EEC 250 Linear Systems and Signals

Lecture 10

Topics: a) Reachability of CT systems with impulsive and finite energy inputs

b) Symmetric matrices, positive definiteness

c) Physical interpretation of the reachability Gramian.

In last lecture we studied DT reachability. We consider now the reachability of the CT system

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \tag{1}$$

with $\underline{x} \in \mathbb{R}^n$ and $\underline{u} \in \mathbb{R}^m$. The reachability problem can be formulated in the same way as in the DT case: starting from a zero initial state $\underline{x}(0_-) = 0$, a state $\underline{v}$ is reachable in finite time $T < \infty$ if we can find an input $\underline{u}(t)$, $0 \le t \le T$ such that

$$\underline{v} = \underline{x}(T) = \int_{0_-}^{T} e^{A(T-t)} B\,\underline{u}(t)\,dt \tag{2}$$

The space of reachable states is denoted as $\mathcal{R}$, and the system (A, B) is said to be reachable if $\mathcal{R} = \mathbb{R}^n$, i.e. if all vectors of the space can be reached in finite time.

However, in order to study precisely the above problem, we must specify the class of allowable inputs. We consider two types of inputs.

(i) Impulsive inputs: In this case

$$\underline{u}(t) = \underline{u}_0\,\delta(t) + \underline{u}_1\,\dot{\delta}(t) + \dots + \underline{u}_{k-1}\,\delta^{(k-1)}(t)$$

where $\underline{u}_i \in \mathbb{R}^m$ for $0 \leq i \leq k-1$ and $\delta^{(i)}(t)$ denotes the i^{th} derivative of an impulse. It is defined through its action on smooth functions, i.e. functions $f(t)$ that admit an infinite number of derivatives. For an arbitrary such function $f(t)$, we have the sifting property

$$\int_a^b f(t)\,\delta^{(i)}(t)\,dt = \begin{cases} (-1)^i f^{(i)}(0) & \text{for } a < 0 < b \\ 0 & \text{otherwise} \end{cases} \qquad (3)$$

For example

$$\int_a^b f(t)\,\delta(t)\,dt = f(0)$$

which is the usual definition of an impulse, and

$$\int_a^b f(t)\,\dot{\delta}(t)\,dt = -\dot{f}(0)$$

which is the definition of a doublet $\dot{\delta}(t)$, i.e. the derivative of an impulse.

(ii) Finite energy inputs: These inputs are such that the energy

$$E = \frac{1}{2}\int_0^T \|\underline{u}(t)\|^2\,dt < \infty$$

Here $\|\underline{u}\|$ denotes the length of the vector $\underline{u}$, i.e. if $\underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_m \end{bmatrix}$

$$\|\underline{u}\|^2 = \underline{u}^T \underline{u} = \sum_{i=1}^{m} u_i^2 .$$

Fortunately, as we shall see below, all the states that can be reached with impulsive inputs can also be reached in finite time with finite energy inputs, so that the reachable space $\mathcal{R}$ is the same for both types of inputs. The only difference is that with impulsive inputs, all the reachable states can be reached instantaneously.

<u>Reachability with impulsive inputs</u>: If we apply the input

$$\underline{u}(t) = \underline{u}_0\,\delta(t) + \underline{u}_1\,\dot{\delta}(t) + \dots + \underline{u}_{k-1}\,\delta^{(k-1)}(t) \qquad (4)$$

to the system (1) with $\underline{x}(0_-) = 0$, the state at time $t = 0_+$ (immediately after $t = 0$) is given by

$$\underline{x}(0_+) = \int_{0_-}^{0_+} e^{-At} B\,\underline{u}(t)\,dt .$$

From the sifting property (3) of $\delta^{(i)}(t)$ we have

$$\int_{0_-}^{0_+} e^{-At} B\,\underline{u}_i\,\delta^{(i)}(t) = (-1)^i \left[\frac{d^i}{dt^i} e^{-At}\right]_{t=0} B\,\underline{u}_i ,$$

where

$$\frac{d^i}{dt^i} e^{-At} = (-1)^i A^i e^{-At} ,$$

so that

$$\int_{0_-}^{0_+} e^{-At} B \underline{u}_i \, \delta^{(i)}(t)\, dt = A^i B \underline{u}_i \, .$$

