V. Conclusion

The results in this note show that the concepts introduced in [6] for solving the minimal communication problem to achieve state disambiguation can be adapted to solve the more general problem of "essential transitions." This adaptation required the introduction of the property of legality, which captures the requirements pertaining to essential transitions. Several issues remain open for future research. The determination of well-posed sets of essential transitions in specific decentralized supervisory control or fault diagnosis problems is of particular interest. This was done in an intuitive manner in the decentralized control example in the note. Systematic procedures for generating these sets when coobservability is violated are currently being investigated. Also, the problem of synthesizing minimal communication maps in multiagent problems (three or more agents) is entirely open and most likely quite challenging.

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PID Stabilization of MIMO Plants

A. N. Gündeş and A. B. Özgüler

***Abstract*—Closed-loop stabilization using proportional–integral–derivative (PID) controllers is investigated for linear multiple-input–multiple-output (MIMO) plants. General necessary conditions for existence of PID-controllers are derived. Several plant classes that admit PID-controllers are explicitly described. Plants with only one or two unstable zeros at or "close" to the origin (alternatively, at or close to infinity) as well as plants with only one or two unstable poles which are at or close to origin are among these classes. Systematic PID-controller synthesis procedures are developed for these classes of plants.**

***Index Terms*—Integral action, proportional–integral–derivative (PID) controllers.**

I. Introduction

Proportional–integral–derivative (PID) controllers are widely used in many control applications and preferred for their simplicity. Due to their integral action, PID-controllers achieve asymptotic tracking of step-input references. The topic of PID-control is treated extensively in every classical control text, e.g., [11]. In spite of the importance and widespread use of these low-order controllers, most PID design approaches lack systematic procedures and rigorous closed-loop stability proofs. Rigorous synthesis methods are explored recently in, e.g., [8]–[10] and [13].

The simplicity of PID-controllers, which is desirable due to easy implementation and from a tuning point-of-view, also presents a major restriction: PID-controllers can control only certain classes of plants. The problem of existence of stabilizing PID-controllers, which is practically very relevant (see [3]), is unfortunately not easy to solve. To gain insight into the problem's difficulty, note that the existence of a stabilizing PID-controller for a plant $G(s)$ is equivalent to that of a *constant stabilizing output feedback* for a transformed plant. Alternatively, the problem can be posed as determining existence conditions of a stable and fixed-order controller for $G(s)((s+1)/s)$, which is also a difficult problem [1], [14]. The restriction on the controller order is a further major difficulty. Strong stabilizability of the plant is a necessary condition for existence of PID-controllers, but it is not sufficient); e.g., $G(s) = 1/(s-p)^3$ cannot be stabilized using a PID-controller for any $p \geq 0$, although the extended plant $G(s)((s+1)/s)$ is stabilizable using a stable controller (whose inverse is also stable).

The goal of this note is to find sufficient conditions on PID stabilizability, and hence, to identify plant classes that admit PID-controllers. Furthermore, explicit construction of the PID parameters for such plant classes is explored, leading to systematic controller synthesis procedures for linear, time-invariant (LTI), multiple-input–multiple-output (MIMO) plants of arbitrarily high order using the standard unity-feedback system shown in Fig. 1. The results obtained here explore conditions for PID stabilizability of general MIMO unstable plants without

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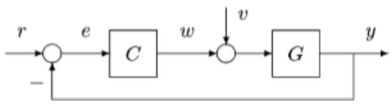

Fig. 1. Unity-feedback system $\mathrm{Sys}(G, C)$.

any restrictions on the plant order. Even in the single-input–single-output (SISO) case, explicit descriptions of high-order plant classes that admit PID-controllers are not available. Computational PID-stabilization methods of "efficient search" in the parameter space were recently developed for SISO delay-free systems (e.g., [12]), and some of these techniques were extended to first-order, scalar, single-delay systems [13]. Although some of the conditions on existence of PID-controllers could be derived *for SISO plants* using root-locus arguments or via a generalization of the Hermite–Biehler theorem [2], [6], [7], [15], they would not extend to the MIMO case and would not lead to explicit synthesis procedures. The results here emphasize systematic designs with freedom in the design parameters.

Section II gives preliminary definitions and the basic necessary conditions for stabilizability using PID-controllers. A two-step construction of stabilizing controllers is used as the basis of our synthesis method, which first constructs a proportional–derivative controller and then adds an integral term. Section III has the main results, where a novel use of the small gain theorem leads to identifying plant classes that are stabilizable using PID-controllers. The plants in Section III-A have restrictions on their blocking zeros in the region of instability, which leaves the pole locations completely free. The plants in Section III-B have restrictions on the unstable poles, which allows complete freedom in the zero locations.

The following notation is used: Let $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{R}_+$ denote complex, real, and positive real numbers, respectively. The extended closed right-half complex plane is $\mathcal{U} = \{s \in \mathbb{C} | \mathcal{R}e(s) \geq 0\} \cup \{\infty\}$; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in $\mathbf{S}$; I_n is the $n \times n$ identity matrix. The H_∞-norm of $M(s) \in \mathcal{M}(\mathbf{S})$ is $\|M\| := \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$. We drop (s) in transfer matrices such as $G(s)$ wherever this causes no confusion. We use coprime factorizations over $\mathbf{S}$; i.e., for $G \in \mathbf{R_p}^{n_y \times n_u}$, $G = Y^{-1}X$ denotes a left-coprime-factorization (LCF), where $X, Y \in \mathcal{M}(\mathbf{S})$, $\det Y(\infty) \neq 0$.

II. PID Stabilization Conditions

Consider the LTI MIMO unity-feedback system $\mathrm{Sys}(G, C)$ shown in Fig. 1, where $G \in \mathbf{R_p}^{n_y \times n_u}$ is the plant's transfer function and $C \in \mathbf{R_p}^{n_u \times n_y}$ is the controller's transfer function. Assume that $\mathrm{Sys}(G, C)$ is well posed, G and C have no unstable hidden modes, and $G \in \mathbf{R_p}^{n_y \times n_u}$ is full (normal) row rank. We consider a realizable form of proper PID-controllers given in (1), where $K_p, K_i, K_d \in \mathbb{R}^{n_u \times n_y}$ are the proportional, integral, derivative constants, respectively, and $\tau \in \mathbb{R}_+$ [3]

$$C_{\mathrm{pid}} = K_p + \frac{K_i}{s} + \frac{K_d s}{\tau s + 1}. \tag{1}$$

For implementation, a (typically fast) pole is added to the derivative term so that C_{pid} in (1) is proper. The integral action in C_{pid} is present when $K_i \neq 0$. The subsets of PID-controllers obtained by setting one or two of the three constants equal to zero are denoted as follows: (1) becomes a PI-controller C_{pi} when $K_d = 0$, an ID-controller C_{id} when $K_p = 0$, a PD-controller C_{pd} when $K_i = 0$, a P-controller C_p when $K_d = K_i = 0$, an I-controller C_i when $K_p = K_d = 0$, and a D-controller C_d when $K_p = K_i = 0$.

Definition 1:

1) $\mathrm{Sys}(G, C)$ is said to be stable iff the transfer function from (r, v) to (y, w) is stable.
2) C is said to stabilize G iff C is proper and $\mathrm{Sys}(G, C)$ is stable.
3) $G \in \mathbf{R_p}^{n_y \times n_u}$ is said to admit a PID-controller iff there exists $C = C_{\mathrm{pid}}$ as in (1) such that $\mathrm{Sys}(G, C_{\mathrm{pid}})$ is stable.

