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Reliable Decentralized PID Stabilization of MIMO Systems

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***Abstract*— Systematic methods are proposed for reliable decentralized PID-controller synthesis. These controllers achieve closed-loop stability and asymptotic tracking of step-input references at each output channel when all controllers are operational, and they maintain stability when one of the controllers fails completely.**

I. INTRODUCTION

In many practical control applications, Proportional + Integral + Derivative (PID) controllers are widely used and preferred due to their simplicity. For multi-input multi-output (MIMO) systems, the decentralized controller structure provides simple implementation. In this paper we propose systematic methods for the synthesis of reliable decentralized PID-controllers for two-channel linear, time-invariant MIMO systems. These controllers achieve closed-loop stability when both channels are operational, and maintained even if one of the controllers fails completely. Due to the integral-action in the PID-controllers, asymptotic tracking of step-input references is achieved at each output channel.

Reliable control has been considered under full-feedback and decentralized controller structures in e.g., [3], [6], [7], [9]. We only consider PID-controllers, which can stailize only certain classes of plants. Although used widely, most PID design approaches lack systematic procedures and rigorous closed-loop stability proofs. Rigorous synthesis methods are studied in recent literature, e.g., [8]. Decentralized PID designs were considered for two-by-two plants in [1].

We present systematic synthesis procedures for several plant classes that can be stabilized using PID-controllers. The proposed designs achieve closed-loop stability and asymptotic tracking. Performance issues are not considered but the freedom offered in the design parameters may be used to satisfy other criteria. The PID designs are (partially) reliable against the failure of one controller. If the plant is stable, (fully) reliable controller designs are also presented against the failure of either one of the two controllers. A controller that fails is set equal to zero; i.e., the failure is recognized and the failed controller is taken out of service. Although the failed channel does not achieve asymptotic tracking with zero steady-state error, closed-loop stability is still maintained.

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We illustrate the design methods with two examples. A partially reliable PID-controller is designed for the linearized model of a sugar mill process, which has poles at the origin and a non-minimum phase zero, [4]. A fully reliable PID-controller is designed to achieve asymptotic tracking of the desired step-input references at two outputs for a simplified model representing a particular patient under anesthesia, [2].

Notation: Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$ denote complex, real, positive real numbers; $\mathcal{U} = \{s \in \mathbb{C} \,|\, \mathcal{R}e(s) \geq 0\} \cup \{\infty\}$ is the extended closed right-half plane; I_n is the $n \times n$ identity matrix; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S}$ is the stable subset with no $\mathcal{U}$-poles; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in $\mathbf{S}$. The H_∞-norm of $M(s) \in \mathcal{M}(\mathbf{S})$ is $\|M\| := \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$; $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$. We drop (s) in transfer functions such as $G(s)$ whenever this causes no confusion. We use coprime factorizations over $\mathbf{S}$; i.e., for $G_2 \in \mathbf{R_p}^{n_2 \times n_2}$, $G_2 = XY^{-1} = \tilde{Y}^{-1}\tilde{X}$ gives a right-coprime-factorization (RCF) and a left-coprime-factorization (LCF); $X, \tilde{X}, Y, \tilde{Y} \in \mathbf{S}^{n_2 \times n_2}$, $\det Y(\infty) \neq 0$, $\det \tilde{Y}(\infty) \neq 0$.

II. PRELIMINARIES

We consider the linear time-invariant (LTI) decentralized feedback system $S(G, C_D)$ with two multi-input multi-output (MIMO) channels as shown in Fig. 1. We assume that the feedback system is well-posed, the plant and controller have no unstable hidden-modes. The plant $G \in \mathbf{R_p}^{n \times n}$ and the decentralized controller $C_D \in \mathbf{R_p}^{n \times n}$ are partitioned as:

$$G = \begin{bmatrix} G_1 & G_{12} \\ G_{21} & G_2 \end{bmatrix} , \quad C_D = \text{diag}\,[\,C_1\,,\; C_2\,], \quad (1)$$

where each channel has as many inputs as outputs, i.e., $G_j \in \mathbf{R_p}^{n_j \times n_j}$, and rank$G = n$. The plants are either stable, or if they are unstable, then one of the input channels or one of the output channels has stable transfer-functions. Since the inputs and outputs can be re-ordered, we assume that these stable transfer-functions are associated with the first channel. Furthermore, the unstable poles of the second channel are reflected in G_2. Let $G_2 = X_2 Y^{-1} = \tilde{Y}^{-1} \tilde{X}_2$ be an RCF and LCF of G_2. Therefore we assume that G satisfies the following *plant assumptions*: i) $G_1 \in \mathbf{S}^{n_1 \times n_2}$; ii) $G_2 \in \mathbf{R_p}^{n_2 \times n_2}$ may be unstable; iii) either $G_{21} \in \mathbf{S}^{n_2 \times n_1}$; and $G_{12} Y \in \mathbf{S}^{n_1 \times n_2}$ *or* $G_{12} \in \mathbf{S}^{n_1 \times n_2}$ and $\tilde{Y} G_{21} \in \mathbf{S}^{n_2 \times n_1}$.

Let $r := [r_1, r_2]^T$, $v := [v_1, v_2]^T$, $y := [y_1, y_2]^T$, $w := [w_1, w_2]^T$ denote the input and output vectors.