This implies

$$\underline{x}(0_+) = \underline{v} = [\, B \;\; AB \;\; \ldots \;\; A^{k-1}B \,] \begin{bmatrix} \underline{u}_0 \\ \underline{u}_1 \\ \vdots \\ \underline{u}_{k-1} \end{bmatrix} \qquad (5)$$

where we recognize the reachability matrix $R_k = [B \; AB \ldots A^{k-1}B]$, so that the space $\mathcal{R}_k$ of states that can be reached with inputs containing impulses of order less or equal to $k-1$ is the column space of R_k. It is thus identical to the set of states that can be reached in k steps for a discrete-time system with the same (A, B) pair.

Using now our DT results, we see that: (i) $\mathcal{R}_k \subseteq \mathcal{R}_{k+1}$, (ii) if $\mathcal{R}_k = \mathcal{R}_{k+1}$ then $\mathcal{R}_\ell = \mathcal{R}_k$ for all $\ell \geqslant k$, and (iii) $\mathcal{R}_n = \mathcal{R}_{n+1}$. Thus we can conclude that the space $\mathcal{R}$ of states that can be reached with impulsive inputs can be reached with inputs containing only impulses of order up to $n-1$, i.e.

$$\underline{u}(t) = \underline{u}_0 \, \delta(t) + \underline{u}_1 \, \dot{\delta}(t) + \ldots + \underline{u}_{n-1} \, \delta^{(n-1)}(t) \, .$$

In other words it is not necessary to employ impulses of order higher than $n-1$, since the use of such impulses does not change the reachable space $\mathcal{R}$, which is spanned by the columns of the reachability matrix $R = [B \; AB \ldots A^{n-1}B]$.

Then we say that a system (A, B) is reachable with impulsive inputs, or impulsively reachable for short, if $\mathcal{R} = \mathbb{R}^n$. All the DT reachability tests are clearly applicable to impulsive reachability, so that (A, B) is impulsively reachable if and only if

$$\text{rank}\,[B \; AB \ldots A^{n-1}B] = n, \tag{6}$$

or equivalently

$$\text{rank}\,[sI - A \,\vdots\, B] = n \tag{7}$$

for all complex values of s.

<u>Reachability with finite energy inputs</u>: Consider now a finite interval $[0, T]$ and an arbitrary vector $\underline{v} \in \mathbb{R}^n$. The state $\underline{v}$ is reachable with a finite energy input $\underline{u}(t)$, $0 \leq t \leq T$ if $\underline{u}(t)$ satisfies

$$\underline{v} = \int_0^T e^{A(T-t)} B \underline{u}(t)\, dt \tag{8}$$

and

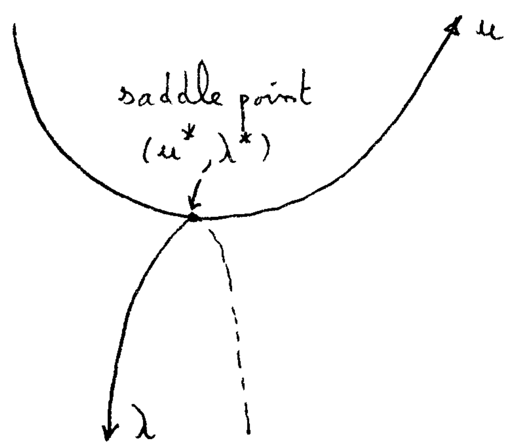
$$J(\underline{u}) = \frac{1}{2} \int_0^T \|\underline{u}(t)\|^2\, dt < \infty .$$

To determine whether $\underline{v}$ is reachable with finite energy inputs, all we need to do is minimize the energy J under the constraint (8). If this minimization problem has no solution, it means that $\underline{v}$ is not reachable over $[0,T]$ with finite energy inputs.