We say that G is stabilizable by a PID-controller and C_{pid} is a stabilizing PID-controller. △

Let $G = Y^{-1}X$ be any LCF and $C = N_c D_c^{-1}$ be any RCF; for $G \in \mathbf{R_p}^{n_y \times n_u}$, $X, Y \in \mathcal{M}(\mathbf{S})$, $\det Y(\infty) \neq 0$, and for $C \in \mathbf{R_p}^{n_u \times n_y}$, $N_c, D_c \in \mathcal{M}(\mathbf{S})$, $\det D_c(\infty) \neq 0$. Then, C stabilizes G if and only if $M := Y D_c + X N_c \in \mathcal{M}(\mathbf{S})$ is unimodular [4], [14]. We now examine necessary conditions for PID stabilizability. Note here that, since a general PID-controller contains a pole at the origin and is hence unstable, the result given in part 2) of Lemma 1 is not obvious.

Lemma 1: (Necessary Conditions for Existence of PID): Let $G \in \mathbf{R_p}^{n_y \times n_u}$. Let $\mathrm{rank} G(s) = n_y$.

1) If G admits a PID-controller such that the integral constant $K_i \in \mathbb{R}^{n_u \times n_y}$ is nonzero, then G has no transmission zeros at $s = 0$ and $\mathrm{rank} K_i = n_y$.
2) If G admits a PID-controller, then G is strongly stabilizable. △

Although several PID-controller synthesis methods exist for stable plants, which obviously admit PID-controllers, Proposition 1 gives a method applicable to MIMO plants.

Proposition 1: (PID-Controller Synthesis for Stable Plants): Let $H \in \mathbf{S}^{n_y \times n_u}$ and $\mathrm{rank} H(s) = n_y \leq n_u$. If the integral term is to be nonzero, also let $\mathrm{rank} H(0) = n_y$ and let $H^I(0)$ be a right inverse of $H(0)$. For any of the P, I, or D terms in C_{pid} to be nonzero, choose the corresponding $\Delta_p, \Delta_d,$ and $\Delta_i = 1$; to make any of these terms zero, choose the corresponding $\Delta_p, \Delta_d,$ and $\Delta_i = 0$. Choose any $\hat{K}_p, \hat{K}_d \in \mathbb{R}^{n_u \times n_y}$, $\tau \in \mathbb{R}_+$. Choose any $\gamma \in \mathbb{R}_+$ satisfying

$$\gamma < \left\| H(s) \left(\Delta_p \hat{K}_p + \Delta_d \frac{\hat{K}_d s}{\tau s + 1} \right) + \Delta_i \frac{H(s) H^I(0) - I}{s} \right\|^{-1}. \tag{2}$$

Let $K_p = \gamma \hat{K}_p$, $K_d = \gamma \hat{K}_d$, and $K_i = \gamma H(0)^I$; then

$$C_{\mathrm{pid}} = \Delta_p \gamma \hat{K}_p + \Delta_i \frac{\gamma H(0)^I}{s} + \Delta_d \frac{\gamma \hat{K}_d s}{\tau s + 1} \tag{3}$$

is a PID-controller that stabilizes H. △

Lemma 2 states that if a stabilizing C_p, C_i, and C_d exist for the plant G, then it is possible to find suitable choices for the remaining constants and extend to stabilizing PI, ID, PD, and PID-controllers.

Lemma 2: Let $G \in \mathbf{R_p}^{n_y \times n_u}$.

1) *Two-step controller synthesis* [14, Th. 5.3.10]: Suppose that C_g stabilizes G and C_h stabilizes $H := G(I + C_g G)^{-1} \in \mathcal{M}(\mathbf{S})$. Then, $C = C_g + C_h$ also stabilizes G.
2) *PID-controllers constructed from subsets*: If G admits a subset of a PID-controller where at least one of the three constants K_p, K_d, and K_i is nonzero, then G admits a PID-controller such that any two or all three of the three constants are nonzero. The integral constant K_i is nonzero only if G has no transmission zeros at $s = 0$.
3) *Two-step PID-controller synthesis*: Let G have no transmission-zeros at $s = 0$. Suppose that there exists a PD-controller C_{pd} stabilizing G. Then, $C_{\mathrm{pid}} = C_{pd} + K_i/s$ also stabilizes G, where $C_{ih} = K_i/s$ is any I-controller that stabilizes $H_{pd} := G(I + C_{pd}G)^{-1}$. In particular, $C_{ih} = K_i/s$ can be chosen as

$$\frac{K_i}{s} = \frac{\gamma H_{pd}(0)^I}{s} = \frac{\gamma \left[G^I(0) + K_p \right]}{s} \tag{4}$$

for any $\gamma \in \mathbb{R}_+$ satisfying

$$0 < \gamma < \left\| \frac{H_{pd}(s)H_{pd}(0)^I - I}{s} \right\|^{-1}. \tag{5}$$

$\triangle$

Condition (5) on the scaling factor γ used in K_i/s proposed in Lemma 2 can also be expressed as

$$\gamma < \left\| (I+GC_{pd})^{-1} \frac{G(s)\left[H_{pd}(0)^I - C_{pd}\right] - I}{s} \right\|^{-1}$$
$$= \left\| (I+GC_{pd})^{-1} \frac{\left[G(s)G^I(0) - I\right]}{s} - \frac{H_{pd}(s)K_d}{\tau s+1} \right\|^{-1}. \tag{6}$$

III. Plant Classes That Admit PID-Controllers

We investigate specific unstable plant classes that admit PID-controllers and propose synthesis methods. By Lemma 1, plants that admit PID-controllers are necessarily strongly stabilizable. Section III-A deals with various plants satisfying the parity-interlacing property, with restrictions on the unstable region blocking zeros but no restrictions on the location of the poles. By contrast, the $\mathcal{U}$-poles are restricted in Section III-B in order to allow complete freedom in the zero locations.

The unstable plant classes considered in this section are all square $(n_y = n_u)$ and full-rank, i.e., G satisfies $G \in \mathbf{R_p}^{n_y \times n_y}$, (normal) $\operatorname{rank} G(s) = n_y$.

1) For unstable plants with no zeros in $\mathcal{U}$ including infinity ($G^{-1} \in \mathbf{S}^{n_y \times n_y}$), there exist P, I, D, (hence, PD, PI, and PID) controllers.
2) For unstable plants with one or two blocking zeros in $\mathcal{U}$ (including $s = 0, \infty$) satisfying certain norm bounds, there exist PD-controllers; when neither one of the zeros is at $s = 0$, there exist PID-controllers.
3) For unstable plants with one or two poles in $\mathcal{U}$ (including $s = 0$) satisfying certain norm bounds, there exist PD and PID-controllers.

A. *Unstable Plants With Restrictions on the $\mathcal{U}$-Zeros*

Unstable plants without blocking zeros in the unstable region $\mathcal{U}$ (including infinity), which are obviously strongly stabilizable, admit PID-controllers. Plants that have (one or two) real-axis blocking zeros in $\mathcal{U}$ also admit PID-controllers under certain sufficient conditions on these zeros.

1) Unstable Plants With No $\mathcal{U}$-Zeros: Let $G \in \mathbf{R_p}^{n_y \times n_y}$ have no transmission-zeros in $\mathcal{U}$ (including infinity); hence, G satisfies the necessary condition in part 1) of Lemma 1 for existence of PID-controllers with nonzero K_i. Therefore, G has an LCF $G = Y^{-1}X = (G^{-1})^{-1}I$, where $\det G(\infty)^{-1} \neq 0$. Proposition 2 shows that G admits P, I, PI, ID, and PID-controllers.

