# REGULAR AND RECIPROCAL MULTIVARIATE STATIONARY GAUSSIAN PROCESSES OVER Z ARE NECESSARILY MARKOV*

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**Abstract**

Following a conjecture of Carmichael, Massé and Theodorescu [5], this paper shows that regular and reciprocal multivariate stationary Gaussian processes over **Z** are necessarily Markov. This result is proved in two steps. We consider first minimal reciprocal processes, i.e. processes which admit a second-order model driven by a full rank noise process [14], [13]. By solving a generalized eigenvalue problem or an equivalent algebraic Riccati equation, a first-order Markov model is constructed for these processes. Observing that if a process is regular and reciprocal, it admits a minimal and reciprocal subsequence, it is then shown that regular reciprocal processes are Markov.

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## 1 Introduction

In this paper we consider multivariate stationary Gaussian reciprocal processes over $\mathbf{Z}$. Reciprocal processes were introduced in 1932 by Bernstein [3]. A process $x(k)$ with $k \in \mathbf{Z}$ is said to be reciprocal if given an arbitrary interval $I = [K, L]$, the values of $x(\cdot)$ in the interior and exterior of $I$ are conditionally independent given $x(K)$ and $x(L)$. From this definition, it is easy to deduce that if a process is Markov, it is necessarily reciprocal. However, the converse is not true in general. The properties of reciprocal processes were studied in detail by Jamison [9]-[10] who, among other results, proposed a classification for all scalar stationary Gaussian reciprocal processes over a finite interval. This classification was subsequently amended first by Chay [6], and then by Carmichael, Massé and Theodorescu [4]. An interesting feature of the resulting classification is that, with a single exception, all scalar stationary Gaussian processes that are reciprocal but not Markov can only exist over finite intervals. The only exception is the process

$$x(k) = X\cos(\theta k) + Y\sin(\theta k), \tag{1.1}$$

with $k \in \mathbf{Z}$, where $X$ and $Y$ are two independent zero-mean Gaussian random variables with variance $\sigma^2$, and $\pi/\theta$ is irrational. This process is reciprocal and stationary with covariance $R(k) = \sigma^2\cos(\theta k)$. However, it is completely predictable from its infinitely remote past and future, so that we can conclude that scalar stationary Gaussian processes that are both regular (i.e. they do not contain any component predictable from the infinite past and future) and reciprocal over $\mathbf{Z}$ are necessarily Markov.

The exception represented by process (1.1) owes its existence to the discrete nature of the index set $\mathbf{Z}$. In the continous-time case, when $k \in \mathbf{R}$, it is easy to check that the maximum interval over which (1.1) remains reciprocal is $[0, \pi/\theta)$. Thus, *all* scalar stationary Gaussian reciprocal processes over $\mathbf{R}$ are Markov. Observing that among the second-order reciprocal processes studied by Miroshin [18] and Abrahams [1], those existing over the entire real line had also the Markov property, Carmichael, Massé and Theodorescu [5] conjectured that all multivariate stationary Gaussian processes that are reciprocal over $\mathbf{R}$ must be Markov. This conjecture is proved here. However, since we consider the case where the index set is $\mathbf{Z}$, we impose the additional constraint that the processes that we study must be regular, so as to exclude processes such as (1.1).

In order to motivate the above problem, it is worth noting that Markov random fields [15, 21, 23] reduce in one dimension to reciprocal processes, not Markov processes. Since Markov random field models have found many applications in image processing, the study of reciprocal processes is a natural stepping stone for developing better representations and estimation algorithms for these fields. In [12], Krener proposed a description of reciprocal diffusion processes "with full rank noise" in terms of second-order stochastic differential equations. In an attempt to clarify some aspects of this theory, discrete- and continuous-time Gaussian reciprocal processes were examined in [14] and [13], respectively, again under the full-rank noise assumption. In these papers, it was shown that while Gauss-Markov processes admit first-order state-space models driven by white noise, Gaussian reciprocal processes admit self-adjoint second-order models driven by locally correlated noise, where the noise correlation structure is specified by the model dynamics. In this context, it became clear that these second-order models are identical to conditional models of Gauss-Markov random fields introduced earlier by Rozanov [19, 21] and Woods [23].

Higher-order and mixed-order reciprocal Gaussian processes defined over a finite interval were studied by Frezza [7], who showed that they can be characterized in terms of self-adjoint models of higher-order with Dirichlet boundary conditions. Models of this type are employed here to characterize the spectral density of regular reciprocal/Markov processes over $\mathbf{Z}$, where the Dirichlet condition is replaced by an asymptotic boundedness condition for the variance of the process of interest.

This paper is organized as follows. In Section 2, the concepts of regularity and minimality for stationary Gaussian processes are reviewed. The minimality property is equivalent to the full-rank noise assumption of [14, 13], but is applicable to general stationary Gaussian processes, instead of being restricted to reciprocal processes. In Section 3, it is shown that minimal reciprocal processes are necessarily Markov. The procedure that we employ is constructive and relies on a spectral factorization of the second-order model describing the reciprocal process of interest. This spectral factorization is closely connected to the solution of algebraic Riccati equations, which are studied in detail in Section 4. Finally, in Section 5, we prove the main result, namely that stationary Gaussian regular reciprocal processes are necessarily Markov.

## 2 Regularity and Minimality

Consider a stationary zero-mean Gaussian process $x(k) \in \mathbf{R}^n$, $k \in \mathbf{Z}$ with covariance $R(k-l) = E[x(k)x^T(l)]$. Its covariance function can be represented as

$$R(k) = \int_{-\pi}^{\pi} e^{ik\lambda} dF(\lambda) \tag{2.1}$$

where the spectral matrix measure $F(\lambda)$ is such that its increments $F(\lambda_2) - F(\lambda_1)$ with $\lambda_2 \geq \lambda_1$ are Hermitian nonnegative definite. It can be decomposed into absolutely continuous and singular components as

$$F(\lambda) = \int_{-\pi}^{\lambda} S(u)du + F_s(\lambda) \tag{2.2}$$

where $S(\lambda)$ is the spectral density matrix of $x(k)$, and $F_s(\lambda)$ is a measure which is singular with respect to the Lebesgue measure on $[-\pi, \pi]$.