To minimize $J(u)$ under the constraint (8), we use the method of Lagrange multipliers, i.e. we introduce the Hamiltonian

$$H(\underline{u},\underline{\lambda}) = \frac{1}{2}\int_0^T \underline{u}^T(t)\,\underline{u}(t)\,dt + \underline{\lambda}^T\left(\underline{v} - \int_0^T e^{A(T-t)} B\,\underline{u}(t)\,dt\right). \qquad (9)$$

where $\underline{\lambda}$ is the multiplier vector. To interpret (9), we note that provided that the constraint (8) is enforced, which is accomplished by setting $\frac{\partial H}{\partial \underline{\lambda}} = 0$, then $H(\underline{u},\underline{\lambda})$ is identical to $J(\underline{u})$, so that it is equivalent to minimize $H(\underline{u},\underline{\lambda})$ and $J(\underline{u})$ over $\underline{u}$. This observation implies that finding a minimum for $J(\underline{u})$ under the constraint (8) is equivalent to finding a saddle-point for $H(\underline{u},\underline{\lambda})$, i.e. a minimum over $\underline{u}$ and a maximum over $\underline{\lambda}$, as shown below.



Next, we introduce

$$\underline{u}^*(t) = B^T e^{A^T(T-t)} \underline{\lambda} \tag{10}$$

so that $H(\underline{u}, \underline{\lambda})$ can be expressed as

$$H(\underline{u}, \underline{\lambda}) = \frac{1}{2} \int_0^T (\underline{u}(t) - \underline{u}^*(t))(\underline{u}(t) - \underline{u}^*(t))\, dt + \underline{\lambda}^T (\underline{v} - W(T)\underline{\lambda}) \tag{11}$$

where

$$W(T) = \int_0^T e^{A(T-t)} B B^T e^{A^T(T-t)} dt$$
$$= \int_0^T e^{As} B B^T e^{A^T s} ds$$

is the <u>reachability Gramian</u> of the system (A, B). From (11) we see that $H(\underline{u}, \underline{\lambda})$ is minimized over $\underline{u}$ and maximized over $\underline{\lambda}$ by selecting

$$\underline{u}(t) = \underline{u}^*(t) \tag{12a}$$

$$\underline{v} = W(T)\underline{\lambda} \,. \tag{12b}$$

The equation (12b) admits a solution for the multiplier vector $\underline{\lambda}$ only if $\underline{v}$ belongs to the column space of $W(T)$. Thus $\underline{v}$ is reachable over $[0,T]$ with finite energy inputs if and only if $\underline{v} \in R(W(T))$. The system (A, B) is reachable over $[0,T]$ with finite energy inputs if and only if $W(T)$ is invertible, in which case $\underline{\lambda} = W^{-1}(T)\underline{v}$ and

$$\underline{u}^*(t) = B^T e^{A^T(T-t)} W^{-1}(T)\, \underline{v} \,. \tag{13}$$

The energy of the input $\underline{u}^*(t)$ is then given by

$$J(\underline{u}^*) = \frac{1}{2}\int_0^T (\underline{u}^*(t))^T \underline{u}^*(t)\, dt = \frac{1}{2}\, \underline{v}^T W^{-1}(T)\, \underline{v} \quad , \tag{14}$$

which will be used below to obtain a physical interpretation of the reachability Gramian and to develop a quantitative measure of the degree of reachability of a linear system.

To prove that the reachable states are the same for impulsive inputs and finite energy inputs over an arbitrary finite interval $[0, T]$, we establish the following result.

<u>Theorem 1</u>: The column spaces of $R = [B\ AB \ldots A^{n-1}B]$ and $W(T)$ are identical.

<u>Proof</u>: Instead of proving directly that the column spaces of R and $W(T)$ are identical, we show that their left null spaces $N(R^T)$ and $N(W^T(T))$ are the same. Then we use the fact that the column space $R(M)$ of a matrix M is the orthogonal complement of its left null space, i.e. $R(M) = N(M^T)^\perp$, so that if two matrices have the same left null space, they have the same column space.