Proposition 2: Choose $K_p, K_d \in \mathbb{R}^{n_y \times n_y}$, $\tau \in \mathbb{R}_+$ such that $\det[G(\infty)^{-1} + K_p + \tau^{-1}K_d] \neq 0$ (K_p and/or K_d may be zero). Let $W_{pd} := G^{-1}(s) + K_p + (K_d s/(\tau s+1))$. Then

$$C_{\text{pid}} = K_p + \frac{\theta\left[G(\infty)^{-1} + K_p + \tau^{-1}K_d\right]}{s} + \frac{K_d s}{\tau s + 1} \tag{7}$$

stabilizes G for any $\theta \in \mathbb{R}_+$ satisfying

$$\theta > \left\| s\left[W_{pd}\left(G(\infty)^{-1} + K_p + \tau^{-1}K_d\right)^{-1} - I\right]\right\|. \tag{8}$$

$\triangle$

2) Unstable Plants With Positive Real Zeros Including Zero and Infinity: Let G have no transmission zeros in $\mathcal{U}$ other than $\ell \in \{1, 2\}$ (one or two) real-axis blocking zeros (at $s = z_j \in \mathbb{R}$, $z_j \geq 0$, $j \in \{1, \ell\}$); G may have any number of transmission zeros in the stable region. The condition in 1) of Lemma 1 for existence of PID-controllers with nonzero K_i is satisfied only when $z_j \neq 0$. The poles of G are completely arbitrary, except that we assume G has no poles at $s = 0$ if there is a zero close to the origin.

We consider two cases where the real-axis zeros at $z_j \geq 0$ are either "small," including $z_j = 0$, or "large," including infinity.

Case 1): Let $z_j \in \mathbb{R}$, $z_j \geq 0$, $z_j \leq z_{j+1}$; with $a_j \in \mathbb{R}_+$, $j \in \{1, \ell\}$, let

$$y := \prod_{j=1}^{\ell} y_j = \prod_{j=1}^{\ell}(a_j s + 1) \quad x := \prod_{j=1}^{\ell} x_j = \prod_{j=1}^{\ell}(s - z_j). \tag{9}$$

Let G have an LCF $G = Y^{-1}X = ((x/y)G^{-1})^{-1}((x/y)I)$. Let G have no poles at $s = 0$. Under these assumptions in Proposition 3, G admits PD-controllers if upper bounds are imposed on the zeros; if $\ell = 1$, G also admits P-controllers; G admits PI and PID-controllers only if $z_j \neq 0$. If $\ell = 1$, some plants (e.g., $G = (s-z)/(s-p)$, $z, p > 0$) do not admit D- and I-controllers. If $\ell = 2$, some plants (e.g., $G = ((s-z_1)(s-z_2))/(s^2 - p)$, $z_1, z_2, p > 0$) do not admit P, D, and I-controllers.

Proposition 3: Let G have no poles at $s = 0$. Let $\ell \in \{1, 2\}$. With x, y as in (9), let $Y = (x/y)G^{-1} \in \mathcal{M}(\mathbf{S})$, where $z_j \in \mathbb{R}$, $z_j \geq 0$. Let $Y(0)^{-1} = x^{-1}G(s)|_{s=0}$. If $\ell = 1$, choose any $k_d \geq 0$, $\tau_1 \in \mathbb{R}_+$. Define $\hat{C}_1 := 1 + (k_d s/(\tau_1 s + 1))$ and define

$$\Phi_1 = \hat{C}_1^{-1} x G^{-1}(s) Y(0)^{-1} - I. \tag{10}$$

If $0 \leq z_1 < \|\Phi_1/s\|^{-1}$, then for any $\alpha \in \mathbb{R}_+$ satisfying (11), the PD-controller C_1 in (12) stabilizes G

$$\alpha < \|\Phi_1/s\|^{-1} - z_1, \tag{11}$$
$$C_1 = K_{p1} + \frac{K_{d1}s}{\tau_1 s + 1} = \frac{1}{z_1 + \alpha}\hat{C}_1 Y(0)$$
$$= (z_1 + \alpha)^{-1}\left[1 + \frac{k_d s}{\tau_1 s + 1}\right] Y(0). \tag{12}$$

If $\ell = 2$, choose any $k_2 \in \mathbb{R}_+$. Define

$$\Phi_2 := (k_2 s + 1)^{-1} x G^{-1}(s) Y(0)^{-1} - I. \tag{13}$$

If $2(z_1 + z_2) < \|\Phi_2/s\|^{-1}$, then for any $\alpha, \beta \in \mathbb{R}_+$ satisfying (14), let $\eta := 2(z_1 + z_2) + \alpha + \beta$, $\rho := \alpha\beta + \alpha z_2 + \beta z_1$, $\tau_2 = \eta\rho^{-1}$, $K_{p2} = \rho^{-1}Y(0)$, and $K_{d2} = \rho^{-1}(k_2 - \tau_2)Y(0)$; the PD-controller C_2 in (15) stabilizes G

$$\alpha + \beta < \|\Phi_2/s\|^{-1} - 2(z_1 + z_2), \tag{14}$$
$$C_2 = K_{p2} + \frac{K_{d2}s}{\tau_2 s + 1} = \frac{\rho^{-1}(k_2 s + 1)}{\tau_2 s + 1} Y(0)$$
$$= \frac{k_2 s + 1}{(2(z_1 + z_2) + \alpha + \beta)s + \alpha\beta + \alpha z_2 + \beta z_1} Y(0). \tag{15}$$

If $z_\ell \neq 0$, let $C_{pd} = C_\ell$ as in (12) or (15) for $\ell = 1$ or $\ell = 2$, respectively. Choose any $\gamma \in \mathbb{R}_+$ satisfying (5). Then, a PID-controller that stabilizes G is given by

$$C_{\text{pid}} = C_\ell + \frac{\gamma\left[G^{-1}(0) + K_{p\ell}\right]}{s}. \tag{16}$$

Equation (16) becomes $C_{\text{pid}} = C_1 + ((z_1 + \alpha)^{-1}\alpha\gamma/s)G^{-1}(0)$ for $\ell = 1$ and $C_{\text{pid}} = C_2 + (\rho^{-1}(\rho + z_1 z_2)\gamma/s)G^{-1}(0)$ for $\ell = 2$. $\triangle$

Case 2): With $z_j \in \mathbb{R}$, $0 < z_j \le \infty$ and $a_j \in \mathbb{R}$, $a_j > 0$, $j \in \{1, \ell\}$, let

$$\tilde{y} := \prod_{j=1}^{\ell} \tilde{y}_j = \prod_{j=1}^{\ell} (s + a_j) \quad \tilde{x} := \prod_{j=1}^{\ell} \tilde{x}_j = \prod_{j=1}^{\ell} (1 - s/z_j). \quad (17)$$

Let G have an LCF $G = Y^{-1}X = ((\tilde{x}/\tilde{y})G^{-1})^{-1}((\tilde{x}/\tilde{y})I)$. Proposition 4 shows that, with lower bounds on the zeros, plants in this class admit PD and PID-controllers; if $\ell = 1$, they also admit P- and PI-controllers. If $\ell = 1$, some plants (e.g., $G = 1/(s-p)$, $p > 0$) do not admit D- and I-controllers. If $\ell = 2$, some plants (e.g., $G = 1/(s^2 - p)$, $p > 0$) do not admit P-, D-, and I-controllers.