Now, let $K$ be an arbitrary finite set of $\mathbf{Z}$, and let $\mathcal{K}$ be the set of such finite sets. Consider the Hilbert spaces of random variables

$$H \quad = \quad H\{x(l), l \in \mathbf{Z}\} \tag{2.3a}$$

$$\begin{aligned} H_K &= H\{x(l), l \in \mathbf{Z} - K\} && (2.3b) \\ H_\infty &= \bigcap_{\mathcal{K}} H_K. && (2.3c) \end{aligned}$$

If $k \in \mathbf{Z}$ and $p \geq 1$ is a positive integer, let

$$H_{p,k} = H_{[k-p,k+p]} = H\{x(l) \text{ for } |l-k| \geq p\} \quad (2.4a)$$

be the Hilbert space spanned by the random vectors $x(l)$ where $l$ is a least at distance $p$ from point $k$. With $k$ fixed, for an arbitrary set $K$ we can always select $p$ sufficiently large so that $K \subset [k-p, k+p]$ and thus $H_{k,p} \subset H_K$. This implies that $H_\infty$ can also be represented as

$$H_\infty = \bigcap_{p \geq 1} H_{k,p}. \quad (2.4b)$$

Consider now the interpolation problem consisting of estimating $x(k)$ from all the $x(l)$'s such that $l$ is at least at distance $p$ from $k$, i.e. with $|l-k| \geq p$. The interpolation error is given by

$$d(k;p) = x(k) - E[x(k)|H_{k,p}] \quad (2.5)$$

and has for covariance matrix $D(p) = E[d(k;p)d^T(k;p)]$. Clearly $D(p+1) \geq D(p)$, i.e. $D(p)$ is an increasing matrix sequence. Then we can introduce the following concepts.

**Definition 2.1** *A stationary Gaussian process $x(k)$ is regular if $H_\infty = \{0\}$, and singular if $H_\infty = H$. Also, $x(k)$ is minimal of order $p$ if $D(p)$ is positive definite.*

In the following, if $x(k)$ is minimal of order 1, it will be called minimal, for short. Note that this minimality concept bears no relation with the minimality property of linear systems. In [17, 20], the following minimality criterion was presented.

**Theorem 2.1** *A stationary Gaussian process $x(k)$ is minimal if and only if $F(\lambda)$ is absolutely continuous and*

$$\int_{-\pi}^{\pi} S^{-1}(\lambda) d\lambda < \infty. \quad (2.6)$$

A consequence of (2.6) is that a necessary condition for $S(\lambda)$ to be the spectral density matrix of a minimal process is that it must be invertible almost everywhere on $[-\pi, \pi]$.

Although simple tests for the reglarity and minimality of order $p$ of scalar stationary Gaussian processes are given in [19, 20], it does not appear that there exist corresponding tests in the multivariate case. For our purposes, we shall need only the fact that if $x(k)$ is regular, it is necessarily minimal of order $p$ for some $p$. To see this, consider the error $d(k,\infty) = x(k) - E[x(k)|H_\infty]$ associated with the estimation of $x(k)$ given $H_\infty$, and let $D(\infty)$ be its covariance matrix. From the representation (2.4b) of $H_\infty$, we can deduce that

$$\lim_{p \to \infty} D(p) = D(\infty). \quad (2.7)$$

Now if $x(k)$ is regular, $H_\infty = \{0\}$ and $d(k,\infty) = x(k)$, so that $D(\infty) = R(0) > 0$, where we have assumed here that $x(k)$ is a full rank process, i.e. that there does not exist a nontrivial linear dependence relation $\sum_{i=1}^n a_i x_i(k) = 0$ between the entries of $x(k)$. Since the increasing sequence $D(p)$ tends to $R(0) > 0$ as $p \to \infty$, we see that for $p$ sufficiently large, $D(p)$ will be positive definite, so that $x(k)$ is minimal of order $p$ for $p$ sufficiently large.

The following example illustrates the concept of minimality of order $p$.

**Example 2.1:** Consider the stationary Gauss-Markov process

$$x(k+1) = Ax(k) + w(k) \tag{2.8}$$

where $w(k)$ is a zero-mean white Gaussian noise process with intensity matrix $Q$, where the rank of $Q$ is $m < n$ and if $Q^{1/2}$ denotes an arbitrary square root of $Q$, $(A, Q^{1/2})$ is reachable. Then

$$S(\lambda) = (e^{j\lambda}I - A)^{-1}Q(e^{-j\lambda}I - A^T)^{-1} \tag{2.9}$$

where $S(\lambda)$ has rank $m < n$. Since $S(\lambda)$ is not invertible anywhere on $[-\pi, \pi]$, $x(k)$ is not minimal of order 1. However, it is minimal of order $p$ provided that $p \geq \mu$, where $\mu$ is the reachability index of $(A, Q^{1/2})$, i.e. $\mu$ is the smallest integer such that

$$R(\mu) = \left[\begin{array}{cccc} Q^{1/2} & AQ^{1/2} & \dots & A^{\mu-1}Q^{1/2} \end{array}\right] \tag{2.10}$$

has full rank.

To see this, note that since $x(k)$ is Markov, it is also reciprocal, so that

$$\hat{x}(k;p) \stackrel{\triangle}{=} E[x(k)|H_{k,p}] = E[x(k)|x(k-p), x(k+p)]. \tag{2.11}$$

To compute this estimate, we can use the $p$-step Markov model

$$x(k+p) = A^p x(k) + w(k;p) \tag{2.12a}$$

where

$$w(k;p) = \sum_{s=0}^{p-1} A^t w(k+p-t-1) \tag{2.12b}$$

is a zero-mean white Gaussian noise with intensity

$$Q(p) = R(p)R^T(p) = \sum_{s=0}^{p-1} A^t Q (A^t)^T, \tag{2.13}$$

where $Q(p) > 0$ provided that $p \geq \mu$. Then, it is shown in Section 4 of [14] that the estimate $\hat{x}(k;p)$ is given by

$$\hat{x}(k;p) = D(p)(Q^{-1}(p)A^p x(k-p) + (A^p)^T Q^{-1}(p) x(k+p)) \tag{2.14a}$$

where the error covariance matrix

$$D(p) = \left(Q^{-1}(p) + (A^p)^T Q^{-1}(p) A^p\right)^{-1} \tag{2.14b}$$

is positive definite, so that $x(k)$ is minimal of order $p$ for $p \geq \mu$. □

Another useful observation is that if $x(k)$ is a minimal stationary Gaussian process of order $p$, the subsampled process $x^{(p)}(l) = x(lp)$ is minimal of order 1.