Assume that $\underline{q}^T \neq 0$ is a vector of the left null space of $W(T)$, i.e.

$\underline{q}^T W(T) = 0$. Then we have

$$0 = \underline{q}^T W(T) \underline{q} = \int_0^T \underline{q}^T e^{At} B B^T e^{A^T t} \underline{q} \, dt$$

$$= \int_0^T \| \underline{z}(t) \|^2 \, dt \tag{14}$$

where $\underline{z}^T(t) = \underline{q}^T e^{At} B$. This implies $\underline{z}(t) \equiv 0$ for $0 \leq t \leq T$, so that all the derivatives of $\underline{z}(t)$ are identically zero, i.e.

$$\frac{d^k}{dt^k} \underline{z}^T(t) = q^T A^k e^{At} B = 0 \quad , \quad 0 \leq t \leq T .$$

Evaluating $\underline{z}(t)$ and its derivatives at $t = 0$ yields $\underline{q}^T A^k B = 0$ for all k, so that

$$\underline{q}^T [B \ AB \ \dots \ A^{n-1} B] = 0 \quad , \tag{15}$$

i.e. $\underline{q}^T$ belongs to the left null space of R.

Conversely, let $\underline{q}^T \neq 0$ be a vector of the left null space of R, so that (15) is satisfied. Using the Cayley-Hamilton identity $A^n = -a_1 A^{n-1} \dots - a_n I$ inside the power series expansion

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \quad ,$$

we can deduce that e^{At} can be expressed as

$$e^{At} = \alpha_1(t) A^{n-1} \dots + \alpha_{n-1}(t) A + \alpha_n(t) I \quad , \tag{16}$$

for some functions $\alpha_i(t)$ $1 \leq i \leq n-1$. This implies

$$\underline{q}^T e^{At} B = \sum_{i=1}^{n} \alpha_i(t)\, \underline{q}^T A^{n-i} B = 0$$

so that

$$\underline{q}^T W(T) = \int_0^T \underline{q}^T e^{At} B B^T e^{A^T t}\, dt = \underline{0} \,, \qquad (17)$$

i.e. $\underline{q}^T$ belongs to the left null space of $W(T)$. This shows that the left null spaces, and therefore the column spaces, of R and $W(T)$ are identical. ∎

As mentioned earlier, Theorem 1 implies that the reachable space $\mathcal{R}$ is the same for impulsive inputs, and for finite energy inputs defined over a finite interval $[0, T]$. Another consequence of Theorem 1 is as follows.

<u>Theorem 2</u>: A system (A, B) is reachable if and only if the reachability Gramian $W(T)$ is invertible for some $T < \infty$.

<u>Symmetric matrices</u>: Before proceeding further, we need to review several useful properties of symmetric matrices. An $n \times n$ matrix M is symmetric if its entries m_{ij} are such that $m_{ij} = m_{ji}$. Such a matrix has the following properties.

1) The eigenvalues and eigenvectors of M are all <u>real</u>. To see this, let $\underline{p} \neq 0$

be an eigenvector of M corresponding to eigenvalue λ, i.e. $M\underline{p} = \lambda \underline{p}$. Conjugating and transposing this identity yields $\underline{p}^H M = \lambda^* \underline{p}^H$ where $\underline{p}^H = (\underline{p}^*)^T$. This implies

$$\underline{p}^H M \underline{p} = \lambda \underline{p}^H \underline{p} = \lambda^* \underline{p}^H \underline{p}$$

so that $\lambda = \lambda^*$, i.e. λ is real. Consequently the eigenvector $\underline{p}$ is also real.