Proposition 4: Let $\ell \in \{1, 2\}$. With $\tilde{x}$, $\tilde{y}$ as in (17); let $Y = (\tilde{x}/\tilde{y})G^{-1} \in \mathcal{M}(\mathbf{S})$, where $z_j \in \mathbb{R}$, $0 < z_j \le \infty$, and $a_j \in \mathbb{R}$, $a_j > 0$, $j \in \{1, \ell\}$. Let $Y(\infty)^{-1} = \tilde{x}^{-1}\tilde{y}G(s)|_{s=\infty}$. If $\ell = 1$, choose any $k_p, \tau_1 \in \mathbb{R}_+$. Define $\hat{C}_1 := (\tau_1 s + k_p)/(\tau_1 s + 1)$ and define

$$\Psi_1 := \hat{C}_1^{-1} \hat{x} G^{-1}(s) Y(\infty)^{-1} - sI. \quad (18)$$

If $\|\Psi_1\| < z_1 \le \infty$, then for any $\alpha \in \mathbb{R}_+$ satisfying (19), the PD-controller C_1 in (20) stabilizes G

$$\alpha > (1 - \|\Psi_1\|/z_1)^{-1} \|\Psi_1\|, \quad (19)$$

$$C_1 = K_{p1} + \frac{K_{d1}s}{\tau_1 s + 1} = \frac{\alpha}{1 + \alpha/z_1} \hat{C}_1 Y(\infty) = \frac{\alpha}{1 + \alpha/z_1} \left[k_p + \frac{(1 - k_p)\tau_1 s}{\tau_1 s + 1} \right] Y(\infty). \quad (20)$$

If $\ell = 2$, choose any $k_2 \in \mathbb{R}_+$. Define

$$\Psi_2 := (s + k_2)^{-1} x G^{-1}(s) Y(\infty)^{-1} - sI. \quad (21)$$

If $2(1/z_1 + 1/z_2) < \|\Psi_2\|^{-1}$, then for any $\alpha, \beta \in \mathbb{R}_+$ satisfying (22), let $\eta := \alpha\beta + \alpha/z_2 + \beta/z_1$, $\rho := \alpha + \beta + 2(1/z_1 + 1/z_2)$, $\tau_2 = \eta\rho^{-1}$, $K_{p2} = \rho^{-1}k_2 Y(\infty)$, and $K_{d2} = \rho^{-1}(1 - \tau_2 k_2)Y(\infty)$; the PD-controller C_2 in (23) stabilizes G

$$\alpha + \beta < \|\Psi_2\|^{-1} - 2\left(\frac{1}{z_1} + \frac{1}{z_2}\right) \quad (22)$$

$$C_2 = K_{p2} + \frac{K_{d2}s}{\tau_2 s + 1} = \frac{\rho^{-1}(s + k_2)}{\tau_2 s + 1} Y(\infty) = \frac{s + k_2}{\left(\alpha\beta + \frac{\alpha}{z_1} + \frac{\beta}{z_2}\right)s + 2\left(\frac{1}{z_1} + \frac{1}{z_2}\right) + \alpha + \beta} Y(\infty). \quad (23)$$

If $z_\ell \ne 0$, let $C_{pd} = C_\ell$ as in (20) or (23) for $\ell = 1$ or $\ell = 2$, respectively. Choose any $\gamma \in \mathbb{R}_+$ satisfying (5). Then, a PID-controller that stabilizes G is given by (16), where $K_{p\ell}$ is as in (20) or (23). △

Remark 1:

1) In Proposition 3, when $\ell = 1$, choosing $k_d = 0$ gives a P-controller $C_p = (z_1 + \alpha)^{-1} Y(0)$ in (12); if $z_1 \ne 0$, then (16) becomes a PI-controller $C_{pi} = (z_1 + \alpha)^{-1}(-z_1 + (\alpha\gamma/s))G^{-1}(0)$. When $\ell = 2$, let $\alpha = \beta = 0$; C_2 in (15) becomes a PI-controller $C_{pi} = 0.5 z_1 z_2 (z_1 + z_2)^{-1}(k_2 + (1/s))G^{-1}(0)$.
2) By Proposition 3 (and by the dual Proposition 4), any unstable plant with (up to) two blocking zeros at $z_1 = z_2 = 0$ (or $z_1 = z_2 = \infty$) and any number of zeros in the stable region can be stabilized using PID-controllers since the norm bounds $z_1 < \|\Phi_1/s\|^{-1}$ or $2(z_1 + z_2) < \|\Phi_2/s\|^{-1}$ (or $1/z_1 < \|\Psi_1\|^{-1}$ or $2(1/z_1 + 1/z_2) < \|\Psi_2\|^{-1}$, respectively) are obviously satisfied.
3) For some insight on the norm bounds of Proposition 3, we observe that when G has only one $\mathcal{U}$-zero z_1 ($\ell = 1$), the bound $0 \le z_1 < \|\Phi_1/s\|^{-1}$ is satisfied only if z_1 is closer to the origin than the smallest positive real pole $p_{\min}$ because $G^{-1}(p_{\min}) = 0$ implies (by the H_∞-norm definition) that $\|\Phi_1/s\| \ge 1/p_{\min}$; hence, $z_1 < p_{\min}$ as claimed. Similarly, the norm bound $z_1 > \|\Psi_1\|$ in Proposition 4 is satisfied only if z_1 is farther to the right of the largest positive real pole $p_{\max}$ because $G^{-1}(p_{\max}) = 0$ implies $z_1 > \|\Psi_1\| \ge p_{\max}$.
4) Consider, for example, the simple scalar plant $G = (s - z_1)(s - z_2)(s + 10)/(s^2 - 81)(s - 7)$, which has two $\mathcal{U}$-zeros $z_1, z_2 > 0$. Following Proposition 3, for an arbitrary choice $k_2 > 0$, say $k_2 = 0.09$, if $2(z_1 + z_2) < 3.0043$, then PD- and PID-controllers exist. A different k_2 choice would result in a different norm bound $2(z_1 + z_2) < \|\Phi_2/s\|^{-1}$. △

3) Unstable Plants With Complex Zeros: Let G have no transmission-zeros in $\mathcal{U}$ other than a complex-conjugate pair of blocking zeros at $z_1 = \bar{z}_2 \in \mathcal{U}$; but G may have any number of transmission zeros in the stable region. The poles of G are completely arbitrary except that we assume G has no poles at $s = 0$ if there is a $j\omega$-axis zero close to the origin. We consider two cases where the complex-conjugate zeros are either "small," including zero, or "large," including infinity. Let

$$y := (s + g)^2 \quad x := \prod_{j=1}^{2} (s - z_j) = s^2 - 2fs + g^2 \quad (24)$$

where $z_1 = \bar{z}_2 \in \mathcal{U}$, $f, g \in \mathbb{R}_+$, $f \ge 0$, and $f < g$. Write G as $G = Y^{-1}X = ((x/y)G^{-1})^{-1}((x/y)I)$. The condition in 1) of Lemma 1 for existence of PID-controllers with nonzero K_i is satisfied since $g \ne 0$, i.e., G has no transmission zeros at $s = 0 \in \mathcal{U}$. Part 1) of Proposition 5 shows that with sufficient conditions that impose upper bounds on the zeros, plants that have no poles at $s = 0$ admit PI-, ID-, and PID-controllers. Some plants (e.g., $G = (s^2 + g^2)/(s^2 - p), p > 0$) do not admit P- and D-controllers. Part 2) of Proposition 5 shows that with sufficient conditions that impose lower bounds on the zeros, plants in this class admit PID-controllers.