# 3 Minimal Reciprocal Processes

In this section we show that if a stationary Gaussian process process $x(k) \in \mathbf{R}^n$ is both reciprocal and minimal, it is necessarily Markov. Without loss of generality, $x(k)$ is assumed to have zero-mean. First, observe that if $x(k)$ is reciprocal, its conditional expectation given $H_{k,1}$ can be expressed as

$$E[x(k)|H_{k,1}] = F_- x(k-1) + F_+ x(k+1). \tag{3.1}$$

The residual

$$d(k) = x(k) - F_- x(k-1) - F_+ x(k+1) \tag{3.2}$$

is then uncorrelated with $x(l)$ for $l \neq k$, and

$$E[d(k)x^T(0)] = R(k) - F_- R(k-1) - F_+ R(k+1) = D\delta(k) \tag{3.3}$$

where $D$ denotes the covariance of $d(k)$, and $\delta(k)$ is Kronecker's delta function. When $x(k)$ is minimal, the covariance $D$ is invertible, and multiplying (3.2) on the left by $D^{-1}$ gives the second-order model

$$M_0 x(k) - M_- x(k-1) - M_+ x(k+1) = e(k) \tag{3.4}$$

with

$$M_0 = D^{-1} \quad , \quad M_\pm = D^{-1} F_\pm \quad , \quad e(k) = D^{-1} d(k), \tag{3.5}$$

which was used in [14] to study Gauss-Markov minimal reciprocal processes. If we denote by $Z$ the forward shift operator $Zf(k) = f(k+1)$, it was shown in [14] that the second-order difference operator

$$\Lambda = M_0 I - M_- Z^{-1} - M_+ Z \tag{3.6a}$$

associated to (3.4) is *self-adjoint*, i.e.,

$$M_0 = M_0^T \quad , \quad M_+ = M_-^T \, , \tag{3.6b}$$

and the covariance $E(k,l) = E[e(k)e^T(l)]$ of the input noise $e(k)$ satisfies

$$\text{(i)} \;\; E(k,l) = 0 \;\; \text{for} \;\; |k-l| > 1 \, , \tag{3.7a}$$

$$\text{(ii)}\ E(k,k) = M_0\ ,\quad E(k,k+1) = -M_+\ . \tag{3.7b}$$

The driving noise $e(k)$ for model (3.4) is therefore not white, but *locally correlated*, in the sense that $e(k)$ is correlated with the noises at neighboring points $k-1$ and $k+1$, but not with noises at points whose distance from $k$ is greater than 1. Also, according to (3.7b), the covariance of $e(k)$ is specified entirely in terms of the matrices $M_0$ and $M_\pm$ defining the dynamics of the second-order model (3.4).

Multiplying (3.2) on the left by $D^{-1}$, we see that the covariance $R(k)$ satisfies the second-order difference equation

$$\Lambda R(k) = I\delta(k). \tag{3.8}$$

This equation leads to the following characterization of minimal reciprocal processes in terms of their spectral density matrix.

**Theorem 3.1** *Let $x(k)$ be a stationary Gaussian reciprocal process over* **Z**. *Then $x(k)$ is minimal iff its spectral measure $F(\lambda)$ is absolutely continuous and*

$$S(\lambda) = (M_0 - M_- e^{-j\lambda} + M_+ e^{j\lambda})^{-1} \tag{3.9}$$

*where the polynomial matrix*

$$M(z) = M_0 - M_- z^{-1} - M_+ z \tag{3.10}$$

*has no zero on the unit circle.*

**Proof:** If $x(k)$ is minimal, we know from Theorem 2.1 that its spectral measure is absolutely continuous. In addition, its covariance satisfies (3.8), which after Fourier transformation yields

$$M(e^{i\lambda})S(\lambda) = I \tag{3.11}$$

so that (3.9) is satisfied. Also, since $S(\lambda)$ must be summable over $[-\pi, \pi]$, the expression (3.9) for $S(\lambda)$ precludes the existence of zeros of $M(z)$ on the unit circle.

Conversely, assume that the spectral measure of $x(k)$ is absolutely continuous and its density $S(\lambda)$ satisfies (3.9), where $M(z)$ has no zero on the unit circle. Then we have

$$\int_{-\pi}^{\pi} S^{-1}(\lambda) d\lambda = M_0, \tag{3.12}$$

i.e., $S^{-1}(\lambda)$ is summable over $[-\pi, \pi]$, and according to Theorem 2.1, $x(k)$ is minimal. □

Thus, to an arbitrary minimal stationary Gaussian reciprocal process $x(k)$, we can associate a second-order Laurent polynomial $M(z)$ without zeros on the unit circle. Our proof of the fact that $x(k)$ is Markov relies on a spectral factorization of

$M(z)$. However, instead of considering $M(z)$ directly, we examine the homogeneous polynomial matrix

$$M(s,t) = -M_- t^2 + M_0 st - M_+ s^2 \tag{3.13}$$

which is related to $M(z)$ through the transformation $M(s,t) = stM(s/t)$. The motivation for considering $M(s,t)$ instead of $M(z)$ is that we want to handle the zeros of $M(z)$ at $z = 0$ and $z = \infty$ simultaneously. The self-adjointness of operator $\Lambda$ implies that $M(s,t)$ has the *parasymmetry property*

$$M(s,t) = M^T(t,s). \tag{3.14}$$

Let $p(s,t) = \det M(s,t)$. If $(s_0, t_0) \neq (0,0)$ is such that $p(s_0,t_0) = 0$, $z_0 = t_0/s_0$ is said to be a *mode* of $M(s,t)$. Note that we may have $z = \infty$ if $s_0 = 0$. Then we have the following result.