2) The eigenvectors of M corresponding to distinct eigenvalues are <u>orthogonal</u>. Let $\underline{p}_i$ and $\underline{p}_j$ be two eigenvectors of M corresponding to eigenvalue λ_i and λ_j with $\lambda_i \neq \lambda_j$. From $M\underline{p}_i = \lambda_i \underline{p}_i$ and $M\underline{p}_j = \lambda_j \underline{p}_j$ we deduce

$$\underline{p}_j^T M \underline{p}_i = \lambda_i \underline{p}_j^T \underline{p}_i = \lambda_j \underline{p}_j^T \underline{p}_i$$

so that

$$(\lambda_i - \lambda_j)\, \underline{p}_j^T \underline{p}_i = 0$$

Since $\lambda_i \neq \lambda_j$, this implies $\underline{p}_j^T \underline{p}_i = 0$.

3) M can be <u>diagonalized</u>, i.e. it does not have Jordan blocks of size greater or equal to two. This requires showing that M does not admit any generalized eigenvector. Assume that $\underline{g} \neq \underline{0}$ is a generalized eigenvector such that $(M - \lambda I)^{k-1} \underline{g} \neq \underline{0}$ and $(M - \lambda I)^k \underline{g} = \underline{0}$ for some $k \geq 2$. Then

$$0 = \underline{g}^T (M - \lambda I)^{k-2} (M - \lambda I)^k \underline{g} = \| (M - \lambda I)^{k-1} \underline{g} \|^2 ,$$

so that $(M - \lambda I)^{k-1} \underline{g} = \underline{0}$, which is a contradiction. Thus M has no generalized eigenvector. It has only ordinary eigenvectors, and can therefore be diagonalized.

Thus M admits n independent eigenvectors $\underline{p}_i$. Without loss of generality, the eigenvectors corresponding to identical eigenvalues can be orthogonalized, and all eigenvectors can be selected with unit length, i.e. $\|\underline{p}_i\|^2 = 1$ for all i. This implies that the matrix $P = [\underline{p}_1 \cdots \underline{p}_n]$ is orthonormal, i.e. $P^T P = I$. Then

$$MP = P\Lambda \quad \text{with} \quad \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \lambda_i & \\ 0 & & & \ddots \\ & & & & \lambda_n \end{bmatrix}$$

so that

$$M = P \Lambda P^T, \tag{18}$$

i.e. the transformation P that brings M to diagonal form is orthonormal.

To a symmetric matrix M, we can associate a quadratic form $\underline{x}^T M \underline{x}$. An important property of this quadratic form is as follows.

<u>Rayleigh's lemma</u>: Let M be a symmetric matrix whose smallest and largest eigenvalues are denoted respectively by λ_{min} and λ_{max}. Then, for $\underline{x} \neq \underline{0}$ we have

$$\lambda_{min} \leq \frac{\underline{x}^T M \underline{x}}{\underline{x}^T \underline{x}} \leq \lambda_{max} \quad . \tag{19}$$

In this inequality, the minimum and maximum are reached for $\underline{x} = \underline{p}_{min}$ and $\underline{x} = \underline{p}_{max}$, respectively, where $\underline{p}_{min}$ and $\underline{p}_{max}$ are the eigenvectors corresponding to λ_{min} and λ_{max}.

<u>Proof</u>: We have

$$\underline{x}^T M \underline{x} = \underline{x}^T P \Lambda P^T \underline{x} = \underline{z}^T \Lambda \underline{z} = \sum_{i=1}^{n} \lambda_i z_i^2 \tag{20}$$

where

$$P^T \underline{x} = \underline{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} .$$

But

$$\lambda_{min} \sum_{i=1}^{n} z_i^2 \leq \sum_{i=1}^{n} \lambda_i z_i^2 \leq \lambda_{max} \sum_{i=1}^{n} z_i^2 \tag{21a}$$

with

$$\sum_{i=1}^{n} z_i^2 = \underline{z}^T \underline{z} = \underline{x}^T P P^T \underline{x} = \underline{x}^T \underline{x} \tag{21b}$$

where we have used the fact that P is orthonormal. Combining (20) and (21a)-(21b) yields (19).