Proposition 5:

1) Let G have no poles at $s = 0$. With x, y as in (24); let $Y = (x/y)G^{-1} \in \mathcal{M}(\mathbf{S})$. Choose any $k_p \ge 0$, $k_d \ge 0$, and $\tau > 0$ (k_p and k_d both are not zero). Define $\hat{C} := k_p + (1/s) + (k_d s/(\tau s + 1))$ and

$$\Phi := \frac{x}{x(0)s\hat{C}} G^{-1}(s) G(0) - I. \quad (25)$$

If $2(f + g) < \|\Phi/s\|^{-1}$, then a PID-controller that stabilizes G is given by $(1/2(f + g))\hat{C}x(0)G(0)^{-1}$, i.e.,

$$C_{\rm pid} = \frac{1}{2(f + g)} \left(k_p + \frac{1}{s} + \frac{k_d s}{\tau s + 1} \right) x(0) G(0)^{-1}. \quad (26)$$

2) Let $Y(\infty)^{-1} = x^{-1}yG(s)|_{s=\infty}$. Choose any $\chi(s) := s^2 + k_2 s + k_3$, $k_2, k_3 \in \mathbb{R}_+$. Define

$$\Psi := \chi^{-1} x G^{-1}(s) Y(\infty)^{-1} - I. \quad (27)$$

If $g^2/2(f + g) > \|s\Psi\|$, then for any $\alpha \in \mathbb{R}_+$ satisfying

$$0 < \alpha < \|s\Psi\|^{-1} - \frac{2(f + g)}{g^2} \quad (28)$$

let $\rho := \alpha g^2 + 2(f + g)$, and let $\tau = \alpha g/\rho$, $K_p = (k_2 - k_3\tau/\rho)Y(\infty)$, $K_i = (k_3/\rho)Y(\infty)$, and $K_d = (1 - k_2\tau + k_3\tau^2/\rho)Y(\infty)$; a PID-controller that stabilizes G is given by

$$C_{\rm pid} = \frac{\chi}{sw} Y(\infty) = \frac{s^2 + k_2 s + k_3}{\rho s(\tau s + 1)} Y(\infty). \quad (29)$$

△

B. Unstable Plants With Restrictions on the $\mathcal{U}$-Poles

The restrictions on the unstable poles are completely dual to the restrictions on the $\mathcal{U}$-zeros in Section III-A. We show that plants that have (one or two) $\mathcal{U}$-poles admit PID-controllers under certain sufficient conditions.

Let $G \in \mathbf{R}_{\mathbf{p}}^{n_y \times n_y}$, $\mathrm{rank} G = n_y$, and let G have no transmission-zeros at $s = 0$. Let G have any number of poles in the stable region. Other than $\ell \in \{1, 2\}$ (one or two) $\mathcal{U}$-poles at $p_1 \in \mathcal{U}$ and $p_2 \in \mathcal{U}$, let G have no poles in the unstable region $\mathcal{U}$. If $\ell = 1$, $p_1 \in \mathbb{R}$, $p_1 \geq 0$. If $\ell = 2$, $p_1, p_2 \in \mathcal{U}$ may be real or complex. The poles at p_1 or p_2 may appear in some or all entries of G. With $a_j \in \mathbb{R}_+$, $j \in \{1, \ell\}$, let

$$y := \prod_{j=1}^{\ell} y_j = \prod_{j=1}^{\ell} (a_j s + 1) \quad n := \prod_{j=1}^{\ell} n_j = \prod_{j=1}^{\ell} (s - p_j). \tag{30}$$

Let G have an LCF $G = Y^{-1} X = ((n/y)I)^{-1}((n/y)G)$, where $\mathrm{rank} X(p_j) = \mathrm{rank} nG(s)|_{s=p_j} = n_y$, $j \in \{1, \ell\}$. Furthermore, since G has no transmission zeros at $s = 0$, $\mathrm{rank} X(0) = \mathrm{rank} nG(s)|_{s=0} = n_y$. We consider the following two cases of real and complex-conjugate pairs of poles.

Case 1) The unstable poles are real, i.e., $p_j \in \mathbb{R}$, $p_j \geq 0$, $j \in \{1, \ell\}$. Part 1) of Proposition 6 shows that under certain assumptions, plants in this class admit PD- and PID-controllers; if $\ell = 1$, they also admit P- and PI-controllers. If at least one $p_j = 0$, then G does not admit D-controllers since the plant pole at $s = 0$ would then cancel the zero in C_d. If $\ell = 1$, some plants (e.g., $G = (1/s(s+e))e \geq 0$) do not admit I-controllers. For $p > 0$, some plants (e.g., $G = (s-z)/(s-p), z, p > 0$) do not admit D- and I-controllers. If $\ell = 2$, some plants (e.g., $G = 1/(s - p_1)(s - p_2)$, $p_1 \geq 0, p_2 \geq 0$) do not admit P- and I-controllers.

Case 2) The two poles are a complex-conjugate pair, i.e., $p_1 = \bar{p}_2$, $n = s^2 - (p_1 + p_2)s + p_1 p_2 = s^2 - 2fs + g^2$, $f \geq 0$, $g > 0$, and $f < g$. In this case, $X(0) = g^2 G(0)$. Part 2) of Proposition 6 shows that under certain assumptions, plants in this class admit D-, PD-, ID-, and PID-controllers. Some plants (e.g., $G = 1/(s^2 + g^2)$, $g \geq 0$) do not admit P- or I-controllers.

Proposition 6: Let $\ell \in \{1, 2\}$. With n, y as in (30), let $X = (n/y)G \in \mathcal{M}(\mathbf{S})$, where $p_j \in \mathcal{U}$. Let $\mathrm{rank} X(p_j) = \mathrm{rank} nG(s)|_{s=p_j} = n_y$, $j = \{1, \ell\}$. Let $X(0) = nG(s)|_{s=0}$ be nonsingular, where $G^{-1}(0) = X(0)^{-1}(-1)^{\ell} \prod_{j=1}^{\ell} p_j$.

Case 1) Let $p_j \in \mathbb{R}$, $p_j \geq 0$, $j \in \{1, \ell\}$. If $\ell = 1$, choose any $F_1 \in \mathbb{R}^{n_y \times n_y}$, $\tau_1 > 0$. Define $\hat{C}_1 := I + (F_1 s/(\tau_1 s + 1))$ and define

$$\Gamma_1 := nG(s)\hat{C}_1 X(0)^{-1} - I. \tag{31}$$

If $0 \leq p_1 < \|\Gamma_1/s\|^{-1}$, then for any $\alpha \in \mathbb{R}_+$ satisfying (32), the PD-controller C_1 in (33) stabilizes G

$$0 < \alpha < \|\Gamma_1/s\|^{-1} - p_1, \tag{32}$$

$$C_1 = K_{p1} + \frac{K_{d1} s}{\tau_1 s + 1} = (p_1 + \alpha)\left(I + \frac{F_1 s}{\tau_1 s + 1}\right) X(0)^{-1}. \tag{33}$$

If $\ell = 2$, choose any $\tau_2 > 0$. Define

$$\Gamma_2 := (\tau_2 s + 1)^{-1} nG(s) X(0)^{-1} - I. \tag{34}$$

If $2(p_1 + p_2) < \|\Gamma_2/s\|^{-1}$, then for any $\alpha, \beta \in \mathbb{R} \cup \{0\}$ satisfying (35), the PD-controller C_2 in (36) stabilizes G, where $\eta := 2(p_1 + p_2) + \alpha + \beta$, $\rho := \alpha\beta + \alpha p_2 + \beta p_1$, $K_{p2} = \rho X(0)^{-1}$, and $K_{d2} = (\eta - \rho\tau_2) X(0)^{-1}$

$$0 \leq \alpha + \beta < \|\Gamma_2/s\|^{-1} - 2(p_1 + p_2) \tag{35}$$

$$\begin{aligned} C_2 &= K_{p1} + \frac{K_{d1} s}{\tau_2 s + 1} = \frac{(\eta s + \rho)}{\tau_2 s + 1} X(0)^{-1} \\ &= \frac{(2(p_1 + p_2) + \alpha + \beta)s + \alpha\beta + \alpha p_2 + \beta p_1}{\tau_2 s + 1} X(0)^{-1}. \end{aligned} \tag{36}$$

Choose any $\gamma \in \mathbb{R}_+$ satisfying (5). Then, a PID-controller that stabilizes G is given by (16), where $C_{pd} = C_\ell$ and $K_{p\ell}$ are as in (33) or (36) for $\ell = 1$ or $\ell = 2$, respectively; (16) becomes $C_{\mathrm{pid}} = C_1 + (\gamma\alpha/s)X(0)^{-1}$ for $\ell = 1$, and $C_{\mathrm{pid}} = C_2 + (\gamma(p_1 p_2 + \rho)/s)X(0)^{-1}$ for $\ell = 2$.