**Lemma 3.1** *If $M(s,t)$ corresponds to a minimal stationary Gaussian reciprocal process in $\mathbf{R}^n$, $p(s,t) = \det M(s,t)$ is a scalar homogeneous polynomial of degree $2n$ such that $p(s,t) = p(t,s)$, and $M(s,t)$ has no mode on the unit circle. This implies that $p(s,t)$ admits a factorization of the form*

$$p(s,t) = c \prod_{i=1}^{n} (s - tz_i)(t - sz_i) \tag{3.15}$$

*where $c$ is a constant and $|z_i| < 1$ for all $i$, i.e. $M(s,t)$ has $n$ modes strictly inside, and $n$ modes strictly outside the unit circle.*

**Proof:** The parasymmetry property of $M(s,t)$ implies $p(s,t) = p(t,s)$. Also, since the entries of the $n \times n$ matrix $M(s,t)$ are homogeneous of degree 2, and $p(s,t) = \det M(s,t)$ is not identically zero, $p(s,t)$ is homogeneous of degree $2n$. The property $p(s,t) = p(t,s)$ implies that if $z_0$ is a mode of $M(s,t)$, so is $z_0^{-1}$. Furthermore, since $M(z)$ has no zero in the unit circle,

$$M(e^{j\lambda/2}, e^{-j\lambda/2}) = M_0 - M_+ e^{j\lambda} - M_- e^{-j\lambda} = S^{-1}(\lambda) \tag{3.16}$$

is invertible for all $\lambda \in [-\pi, \pi]$, so that $M(s,t)$ has no mode on the unit circle. These two properties imply the existence of the factorization (3.15). $\square$

As a side comment, note that since $M(s,t)$ has real matrix coefficients, if $z_0$ is a mode of $M(s,t)$, $z_0^*$ must also be a mode. Thus, the complex modes of $M(s,t)$ occur in groups of four: $z_0$, $z_0^*$, $z_0^{-1}$, $(z_0^*)^{-1}$; and the real modes in groups of two: $z_0$ and $z_0^{-1}$.

Next, observing that the second-order stochastic model (3.4) can be rewritten as the first-order descriptor system

$$\mathcal{E} \begin{bmatrix} x(k+1) \\ r(k+1) \end{bmatrix} = \mathcal{A} \begin{bmatrix} x(k) \\ r(k) \end{bmatrix} + \begin{bmatrix} 0 \\ e(k) \end{bmatrix} \tag{3.17a}$$

with

$$\mathcal{E} = \begin{bmatrix} 0 & -I \\ M_+ & 0 \end{bmatrix} \ , \ \mathcal{A} = \begin{bmatrix} -M_- & 0 \\ M_0 & -I \end{bmatrix}, \tag{3.17b}$$

we are led to the study of the matrix pencil $\{\mathcal{E},\ \mathcal{A}\}$. As a first step, note that

$$\det(s\mathcal{E} - t\mathcal{A}) = (-1)^n \det M(s,t) \ , \tag{3.18}$$

so that the modes of $s\mathcal{E} - t\mathcal{A}$ and $M(s,t)$ are the same. Also, there exists a one-to-one correspondence between the generalized right eigenvectors of $s\mathcal{E} - t\mathcal{A}$ and $M(s,t)$. To see this, note that if $[x_0^T, u_0^T]^T \neq 0$ is a generalized right eigenvector of $s\mathcal{E} - t\mathcal{A}$ associated to the mode $(s_0, t_0) \neq (0,0)$ ,i.e.

$$s_0 \begin{bmatrix} 0 & -I \\ M_+ & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = t_0 \begin{bmatrix} -M_- & 0 \\ M_0 & -I \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \tag{3.19}$$

by eliminating $u_0$ from (3.19), we obtain

$$M(s_0, t_0)x_0 = 0. \tag{3.20}$$

In this identity we have $x_0 \neq 0$, since according to (3.19) $x_0 = 0$ would imply $s_0 u_0 = t_0 u_0 = 0$, and thus $u_0 = 0$, a contradiction since the vector $[x_0^T, u_0^T]^T$ is nonzero. Thus, $x_0 \neq 0$ is a generalized right eigenvector of $M(s,t)$ corresponding to $(s_0, t_0)$.

Then, we compute the *stable generalized eigenspace* of $\{\mathcal{E}, \mathcal{A}\}$, i.e. we find

$$S = \begin{bmatrix} X \\ U \end{bmatrix} \in \mathbf{R}^{2n \times n} \quad , \quad J \in \mathbf{R}^{n \times n}$$

such that

$$\begin{bmatrix} 0 & -I \\ M_+ & 0 \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} J = \begin{bmatrix} -M_- & 0 \\ M_0 & -I \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix}, \tag{3.21}$$

where $S$ has full column rank, and $J$ has all its eigenvalues stricly inside the unit circle. The matrices $S$ and $J$ can be obtained by computing the generalized real Schur decomposition ([8], p. 396)

$$Q^T \mathcal{E} Z = F \quad , \quad Q^T \mathcal{A} Z = G \tag{3.22}$$

of the pencil $\{\mathcal{E}, \mathcal{A}\}$, where $Q$ and $Z$ are orthogonal matrices, $G$ is upper triangular, and $F$ is quasi-upper-triangular, i.e. it has the structure

$$F = \begin{bmatrix} F_{11} & F_{12} & & F_{1m} \\ & F_{22} & & \cdot \\ & & \cdot & \cdot \\ & 0 & \cdot & \cdot \\ & & & F_{mm} \end{bmatrix}, \tag{3.23}$$

where each diagonal block $F_{ii}$ is either a $1 \times 1$ matrix or a $2 \times 2$ matrix with complex conjugate eigenvalues. In this decomposition, it is always possible to guarantee that the $n \times n$ blocks $F_s$ and $G_s$ in the partition

$$F = \begin{bmatrix} F_s & F_{s\bar{s}} \\ 0 & F_{\bar{s}} \end{bmatrix} \quad , \quad G = \begin{bmatrix} G_s & G_{s\bar{s}} \\ 0 & G_{\bar{s}} \end{bmatrix} \tag{3.24}$$

correspond to the stable eigenmodes of the pencil $\{\mathcal{E}, \mathcal{A}\}$, i.e. $J = G_s^{-1} F_s$ is stable. Then $S$ is the matrix formed by the first n columns of $Z$.

Given the matrices $X$, $U$ and $J$, we can prove the following results.

