that

$$\det(sI - \bar{A} + \bar{B}\bar{K}) = \alpha(s) = s^n + \alpha_1 s^{n-1} \cdots + \alpha_n , \tag{63}$$

where $\alpha(s)$ is arbitrary. In terms of the original system (1a), the desired feedback law takes therefore the form

$$\underline{u} = \underline{v} - \underbrace{V^{-1}(L + \bar{K}\,T^{-1})}_{K}\,\underline{x} \ . \tag{64}$$

Example 4: For the system (A, B) of Example 3, suppose we want to locate all four closed-loop poles at $s = -1$, so that

$$\alpha(s) = (s+1)^4 = s^4 + 4s^3 + 6s^2 + 4s + 1 .$$

Then

$$\bar{K} = \begin{bmatrix} 4 & 6 & 4 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

and the desired closed-loop gain matrix is given by

$$K = L + \bar{K} S = \begin{bmatrix} 4 & 12 & 28 & 6 \\ 1 & 2 & -10 & 8 \end{bmatrix} .$$

<u>Remark</u>: For a reachable system (A, B) with more than one input, the feedback gain matrix K satisfying

$$\det(sI - A + BK) = \alpha(s) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n \qquad (65)$$

is not unique. In fact, there exists an infinite number of such matrices. This is due to the fact that K has nm entries, and thus nm degrees of freedom, whereas (65) imposes only n constraints on K, so that K has $n(m-1)$ remaining degrees of freedom.

To verify that K is not unique, note for the system of Example 3 that if b_1 denotes the first column of B, the pair (A, b_1) is reachable. Consequently, the system poles can be moved arbitrarily by applying state feedback only to the first input, which corresponds to a feedback gain matrix of the form

$$K = \begin{bmatrix} \underline{k}_1^T \\ 0 \end{bmatrix}$$

which is clearly different from the matrix employed in Example 4.