Case 2) Let $p_1 = \bar{p}_2 \in \mathbb{C}$, $n = s^2 - (p_1 + p_2)s + p_1 p_2 = s^2 - 2fs + g^2$, $f \geq 0$, $g > 0$, and $f < g$. Choose any $\tau_2 \in \mathbb{R}_+$. Define Γ_2 as in (34). If $f + 2g < \|\Gamma_2/s\|^{-1}$, then for any $\beta \in \mathbb{R}_+$ satisfying (37), the PD-controller in (38) stabilizes G, where $K_p = (\beta - f)gX(0)^{-1}$, $K_d = [\beta + f + 2g - \tau_2(\beta - f)g]X(0)^{-1}$

$$0 \leq \beta < \|\Gamma_2/s\|^{-1} - (f + 2g) \tag{37}$$

$$\begin{aligned} C_{pd} &= K_p + \frac{K_d s}{\tau_2 s + 1} \\ &= \left[\frac{(\beta + f + 2g)s + (\beta - f)g}{\tau_2 s + 1}\right] \frac{G(0)^{-1}}{g^2}. \end{aligned} \tag{38}$$

Choose any $\gamma \in \mathbb{R}$ satisfying (6). With $H_{pd}(0)^{-1} = g(\beta + g - f)X(0)^{-1}$, a PID-controller that stabilizes G is given by

$$C_{\mathrm{pid}} = C_{pd} + \frac{\gamma(\beta + g - f)}{s} \frac{G(0)^{-1}}{g}. \tag{39}$$

△

Remarks 2:

1) In part 2) of Proposition 6, if $2(f + g) < \|\Gamma_1/s\|^{-1}$, then choosing $\beta = f$ in (37), C_{pd} in (38) becomes a D-controller $C_d = (K_d s/(\tau_2 s + 1)) = 2(f + g)G(0)^{-1} s/g^2(\tau_2 s + 1)$ and (38) becomes an ID-controller $C_{id} = C_d + (\gamma G(0)^{-1}/s)$.
2) By Proposition 6, any plant with (up to) two poles at $p_1 = p_2 = 0$ and any number of poles in the stable region, with no restrictions on the location of the zeros, can be stabilized using PID-controllers since the norm bounds $p_1 < \|\Gamma_1/s\|^{-1}$ or $2(p_1 + p_2) < \|\Gamma_2/s\|^{-1}$ are obviously satisfied.
3) The norm bounds of Proposition 6 can be interpreted as follows: When G has only one $\mathcal{U}$-pole p_1 ($\ell = 1$), the bound $0 \leq p_1 < \|\Gamma_1/s\|^{-1}$ is satisfied only if p_1 is closer to the origin than the smallest positive real blocking zero $z_{\min}$ since $G(z_{\min}) = 0$ implies that $\|\Gamma_1/s\| \geq 1/z_{\min}$.
4) The time constant τ in the derivative term $K_d s/(\tau s + 1)$ is completely free in most of the propositions (Propositions 1 and 2; one unstable blocking-zero case of Propositions 3 and 4; Proposition 5; both real and complex unstable pole cases of Proposition 6).△

IV. Conclusion

In this note, we showed the existence of stabilizing PID-controllers for several LTI MIMO plant classes. We proposed systematic PID-controller synthesis procedures that guarantee robust closed-loop stability. We achieved stabilizing PID-controller designs with freedom in the design parameters that can be used towards satisfaction of performance criteria. Some of these results were recently extended to delay differential systems in [5]. Other future goals of this study include identifying other classes of PID stabilizable plants and incorporation of performance issues into design.

APPENDIX
PROOFS

Proof of Lemma 1:

1) Writing $C_{\rm pid} = N_c D_c^{-1} = [(K_p + (K_d s/(\tau s+1)))(s/(s+e)) + (K_i/(s+e))][sI/(s+e)]^{-1} = [(s/(s+e))C_{\rm pid}][sI/(s+e)]^{-1}$ (for any $e \in \mathbb{R}_+$) and $G = Y^{-1}X$, $M_c = YD_c + XN_c$ unimodular implies ${\rm rank} M_c(0) = n_y = {\rm rank} X(0)K_i$; hence, ${\rm rank} X(0) = n_y$ (equivalently, G has no transmission zeros at $s = 0$), and ${\rm rank} K_i = n_y$.
2) For all $z_i > 0$, $\det D_c(z_i) = \det(z_i/(z_i+e))I > 0$. Now M_c unimodular implies $\det M_c(z_i) = \det Y(z_i) \det D_c(z_i)$ has the same sign for all $z_i \in \mathcal{U}$ such that $X(z_i) = 0$; equivalently, $\det Y(z_i)$ has the same sign at all blocking zeros of G; hence, G is strongly stabilizable [14].

Proof of Proposition 1: Write $C_{\rm pid} = N_c D_c^{-1}$, with $D_c = (1 - (\gamma/(s+\gamma))\Delta_i)I$ and $N_c = C_{\rm pid} D_c$ ($D_c = I$ if the integral term is absent). Then, $C_{\rm pid}$ in (3) stabilizes H since $M_{\rm pid} := D_c + HN_c = D_c + HC_{\rm pid}D_c = I + \gamma[H(\Delta_p \hat{K}_p + \Delta_d(\hat{K}_d s/(\tau s+1)))D_c + (s/(s+\gamma))\Delta_i((H(s)H(0)^I - I)/s)]$ is unimodular.

Proof of Lemma 2:

1) It follows from [14, Th. 5.3.10].
2) Suppose $C_g = C_p$ or $C_g = C_d$ or $C_g = C_i$ is a P-, or D-, or I-controller stabilizing G; equivalently, $H = G(I + C_g G)^{-1} \in \mathcal{M}(\mathbf{S})$. The rank of H is equal to ${\rm rank} G = n_y$. When $C_g = C_p$ or $C_g = C_d$, if G has no transmission zeros at $s = 0$, i.e., ${\rm rank} X(0) = n_y$, then ${\rm rank} H(0) = {\rm rank}(Y + XC_g)^{-1}(0)X(0) = {\rm rank} X(0) = n_y$. By Proposition 1, there exists a P-, D-, I-, PI-, ID-, and PID-controller C_h for $H \in \mathcal{M}(\mathbf{S})$. When $C_g = C_i$, by Proposition 1, there exists a P-, D-, or PD-controller C_h for H. By 1), $C = C_g + C_h$ stabilizes G and is a PI-, PD-, ID-, or PID-controller.
3) By 1), if C_{pd} stabilizes G, then $C_{pd} + C_{ih}$ also stabilizes G. By Proposition 1, choosing $\hat{K}_p = \hat{K}_d = 0$, an I-controller that stabilizes H_{pd} is given by (3), where $\gamma > 0$ satisfies (2), equivalently, (5).