**Lemma 3.2** *$U^T X$ is a symmetric matrix.*

**Proof:** The identity (3.21) yields

$$UJ = M_- X \tag{3.25a}$$
$$M_+ X J = M_0 X - U. \tag{3.25b}$$

Premultiplying (3.25b) by $X^T$ and taking (3.25a) into account gives

$$J^T U^T X J = X^T M_0 X - X^T U. \tag{3.26}$$

Taking the transpose of (3.26) and subtracting it from (3.26), we find

$$\Delta - J^T \Delta J = 0 \tag{3.27}$$

where $\Delta = X^T U - U^T X$. Since $J$ is stricly stable, the unique solution of the Lyapunov equation (3.27) is $\Delta = 0$, so that $U^T X$ is symmetric. □

**Lemma 3.3** *$X$ is an invertible matrix.*

**Proof:** From (3.25a)-(3.25b) we have

$$\begin{aligned} X^T M(s,t) X &= X^T(-M_+ X s^2 + M_0 X st - M_- X t^2) X \\ &= X^T(-sM_+ X + tU)(sI - tJ). \end{aligned} \tag{3.28}$$

Next, taking into account (3.25a) and the symmetry of $U^T X$, we find

$$X^T M(s,t) X = (tI - sJ^T) U^T X (sI - tJ), \tag{3.29}$$

so that

$$(tI - sJ^T)^{-1} X^T M(s,t) X (sI - Jt)^{-1} = U^T X. \tag{3.30}$$

Now if $v$ is a vector in the null-space of $X$, i.e. $Xv = 0$, by pre- and post-multiplying (3.30) by $v^T$ and $v$, respectively, we get

$$v^T (tI - sJ^T)^{-1} X^T M(s,t) X (sI - tJ)^{-1} v = 0. \tag{3.31}$$

Setting $s = e^{j\lambda/2}$ and $t = e^{-j\lambda/2}$ in the above identity and noting that $M(e^{j\lambda/2}, e^{-j\lambda/2}) = S^{-1}(\lambda) > 0$, we conclude that

$$X(e^{j\lambda}I - J)^{-1}v \equiv 0 \tag{3.32a}$$

or equivalently

$$XJ^k v \equiv 0. \tag{3.32b}$$

In this case $Uv = M_0Xv - M_+XJv = 0$. But $S$ has full rank, so that we must have $v = 0$. This shows that $X$ is invertible. □

**Lemma 3.4** *Let*

$$A = XJX^{-1} \quad , \quad N = UX^{-1}, \tag{3.33}$$

*Then, $A$ is strictly stable, $N$ is symmetric positive definite, and $M(s,t)$ admits the spectral factorization*

$$M(s,t) = (tI - sA^T)N(sI - tA) \tag{3.34}$$

**Proof:** Multiplying (3.29) on the left by $X^{-T}$ and on the right by $X^{-1}$ gives (3.34). Clearly $A$ is strictly stable since it is related to the strictly stable matrix $J$ via a similarity transform. The symmetry and positive definiteness of $N$ is a consequence of (3.34), since in this case

$$S^{-1}(\lambda) = M(e^{j\lambda/2}, e^{-j\lambda/2}) = (e^{-j\lambda}I - A^T)N(e^{j\lambda}I - A) \tag{3.35}$$

which shows that $N$ is related to $S^{-1}(\lambda) > 0$ via an invertible congruence transformation. □

The factorization (3.34) provides the key to showing that a minimal stationary Gaussian reciprocal process $x(k)$ is necessarily Markov. Specifically, according to Theorem 3.1, the polynomial matrix $M(s,t)$ corresponding to $x(k)$ has no mode on the unit circle. In this case, the stable eigenspace of the pencil (3.17b) has dimension $n$, and we can construct a factorization of the form (3.35) for $S^{-1}(\lambda)$, where $N$ is positive definite and $A$ has all its eigenvalues indide the unit circle. Consequently

$$S(\lambda) = (e^{j\lambda}I - A)^{-1}N^{-1}(e^{-j\lambda}I - A^T)^{-1}, \tag{3.36}$$

so that $x(k)$ is a Markov process with state-space model

$$x(k+1) = Ax(k) + w(k) \tag{3.37}$$

where $w(k)$ is a white Gaussian noise with intensity $Q = N^{-1}$. This proves the following result.

**Theorem 3.2** *If $x(k)$ is a minimal stationary Gaussian reciprocal process over $\mathbf{Z}$ it is neccessarily Markov.*

# 4 Algebraic Riccati Equations

As one might expect, the spectral factorization (3.34) for $M(s,t)$ is closely related to the solution of algebraic Riccati equations (AREs) for $A$ and $N$. Specifically, identifying coefficients of $s^2$, $st$ and $t^2$ on both sides of (3.34) gives

$$M_0 = N + A^T N A \quad , \quad M_+ = A^T N \quad , \quad M_- = NA \, . \tag{4.1}$$

Depending on whether $N$ or $A$ are eliminated from the above identities, we obtain two different AREs.

Eliminating $N$ gives

$$-M_+ A^2 + M_0 A - M_- = 0 \tag{4.2}$$

which is a *quadratic matrix equation* for $A$. The problem is to find a strictly stable solution of this equation. Given $A$, $N$ is then obtained from

$$N = M_0 - M_+ A. \tag{4.3}$$

Eliminating $A$ yields the ARE

$$M_0 = N + M_+ N^{-1} M_- \, , \tag{4.4}$$

for which we seek to find a solution such that

$$A = N^{-1} M_- \tag{4.5}$$

is strictly stable.

We first focus on the quadratic equation (4.2).

**Theorem 4.1** *Let $M(z)$ be a second-order Laurent polynomial of the form (3.10) which is Hermitian positive definite on the unit circle. Then the ARE (4.2) admits a unique strictly stable solution $A$, and the corresponding $N$ given by (4.3) is symmetric positive definite.*

**Proof:** The Hermitian positive definiteness of $M(z)$ on the unit circle guarantees it has no zero there. The existence of a strictly stable solution to (4.2) is then proved by construction, by computing the stable eigenspace of the pencil $\{\mathcal{E}, \mathcal{A}\}$ given by (3.17b), and setting $A = XJX^{-1}$.