For the case when $\underline{x} = \underline{p}_{min}$ we have $M \underline{p}_{min} = \lambda_{min} \underline{p}_{min}$, so that $\underline{p}_{min}^T M \underline{p}_{min} = \lambda_{min} \underline{p}_{min}^T \underline{p}_{min}$. Similarly for $\underline{x} = \underline{p}_{max}$ we have $M \underline{p}_{max} = \lambda_{max} \underline{p}_{max}$ so that $\underline{p}_{max}^T M \underline{p}_{max} = \lambda_{max} \underline{p}_{max}^T \underline{p}_{max}$. ∎

Among symmetric matrices, an important class is as follows.

Definition: A symmetric matrix M is nonnegative definite if $\underline{x}^T M \underline{x} \geq 0$ for all $\underline{x}$. It is positive definite if $\underline{x}^T M \underline{x} > 0$ for all $\underline{x} \neq \underline{0}$.

The nonnegative or positive definiteness of a symmetric matrix M is usually denoted by $M \geq 0$, or $M > 0$, respectively. From Rayleigh's lemma, we obtain the following characterization of nonnegative and positive definite matrices.

Theorem 3: A symmetric matrix M is nonnegative definite if and only if all its eigenvalues are nonnegative, i.e. $\lambda_i \geq 0$ for all i. It is positive definite if and only if all its eigenvalues are positive, i.e. $\lambda_i > 0$ for all i

Another useful observation is that if M is positive definite, M^{-1} is also positive definite. If $M = P \Lambda P^T$ is the eigenvalue/eigenvector decomposition of M, $M^{-1} = P \Lambda^{-1} P^T$ is the eigenvalue/eigenvector decomposition of M^{-1}. This implies that if λ_{min} and λ_{max} are the minimum and maximum eigenvalues of M, and $\underline{p}_{min}$ and $\underline{p}_{max}$ are the corresponding eigenvectors, then

$$\frac{1}{\lambda_{max}} \leqslant \frac{\underline{x}^T M^{-1} \underline{x}}{\underline{x}^T \underline{x}} \leqslant \frac{1}{\lambda_{min}} \qquad (22)$$

where the minimum and maximum are reached for $\underline{x} = \underline{p}_{max}$ and $\underline{x} = \underline{p}_{min}$ respectively.

<u>Physical interpretation of the reachability Gramian</u>: The reachability Gramian

$$W(T) = \int_0^T e^{At} B B^T e^{A^T t} \, dt$$

is obviously symmetric. In addition

$$\underline{x}^T W(T) \underline{x} = \int_0^T \underline{z}^T(t) \, \underline{z}(t) \, dt = \int_0^T \|\underline{z}(t)\|^2 dt \geqslant 0 \qquad (23)$$

with $\underline{z}^T(t) = \underline{x}^T e^{At} B$, so that $W(T)$ is nonnegative definite, i.e. all its eigenvalues are nonnegative. From Theorem 2 we can conclude that (A, B) is reachable if and only if $W(T)$ is positive definite. This is due to the fact that a nonnegative definite matrix M is invertible if and only if it is positive definite (if it has zero eigenvalues, it is not invertible).

Assuming now that (A, B) is reachable, i.e. that $W(T)$ is positive definite, the minimum energy required to reach an arbitrary state $\underline{v}$ is given by

$$J = \frac{1}{2} \underline{v}^T W^{-1}(T) \underline{v} . \qquad (24)$$

We can now ask the question: among all state vectors $\underline{v}$ of unit length i.e. such that $\underline{v}^T\underline{v} = 1$, which one is the most difficult to reach, or equivalently which one requires the most energy to reach it? The motivation for normalizing the length of the vector $\underline{v}$ is that as $\underline{v}$ is scaled by a factor a, yielding $\underline{v}' = a\underline{v}$, the energy necessary to reach $\underline{v}'$ is $J' = a^2 J$. Since the energy required to reach $\underline{v}'$ is scaled by a factor a^2, we could make a vector $\underline{v}$ more difficult to reach by just scaling its length. Thus, by requiring $\underline{v}$ to have unit length, we are really examining which directions of the space $\mathbb{R}^n$ are the most difficult to reach.