Proof of Proposition 2: By assumption, K_p, K_d, and τ are such that $W_{pd}^{-1}(\infty)$ exists. By (8), $C_{\rm pid}$ stabilizes G since $M_{\rm pid} := (s/(s+\theta))G^{-1} + (s/(s+\theta))C_{\rm pid} = (s/(s+\theta))W_{pd} + (\theta/(s+\theta))W_{pd}(\infty) = [I + (1/(s+\theta))s(W_{pd}(s)W_{pd}^{-1}(\infty) - I)]W_{pd}(\infty)$ is unimodular.

Proof of Proposition 3: If $\ell = 1$, define $y_\alpha := (s+\alpha)$ and $M_1 := XC_1 + Y = [(x_1/y_1)I + YC_1^{-1}]C_1 = [(x_1/y_\alpha)I + ((z_1 + \alpha)/y_\alpha)(y_1 YY(0)^{-1}/\hat{C}_1)](y_\alpha/y_1)C_1 = [I + ((z_1+\alpha)/y_\alpha)((y_1 YY(0)^{-1}/\hat{C}_1) - I)](y_\alpha/y_1)C_1 = [I + ((z_1+\alpha)s/y_\alpha)(\Phi_1/s)](y_\alpha/y_1)C_1$. If $\ell = 2$, let $y_\alpha := (s+\alpha+z_1)$, $y_\beta := (s+\beta+z_2)$, and $y_\alpha y_\beta - x = \eta s + \rho$. Define $M_2 := XC_2 + Y = [(x/y_\alpha y_\beta)I + (y\rho(\tau_2 s+1)/y_\alpha y_\beta(k_2 s+1))YY(0)^{-1}](y_\alpha y_\beta/y)C_2 = [I + ((\eta s+\rho)/y_\alpha y_\beta)((yYY(0)^{-1}/(k_2 s+1)) - I)](y_\alpha y_\beta/y)C_2 = [I + ((\eta s+\rho)s/y_\alpha y_\beta)(\Phi_2/s)](y_\alpha y_\beta/y)$. Since C_ℓ is unimodular and $\Phi_\ell(0) = 0$ implies $\Phi_\ell/s \in \mathcal{M}(\mathbf{S})$, (11) and (14) imply M_ℓ is unimodular for $\ell = 1, 2$; hence, C_ℓ in (12) or (15) stabilizes G. Therefore, $H_{pd} := M_\ell^{-1}X = G(I + C_\ell G)^{-1} \in \mathcal{M}(\mathbf{S})$, where $H_{pd}(0)^{-1} = G^{-1}(0) + K_{p\ell}$, $K_{p1} = -z_1(z_1+\alpha)^{-1}G^{-1}(0)$, and $K_{p2} = z_1 z_2 \rho^{-1} G^{-1}(0)$. The conclusion follows from part 3) of Lemma 2.

Proof of Proposition 4: If $\ell = 1$, define $y_\alpha := (s+\alpha)$ and $M_1 := XC_1 + Y = [\alpha(\tilde{x}_1/\tilde{y}_1)I + Y\alpha C_1^{-1}]\alpha^{-1}C_1 = [\alpha(\tilde{x}_1/y_\alpha)I + ((1+\alpha/z_1)/y_\alpha)(\tilde{y}_1/\hat{C}_1)YY(\infty)^{-1}](y_\alpha/\tilde{y}_1)\alpha^{-1}C_1 = [I + ((1+\alpha/z_1)/y_\alpha)((\tilde{y}_1/\hat{C}_1)YY(\infty)^{-1} - sI)](y_\alpha/\tilde{y}_1)\alpha^{-1}C_1 = [I + ((1+\alpha/z_1)/y_\alpha)\Psi_1](y_\alpha/\tilde{y}_1)\alpha^{-1}C_1$. If $\ell = 2$, let $y_\alpha := (1 + (\alpha + (1/z_1))s)$, $y_\beta := (1 + (\beta + (1/z_2))s)$, and $y_\alpha y_\beta - \tilde{x} = (\eta s + \rho)s$. Define $M_2 := XC_2 + Y = [(\tilde{x}/y_\alpha y_\beta)I + (\tilde{y}\rho(\tau_2 s+1)/y_\alpha y_\beta(s+k_2))YY(\infty)^{-1}](y_\alpha y_\beta/\tilde{y})C_2 = [I + ((\eta s+\rho)s/y_\alpha y_\beta)((\tilde{y}YY(\infty)^{-1}/(s+k_2)) - sI)](y_\alpha y_\beta/\tilde{y})C_2 = [I + ((\eta s+\rho)/y_\alpha y_\beta)(\Psi_2/s)](y_\alpha y_\beta/\tilde{y})C_2$. Since C_ℓ is unimodular and $\Psi_\ell(\infty) = \prod_{j=1}^{\ell} a_j I$ implies $\Psi_\ell \in \mathcal{M}(\mathbf{S})$, (19) and (22) imply M_ℓ is unimodular for $\ell = 1, 2$; hence, C_ℓ in (20) or (23) stabilizes G. Therefore, $H_{pd} := M_\ell^{-1}X = G(I + C_\ell G)^{-1} \in \mathcal{M}(\mathbf{S})$, where $H_{pd}(0)^{-1} = G^{-1}(0) + K_{p\ell}$, $K_{p1} = (1 + \alpha/z_1)^{-1}\alpha k_p Y(\infty)^{-1}$, and $K_{p2} = \rho^{-1}k_2 Y(\infty)^{-1}$. The conclusion follows from part 3) of Lemma 2.

Proof of Proposition 5: With $N_c = (g^2/2(f+g))\hat{C}Y(0)(s/(s+e))$ unimodular, write $C_{\rm pid} = N_c D_c^{-1} = [(s/(s+e))C_{\rm pid}][sI/(s+e)]^{-1}$, where the following hold.

1) Define $M_{\rm pid} := [XN_c + YD_c] = [X + YC_{\rm pid}^{-1}]N_c$. Then, $C_{\rm pid}$ in (26) stabilizes G since $\|(2(f+g)s^2/(s+g)^2)(\Phi/s)\| \le \|2(f+g)(\Phi/s)\| < 1$ implies $M_{\rm pid} = [(x/y)I + (2(f+g)/g^2\hat{C})YY(0)^{-1}]N_c = [I + (2(f+g)s/(s+g)^2)((y/g^2 s\hat{C})YY(0)^{-1} - I)]N_c = [I + (2(f+g)s^2/(s+g)^2)(\Phi/s)]N_c$ is unimodular.
2) Define $v := ((\alpha g+1)s + g)(s+g)$. Then, $v - x = sw = s[\alpha g s + \alpha g^2 + 2(f+g)] = s(\alpha g s + \rho)$ and $\|w/v\| \le \rho/g^2$. Then, $C_{\rm pid}$ in (29) stabilizes G since $\|(w/v)s\Psi\| \le (\rho/g^2)\|s\Psi\| < 1$ implies $M_{\rm pid} := [X + YC_{\rm pid}^{-1}]N_c = [(x/v)I + (y/v)YC_{\rm pid}^{-1}](v/y)N_c = [(1 - (sw/v)) I + (sw/v)(y/\chi)YY(\infty)^{-1}](v/y)N_c = [I + (w/v)s\Psi](v/y)N_c$ is unimodular.