Conversely, if $A$ is a strictly stable solution of (4.2), and $N$ is given by (4.3), $M(s,t)$ admits the factorization (3.34). From

$$M(e^{j\lambda}) = (e^{-j\lambda} I - A^T) N (e^{j\lambda} I - A) \tag{4.6}$$

we see that $N$ must be positive definite, since it is related to $M(e^{j\lambda}) > 0$ through an invertible congruence transformation. If $A_1$ and $A_2$ are two strictly stable solutions of (4.2) we have

$$(tI - sA_1^T) N_1 (sI - tA_1) = (tI - sA_2^T) N_2 (sI - tA_2) \tag{4.7}$$

so that

$$\begin{aligned} G(e^{j\lambda}) &= N_1(e^{j\lambda}I - A_1)(e^{j\lambda}I - A_2)^{-1} && (4.8a)\\ &= (e^{-j\lambda}I - A_1^T)^{-1}(e^{-j\lambda}I - A_2^T)N_2\,. && (4.8b)\end{aligned}$$

According to (4.8a), $G(e^{j\lambda})$ is the discrete-time Fourier transform of a causal matrix sequence, and according to (4.8b) it is the transform of an anticausal sequence. This implies

$$G(e^{j\lambda}) = G, \tag{4.9}$$

where $G$ is a constant matrix, so that

$$G(e^{j\lambda}I - A_2) = N_1(e^{j\lambda}I - A_1)\,. \tag{4.10}$$

Identifying coefficients and observing that $N_1 > 0$, we obtain $G = N_1$ and $A_1 = A_2$. The strictly stable solution of (4.2) is therefore unique. □

Next, we consider the ARE (4.4). Somewhat surprisingly, except for the results of [2], little seems to be known about this equation. An interesting observation, due to Anderson et al. [2] is that if we consider the matrix

$$Z(N) = \begin{bmatrix} M_0 & -M_+ \\ -M_- & N \end{bmatrix} \tag{4.11a}$$

and if

$$S(N) = M_0 - M_+N^{-1}M_- \tag{4.11b}$$

denotes the Schur complement of $N$ inside $Z(N)$ (see [11], p. 656 for a definition of the Schur complement of a matrix), the ARE (4.4) can be rewritten as

$$N = S(N)\,. \tag{4.12}$$

Employing standard matrix inversion identities, we find also

$$N^{-1} = \begin{bmatrix} I & 0 \end{bmatrix} Z^{-1}(N) \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{4.13}$$

As a starting point, note that all the solutions of (4.4) must be *positive definite*, since for each solution $N$, we obtain a factorization of the form (4.6) for $M(e^{j\lambda})$, which indicates that $N$ must be positive definite since it is related to $M(e^{j\lambda}) > 0$ through a congruence transformation.

From Theorem 4.1, we know that there exists one solution of (4.4) such that $A = N^{-1}M_-$ is stable. We now prove that it is the largest among all the solutions of (4.4).

**Theorem 4.2** *The set of solutions of (4.4) admits a largest element $N_s$ with respect to the partial ordering $N \leq N'$ of square symmetric matrices, and the associated matrix $A_s = (N_s)^{-1}M_-$ is stable.*

**Proof:** Let $N_s$ be the solution of (4.4) for which $A_s = M_s^{-1}M_-$ is stable, and let $N$ be another solution with associated matrix $A$. Let $N_s^{1/2}$ and $N^{1/2}$ be two arbitrary matrix square roots of $N_s$ and $N$, respectively, and denote

$$W_s(z) = N_s^{1/2}(zI - A_s) \quad , \quad W(z) = N^{1/2}(zI - A) \,. \tag{4.14}$$

For $z = e^{j\lambda}$ on the unit circle, we have

$$M(z) = W_s^T(z^*)W_s(z) = W^T(z^*)W(z) \,. \tag{4.15}$$

Thus, if we define

$$V(z) = W(z)W_s^{-1}(z) \,, \tag{4.16}$$

$V(z)$ is analytic outside the unit circle, and $V^T(z^*)V(z) = I$ on the unit circle. By the maximum modulus theorem, this implies

$$V^T(z^*)V(z) \leq I \tag{4.17}$$

outside the unit circle. In particular for $z = \infty$, we have $N_s^{-T/2}NN_s^{-1/2} \leq I$, so that $N \leq N_s$. Since $N$ is an arbitrary solution of (4.4), this implies that $N_s$ is a maximum element for the set of solutions of (4.4). □

The following procedure, which is adapted from Anderson et al. [2] can be employed for computing $N_s$.

**Theorem 4.3** *The matrix sequence*

$$N_{k+1} = S(N_k) \quad , \quad N_0 = M_0 \,. \tag{4.18}$$

*is monotone decreasing and converges to $N_s$ as $k \to \infty$.*

**Proof:** By induction, we show that

$$N_s \leq N_{k+1} \leq N_k \,, \tag{4.19}$$

for all $k$. According to (4.4), $N_s \leq M_0$, so that (4.19) is satisfied for $k = 0$. Also, the matrix $Z(N_s)$ is positive definite since both $N_s$ and $S(N_s) = N_s$ are positive definite. Now if (4.19 holds for index $k$, we have $Z(N_s) \leq Z(N_{k+1}) \leq Z(N_k)$ where $Z(N_s)$ is positive definite, so that both $Z(N_{k+1})$ and $Z(N_k)$ are positive definite. Thus $Z^{-1}(N_k) \leq Z^{-1}(N_{k+1}) \leq Z^{-1}(N_s)$, and

$$\begin{aligned} N_{k+1}^{-1} = \begin{bmatrix} I & 0 \end{bmatrix} Z^{-1}(N_k) \begin{bmatrix} I \\ 0 \end{bmatrix} \;&\leq\; N_{k+2}^{-1} = \begin{bmatrix} I & 0 \end{bmatrix} Z^{-1}(N_{k+1}) \begin{bmatrix} I \\ 0 \end{bmatrix} \\ &\leq\; N_s^{-1} = \begin{bmatrix} I & 0 \end{bmatrix} Z^{-1}(N_s) \begin{bmatrix} I \\ 0 \end{bmatrix} , \end{aligned} \tag{4.20}$$

which implies $N_s \leq N_{k+2} \leq N_{k+1}$, so that (4.19) is satisfied for all $k$.

Since the sequence $N_k$ is monotone decreasing and bounded from below by $N_s$, it tends to a limit $N_* \geq N_s$ which satisfies

$$N_* = S(N^*) , \tag{4.21}$$

i.e. it is a solution of the ARE (4.4). But $N_s$ is the largest solution of (4.4), so that $N_* = N_s$. □

From the results of [2], it is easy to check that the recursions (4.18) perform the Cholesky factorization of the semi-infinite block tridiagonal Toeplitz matrix generated by $M(z)$.