The answer to the above question is provided by Rayleigh's lemma. Let λ_{min} be the smallest eigenvalue of $W(T)$, and $\underline{p}_{min}$ the corresponding eigenvector normalized such that $\underline{p}_{min}^T \underline{p}_{min} = 1$, so that

$$W(T)\,\underline{p}_{min} = \lambda_{min}\,\underline{p}_{min} .$$

Then, the vector that maximizes the energy J is $\underline{v} = \underline{p}_{min}$, in which case

$$J = \frac{1}{2\lambda_{min}} . \qquad (25)$$

The energy can be quite large if the smallest eigenvalue of $W(T)$ is very small. In other words, a measure of the degree of reachability of a system is provided by the smallest eigenvalue λ_{min} of $W(T)$. If it is very small, particularly in comparison to the largest eigenvalue λ_{max}, the system is almost not reachable.

<u>Example</u>: Let $A = \begin{bmatrix} -\frac{1}{2}(1+\epsilon) & 0 \\ 0 & -\frac{1}{2}(1-\epsilon) \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\epsilon \neq 0$, the reachability matrix

$$R = [b \;\; Ab] = \begin{bmatrix} 1 & -\frac{1}{2}(1+\epsilon) \\ 1 & -\frac{1}{2}(1-\epsilon) \end{bmatrix}$$

has rank two, but for $\epsilon = 0$ it has only rank one, so that in this case the system is not reachable. Thus for small ϵ, we would expect that the system is almost not reachable. We have

$$e^{At} b b^T e^{A^T t} = \begin{bmatrix} e^{-(1+\epsilon)t} & e^{-t} \\ e^{-t} & e^{-(1-\epsilon)t} \end{bmatrix}$$

and

$$W(T) = \int_0^T e^{At} b b^T e^{A^T t} dt = \begin{bmatrix} \frac{1}{1+\epsilon}(1 - e^{-(1+\epsilon)T}) & 1 - e^{-T} \\ 1 - e^{-T} & \frac{1}{1-\epsilon}(1 - e^{-(1-\epsilon)T}) \end{bmatrix}$$

Thus $\operatorname{tr} W(T) = \frac{1}{1+\epsilon}(1 - e^{-(1+\epsilon)T}) + \frac{1}{1-\epsilon}(1 - e^{-(1-\epsilon)T})$

$$= \frac{2}{1-\epsilon^2}\left[1 - e^{-T}(\cosh(\epsilon T) + \epsilon \sinh(\epsilon T))\right] \simeq 2(1 - e^{-T}) \qquad (26a)$$

and $$\det W(T) = \frac{1}{1-\epsilon^2}\left(1 - e^{-(1+\epsilon)T}\right)\left(1-e^{-(1-\epsilon)T}\right) - \left(1-e^{-T}\right)^2$$

$$= \frac{4e^{-T}}{1-\epsilon^2}\left[\epsilon^2 \sinh^2\left(\frac{T}{2}\right) - \sinh^2\left(\frac{\epsilon T}{2}\right)\right]$$

$$\simeq 4\epsilon^2 e^{-T} \sinh^2\left(\frac{T}{2}\right) \qquad (26b)$$

where in deriving (26a) and (26b), we have used the approximations $\frac{T}{2} \gg 1$ and $\frac{\epsilon T}{2} \ll 1$, i.e. the interval T is large, yet ϵT is small. Combining (26a) and (26b) we see that

$$\lambda_{max} \simeq 2(1-e^{-T})$$

$$\lambda_{min} = \frac{\det W(T)}{\lambda_{max}} = \epsilon^2 e^{-T/2} \sinh\left(\frac{T}{2}\right) = \frac{\epsilon^2}{2}(1-e^{-T}) .$$

Since $\lambda_{max}/\lambda_{min} = 4/\epsilon^2$, we see that λ_{min} is very small when compared to λ_{max}, so that the system is barely reachable, as expected.

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