Proof of Proposition 6:

1) If $\ell = 1$, define $y_\alpha := (s+\alpha)$ and $M_1 := XC_1 + Y = ((y_1/y_\alpha)XC_1 + (n_1/y_\alpha)I)(y_\alpha/y_1) = [I + ((p_1+\alpha)/y_\alpha)(y_1 X\hat{C}_1 X(0)^{-1} - I)](y_\alpha/y_1) = [I + ((p_1+\alpha)s/y_\alpha)(\Gamma_1/s)](y_\alpha/y_1)$. If $\ell = 2$, let $y_\alpha := (s+\alpha+p_1)$, $y_\beta := (s+\beta+p_2)$, and $y_\alpha y_\beta - x = \eta s + \rho$. Define $M_2 := XC_2 + Y = ((y/y_\alpha y_\beta)XC_2 + (n/y_\alpha y_\beta)I)(y_\alpha y_\beta/y) = [I + ((\eta s+\rho)/y_\alpha y_\beta)((yXX(0)^{-1}/(\tau_2 s+1)) - I)](y_\alpha y_\beta/y) = [I + ((\eta s+\rho)s/y_\alpha y_\beta)(\Gamma_2/s)](y_\alpha y_\beta/y)$. Since $\Gamma_\ell(0) = 0$ implies $\Gamma_\ell/s \in \mathcal{M}(\mathbf{S})$, (32) and (35) imply M_ℓ is unimodular for $\ell = 1, 2$; hence, C_ℓ in (33) or (36) stabilizes G. Therefore, $H_{pd} := M_\ell^{-1}X = G(I + C_\ell G)^{-1} \in \mathcal{M}(\mathbf{S})$, where $H_{pd}(0)^{-1} = G^{-1}(0) + K_{p\ell}$ and $G^{-1}(0) = nX(0)^{-1} = \prod_{j=1}^{\ell}(-1)^j p_j$. The conclusion follows from part 3) of Lemma 2.
2) Define $v := (s+g)(s+\beta+g-f)$; by assumption, $g - f > 0$. Let $w := v - n = (\beta + f + 2g)s + (\beta - f)g$. Then, $\|sw/v\| \le (\beta + f + 2g)$, where $((p_1+p_2)/2) + 2\sqrt{p_1 p_2} = f + 2g$. If (37) holds, since $\|(sw/v)(\Gamma_2/s)\| \le (\beta + f + 2g)\|\Gamma_2/s\| < 1$ implies $M_{pd} := Y + XC_{pd} = (n/y)[I + GC_{pd}] = (v/y)[I + (w/v)\Gamma_2]$ is unimodular, $C_{pd} = (w/(\tau_2 s+1))X(0)^{-1}$ in (38) stabilizes G. Then, $H_{pd} := M_{pd}^{-1}X = G(I + C_{pd}G)^{-1} \in \mathcal{M}(\mathbf{S})$ and $K_i/s = \gamma H_{pd}(0)^{-1}/s$ stabilizes H_{pd}, where $H_{pd}(0)^{-1} = G^{-1}(0) + K_p$. By part 3) of Lemma 2, $C_{\rm pid} = C_{pd} + K_i/s$ in (39) stabilizes G.

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Optimal $\mathcal{H}_2$ Filtering in Networked Control Systems With Multiple Packet Dropout

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***Abstract*—This note studies the problem of optimal $\mathcal{H}_2$ filtering in networked control systems (NCSs) with multiple packet dropout. A new formulation is employed to model the multiple packet dropout case, where the random dropout rate is transformed into a stochastic parameter in the system's representation. By generalization of the $\mathcal{H}_2$-norm definition, new relations for the stochastic $\mathcal{H}_2$-norm of a linear discrete-time stochastic parameter system represented in the state–space form are derived. The stochastic $\mathcal{H}_2$-norm of the estimation error is used as a criterion for filter design in the NCS framework. A set of linear matrix inequalities (LMIs) is given to solve the corresponding filter design problem. A simulation example supports the theory.**

***Index Terms*—$\mathcal{H}_2$-norm, networked control system (NCS), optimal filtering, packet dropout, stochastic systems.**

I. Introduction

Many modern control methods employing the state feedback strategy use state–space formulation. State feedback is applicable under the implicit assumption that all state variables are measurable. However, in practice, some state variables may not be directly accessible or the corresponding sensing devices may be unavailable or very expensive. In such cases, state filters or state estimators are used to give an estimate of the unavailable states.

Networked control systems (NCSs) have gained attention during last few years (e.g., see [7], [8], [12], [14], [19], and references therein). Compared to using the conventional point-to-point system connection, using an NCS has advantages like easy installation and reduced setup, wiring, and maintenance costs. In an NCS, data travel through the communication channels from the sensors to the controller and from the controller to the actuators. Data packet dropout, a kind of uncertainty that may happen due to node failures or network congestion, is a common problem in networked systems. The dropouts happen randomly. Because of random dropout, classical estimation and control methods cannot be used directly. Dropouts can degrade system performance and increase the difficulty of filtering and estimation.

Even though most research conducted on NCSs considers random delay, the closely related random packet dropout has not been well studied and only in last few years has been the focus of some research studies. In fact, systems with packet dropout, uncertain observation, or missing measurements have been studied for a long time (e.g., see [5], [6], [11], [13], [17], [18], and references therein). All of these studies consider the case with uncertainty only in one link and it is not straightforward to extend it to an NCS framework where uncertainty is present both from the sensors to the controller and from the controller to the actuators. Also, in most studies (e.g., see [11], [17], and [18]), the main derivations are given for the case when previous dropout information is given. To the best of our knowledge, no work has been conducted regarding filtering in NCS with multiple packet dropouts, but the problem of stabilization and control has been studied recently in packet dropout systems (e.g., see [9], [10], [20], and references therein). In some of these studies, only sensor data dropouts are studied [9], [20]. While [9] considers adaptive genetic algorithms and simulated annealing algorithms, guaranteed cost control, and the state feedback controller, other references consider switched systems and Markov chains to solve the problem. The main problem in working with Markov chains is the unknown Markov states. Identifying the number of states of the Markov chain and their transient probability by using hidden Markov models are other issues in the research on NCSs.

The problem of optimal $\mathcal{H}_2$ filtering has been tackled in deterministic cases (see, e.g., [4] and [15]). The problem of stochastic packet dropout has also been studied in sensor delay system [16], but, to the best of our knowledge, optimal $\mathcal{H}_2$ filtering has not been studied in NCSs with multiple packet dropout.

In this note, we consider the problem of optimal $\mathcal{H}_2$ filtering in an NCS with multiple packet dropout. A new formulation is proposed to formulate the NCS with multiple random packet dropout. By generalization of the $\mathcal{H}_2$-norm definition, new relations for the stochastic $\mathcal{H}_2$-norm of a linear discrete-time stochastic parameter system represented in the state–space form are derived. The new derivations enable us to consider estimation and filtering of the NCS as a generalization of the classical case. To solve the filtering problem, the filter gains are designed so that the $\mathcal{H}_2$-norm of the estimation error is minimized. As dropout rates are stochastic, the problem formulation leads to a system with stochastic parameters. Thus, the stochastic $\mathcal{H}_2$-norm ($\mathcal{H}_{2s}$-norm) of the estimation error is considered as a measure to minimize. With both deterministic and stochastic inputs present in the NCS framework, a weighted $\mathcal{H}_2$-norm is defined and used. The filtering problem is transformed into a convex optimization problem through a set of linear matrix inequalities (LMIs) that can be solved by using existing numerical

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