# 5 Regular Reciprocal Processes

We are now ready to prove the main result of the paper.

**Theorem 5.1** *If $x(k)$ is a regular stationary Gaussian reciprocal process, it is necessarily Markov.*

**Proof:** Since the reciprocal process $x(k)$ is regular, it must be minimal of order $p$ for some $p$. In this case, the subsampled process $x^{(p)}(l) = x(lp)$ is reciprocal and minimal of order 1. But, according to Theorem 3.2, $x^{(p)}(l)$ is Markov and admits a state-space model of the form

$$x^{(p)}(l+1) = A(p)x^{(p)}(l) + w^{(p)}(l) \tag{5.1}$$

where $A(p)$ has its eigenvalues inside the unit circle and $w^{(p)}(l)$ is a white Gaussian noise sequence with intensity $Q(p) > 0$. The positive definiteness of $Q(p)$ is due to the fact that $x^{(p)}(l)$ is minimal of order 1.

We are left with the problem of going from the Markov model (5.1) for the subsequence $x^{(p)}(l) = x(lp)$ to a model for the complete sequence $x(k)$. To do so, let us define the process

$$w(k;p) \triangleq x(k+p) - A(p)x(k) . \tag{5.2}$$

From the Markov property of (5.1) we can deduce that

$$w(k;p) \perp x(k-lp) \text{ for } l \in \mathbf{N}. \tag{5.3}$$

Now, consider the interval $I = [k, k+p]$ and the problem of estimating $x(k+1)$ given $x(s)$ in the exterior and the end points of $I$. Since $x(k)$ is reciprocal, the estimate $\hat{x}(k+1; I)$ depends only on the values of $x(s)$ at the end points of $I$, so that there exists matrices $F_{\pm}(p)$ such that

$$\hat{x}(k+1; I) = F_{-}(p)x(k) + F_{+}(p)x(k+p) \tag{5.4}$$

and the corresponding residual

$$\begin{aligned} d(k;I) &= x(k+1) - \hat{x}(k+1;I) \\ &= x(k+1) - F_-(p)x(k) - F_+(p)x(k+p) \end{aligned} \tag{5.5}$$

is uncorrelated with $x(s)$ for $s$ outside or on the boundary of $I$. Eliminating $x(k+p)$ from (5.2) and (5.5) we find

$$x(k+1) = Ax(k) + w(k) \tag{5.6}$$

with

$$A \triangleq F_-(p) + F_+(p)A(p), \tag{5.7a}$$

where

$$w(k) \triangleq d(k;I) + F_+(p)w(k+p) \tag{5.7b}$$

is uncorrelated with $x(k-lp)$ for $l \in \mathbf{N}$. The relation (5.6) is in state-space form. However, to ensure it is a valid first-order Gauss-Markov model, we must show that the driving noise $w(k)$ is uncorrelated with $x(s)$ for all $s \leq k$. For a fixed $s$ , let $l = \lfloor (k-s)/p \rfloor$ be the integer part of $(k-s)/p$, i.e. the largest integer smaller or equal to $(k-s)/p$. Then $s$ is in the interior of $J = [k-(l+1)p, k-lp]$. Let $H_J$ and $H_E$ be the Hilbert spaces spanned by the $x(l)$s such that $l$ is in the interior, or the exterior of $J$, respectively. Let also $H_B$ be the Hilbert space spanned by the values of $x$ at the boundary points of $J$. Clearly $w(k) \in H_E$, since it is a linear combination of $x(k)$ and $x(k+1)$. Since $x(s) \in H_J$, and the process $x(k)$ is reciprocal, we have

$$E[w(k)x^T(s)|H_B] = E[w(k)|H_B]E[x^T(s)|H_B], \tag{5.8}$$

where $E[w(k)|H_B] = 0$ because $w(k)$ is uncorrelated with $x(k-lp)$ for $l \in \mathbf{N}$. Thus

$$E[w(k)x^T(s)] = E[E[w(k)x^T(s)|H_B]] = 0 \tag{5.9}$$

so that $w(k)$ is uncorrelated with $x(s)$ for $s \leq k$.

One final point that needs to be checked in order to ensure that (5.6) is a valid Markov model is whether $A$ is a strictly stable matrix. To do so, note that if $w(k)$ has variance $Q$, the $p$-step Markov model (5.1) is related to (5.6) through

$$A(p) = A^p \quad , \quad Q(p) = \sum_{t=0}^{p-1} A^t Q (A^t)^T \, . \tag{5.10}$$

Then, since $A(p)$ has all its eigenvalues strictly inside the unit circle, $A$ has also its eigenvalues strictly inside the unit circle. □

When $x(k)$ is a stationary Gaussian reciprocal process which is minimal of order $p$, it would be desirable to obtain a characterization of its spectral density similar to that of Theorem 3.1 for minimal processes. This task is simplified by the fact that we know that $x(k)$ is Markov, i.e. it admits a state-space model of the form (5.6). If the noise variance $Q$ has rank $m$, $Q$ can be factored as $Q = BB^T$ with $B$ of dimension $n \times m$. The spectral density matrix of $x(k)$ can therefore be expressed as

$$S(\lambda) = (e^{j\lambda}I - A)^{-1}BB^T(e^{-j\lambda}I - A^T)^{-1}\,, \tag{5.11}$$

and since $x(k)$ is minimal of order $p$, $Q(p)$ given by (5.10) is positive definite, so that the truncated reachability matrix

$$R(p) = \begin{bmatrix} B & AB & \dots & A^{p-1}B \end{bmatrix} \tag{5.12}$$

has full rank. This implies that the reachability indices $p_1 \leq p_2 ... \leq p_m$ of the pair $(A, B)$ are smaller than $p$ (see [11] p. 431 for a definition of the reachability indices of a linear system). We can use a similarity transformation

$$x(k) = T\xi(k) \tag{5.13}$$

with $T$ invertible to bring $(A, B)$ to its *controller canonical form* $(A_c, B_c)$. This canonical form has the feature that if we consider the right coprime factorization

$$(zI - A_c)^{-1}B_c = \Psi(z)D^{-1}(z) \tag{5.14}$$

we have

$$\Psi(z) = \text{block diag}\{\begin{bmatrix} z^{p_1-1} \\ \cdot \\ \cdot \\ \cdot \\ z \\ 1 \end{bmatrix}, \dots, \begin{bmatrix} z^{p_m-1} \\ \cdot \\ \cdot \\ \cdot \\ z \\ 1 \end{bmatrix}\} \tag{5.15}$$

where $D(z)$ has column degrees $p_1 \leq p_2 ... \leq p_m$. Since $A_c$ has all its eigenvalues inside the unit circle (it is similar to $A$), and $\det(zI - A_c) = c \det D(z)$ with $c$ constant, $D(z)$ has all its zeros inside the unit circle. Then the spectral density of $\xi(k)$ is given by

$$S_\xi(\lambda) = \Psi(e^{j\lambda})M^{-1}(e^{j\lambda})\Psi^H(e^{j\lambda})\,, \tag{5.16a}$$

with

$$M(z) \triangleq D^T(z^{-1})D(z)\,. \tag{5.16b}$$

$M(z)$ is a matrix Laurent polynomial with the parasymmetry property $M(z) = M^T(z^{-1})$. It has no zero on the unit circle since $D(z)$ has no zero there, so that $M(z)$ is Hermitian positive definite on the unit circle. Finally, since the column degrees of $D(z)$ are all less or equal to $p$, the entries of $M(z)$ have at most degree $2p$.

The representation (5.16a) provides the desired characterization of the spectral density matrix of a reciprocal process which is minimal of order $p$. This representation admits the following stochastic interpretation. Let

$$M(z) = \sum_{s=-p}^{p} M_s z^s\,. \tag{5.17}$$

Then consider the stochastic process $\rho(k) \in \mathbf{R}^m$ specified by the higher-order self-adjoint stochastic model

$$\sum_{s=-p}^{p} M_s \rho(k+s) = e(k) \tag{5.18}$$

where the covariance $E(k,l) = E[e(k)e^T(l)]$ of the zero-mean Gaussian driving noise $e(k)$ satisfies

$$\text{(i)} \;\; E(k,k+s) = 0 \;\; \text{for} \;\; |s| > p \,, \tag{5.19a}$$

$$\text{(ii)} \;\; E(k,k+s) = M_s \;\; \text{for} \;\; |s| \leq p, \tag{5.19b}$$

and where we impose the asymptotic condition that the variance of $\rho(k)$ must be bounded as $k \to \infty$. This condition plays the same role as boundary conditions over a finite interval. It is introduced to ensure that the solution of (5.18)-(5.19b) is a stationary process with spectral density $M^{-1}(e^{j\lambda})$. Then from (5.15)-(5.16b), we see that if

$$\rho^T(k) = [\; \rho_1(k) \;\; \ldots \;\; \rho_i(k) \;\; \ldots \;\; \rho_m(k) \;] \,, \tag{5.20}$$

i.e. $\rho_i(k)$ denotes the $i$th entry of $\rho(k)$, the process $\xi(k)$ can be expressed as

$$\xi^T(k) = [\; {\xi_1}^T(k) \;\; \ldots \;\; {\xi_i}^T(k) \;\; \ldots \;\; {\xi_m}^T(k) \;] \tag{5.21a}$$

with

$${\xi_i}^T(k) = [\; \rho_i(k+p_i-1) \;\; \ldots \;\; \rho_i(k+1) \;\; \rho_i(k) \;] \,, \tag{5.21b}$$

which indicates that $\xi(k)$ is obtained by taking successive lagged values of the entries of $\rho(k)$. Furthermore the lengths $p_1 \leq p_2 ... \leq p_m$ of the lagged chains of all entries of $\rho(k)$ are less than $p$. In summary, a reciprocal (and thus Markov) process that is minimal of order $p$ can be obtained by first constructing a process $\rho(k)$ satisfying a self-adjoint stochastic model (5.18) of order less or equal to $2p$, then taking successive lagged values of the entries of $\rho(k)$ where the lag chains have at most length $p$, and applying a similarity transform $T$ to the resulting process $\xi(k)$.

In the above discussion, the stochastic model (5.18)-(5.21b) was used to characterize the spectral density of reciprocal (and thus Markov) processes over $\mathbf{Z}$. However, as shown by Frezza [7], models of this type can also be employed to describe all higher-order and mixed-order reciprocal processes defined over a *finite interval*, provided that the asymptotic boundedness condition for the variance of $\rho(k)$ is replaced by Dirichlet boundary conditions at both ends of the interval.

# 6 Conclusions

In this paper, following a conjecture of Carmichael, Masse and Theodorescu [5], it has been shown that all regular and reciprocal multivariate stationary Gaussian processes over $\mathbf{Z}$ are necessarily Markov. The proof of this result was broken into two steps, where in the first step it was shown that minimal reciprocal processes are Markov, and in the second step, this result was extended to regular reciprocal processes. The proof of the fact that minimal and reciprocal stationary Gaussian processes are Markov was constructive and relied on the construction of a state-space model for the process of interest through the computation of the stable eigenspace of a matrix pencil. This computation was shown to be equivalent to performing a spectral factorization of a parasymmetric second-order Laurent polynomial which is Hermitian positive definite on the unit circle. This factorization was also related to the solution of two types of algebraic Riccati equations which were studied in detail.

A consequence of the result proved here is that the reciprocal and Markov stochastic realization problems [16], [22] over $\mathbf{Z}$ are roughly equivalent. For the reciprocal case, this problem can be stated as follows: given a stationary Gaussian process $y(k) \in \mathbf{R}^p$ defined over $\mathbf{Z}$, find a reciprocal stationary Gaussian process $x(k) \in \mathbf{R}^n$ such that

$$y(k) = Cx(k) \; . \tag{6.1}$$

Provided that $x(k)$ is regular, it will also be Markov, so that the reciprocal realization problem reduces to the Markov case. The only difference is that, when constructing an internal model for $x(k)$, we are not forced to use a first-order state-space model, but can employ a higher-order noncausal model of the form (5.18)-(5.21b), which is in fact easier to construct. Also, when $y(k)$ is defined only over a finite-interval, it is expected that the reciprocal and Markov realization problems will differ significantly.

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