Definition 2.1: [5] a) The feedback system $S(G, C_D)$ is stable iff the transfer-function from (r, v) to (y, w) is stable. *b)* The controller C_D stabilizes G iff C_D is proper and $S(G, C_D)$ is stable. c) The controller C_D that stabilizes G is partially reliable iff the system $S(G, 0, C_2)$ is stable, i.e., the transfer-function from (r_2, v) to (y, w_2) is stable. d) The controller C_D that stabilizes G is fully reliable iff the system $S(G, 0, C_2)$ is stable, i.e., the transfer-function from (r_2, v) to (y, w_2) is stable, and the system $S(G, C_1, 0)$ is stable, i.e., the transfer-function from (r_1, v) to (y, w_1) is stable. ▪

Fully reliable decentralized controllers exist if and only if G is stable. Partially reliable decentralized controllers exist for the two unstable plant cases, where the transfer-functions of the first channel inputs or outputs are stable.

Theorem 2.1: [6] **a)** Let $G \in \mathcal{M}(\mathbf{R_p})$ as in (1) satisfy the *plant assumptions*. Let rank$G = n$, rank$G_2 = n_2$. The decentralized $C_D = \text{diag}\,[\,C_1,\, C_2\,]$ stabilizes G and is partially reliable if and only if *i)* C_2 stabilizes G_2 *and ii)* C_1 stabilizes $W = G_1 - G_{12} C_2 (I + G_2 C_2)^{-1} G_{21} \in \mathcal{M}(\mathbf{S})$. **b)** Let $G \in \mathcal{M}(\mathbf{S})$, rank$G_j = n_j$. The decentralized $C_D = \text{diag}\,[\,C_1,\, C_2\,]$ stabilizes G and is fully reliable if and only if *i)* C_2 stabilizes G_2 *and ii)* C_1 simultaneously stabilizes G_1 and $W = G_1 - G_{12} C_2 (I + G_2 C_2)^{-1} G_{21} \in \mathcal{M}(\mathbf{S})$. ▪

By Theorem 2.1, partially reliable decentralized controllers can be designed for the class of plants under consideration. The controllers $C_j \in \mathbf{R_p}^{n_j \times n_j}$, $j = 1, 2$, will be designed in the following proper PID-controller form, [4]:

$$C_j = K_{pj} + \frac{K_{ij}}{s} + \frac{K_{dj}\, s}{\tau_j\, s + 1}\,, \qquad (2)$$

where, for $j = 1, 2$, K_{pj}, K_{ij}, $K_{dj} \in \mathbb{R}^{n_j \times n_j}$ are called the proportional, the integral, and the derivative constants, respectively, and $\tau_j \in \mathbb{R}$, $\tau_j > 0$. The integral-action in C_j is present when $K_{ij} \neq 0$. Subsets of PID-controllers are obtained by setting one or two of the three constants equal to zero; (2) becomes a PI-controller when $K_{dj} = 0$ and an I-controller when $K_{pj} = K_{dj} = 0$.

III. PID STABILIZATION CONDITIONS AND DESIGN

PID-controllers cannot stabilize certain unstable plants. We now give conditions for reliable PID stabilizability.

Lemma 3.1: (Existence conditions for partially reliable decentralized PID-controllers): **a)** Let $G \in \mathcal{M}(\mathbf{R_p})$ as in (1) satisfy the *plant assumptions*. Let rank$G = n$, rank$G_2 = n_2$. If there exist partially reliable decentralized PID-controllers $C_D = \text{diag}[\,C_1\,,\ C_2\,]$ with nonzero integral constants $K_{ij} \in \mathbb{R}^{n_j \times n_j}$, then G and G_2 have no transmission-zeros at $s = 0$ and G_2 is strongly stabilizable. **b)** Let $G \in \mathcal{M}(\mathbf{S})$, rank$G = n$, rank$G_2 = n_2$. There exist partially reliable decentralized PID-controllers $C_D = \text{diag}[\,C_1\,,\ C_2\,]$ with nonzero integral constants $K_{ij} \in \mathbb{R}^{n_j \times n_j}$ if and only if G and G_2 have no transmission-zeros at $s = 0$. ▪

Lemma 3.2: (Existence conditions for fully reliable decentralized PID-controllers): Let $G \in \mathcal{M}(\mathbf{S})$, rank$G_j = n_j$. **a)** There exist fully reliable decentralized PID-controllers $C_D = \text{diag}[\,C_1, C_2\,]$ with nonzero integral constants $K_{ij} \in \mathbb{R}^{n_j \times n_j}$ only if G, G_1, G_2 have no transmission-zeros at $s = 0$. **b)** There exist fully reliable decentralized PID-controllers $C_D = \text{diag}[\,C_1, C_2\,]$ with nonzero integral constants $K_{ij} \in \mathbb{R}^{n_j \times n_j}$ if G, G_1, G_2 have no transmission-zeros at $s = 0$ and $W(0) G_1(0)^{-1} = [\,I - G_{12}(0) G_2(0)^{-1} G_{21}(0) G_1(0)^{-1}\,]$ is symmetric, positive-definite. ▪

Remark: If at least one of C_1, C_2, G_{12}, G_{21} has blocking-zeros in $\mathcal{U}$ (including infinity), then fully reliable decentralized PID-controllers exist only if G and G_j, $j = 1, 2$, have no transmission-zeros at $s = 0$ and $\det W(0) G_1(0)^{-1} = \det\,[\,I - G_{12}(0) G_2(0)^{-1} G_{21}(0) G_1(0)^{-1}\,] > 0$.

By Lemma 3.1, a necessary condition for existence of PID-controllers is that G_2 is strongly stabilizable. Obviously then, stable plants admit PID-controllers. A PID-controller synthesis method applicable to MIMO plants based on the small-gain approach is given in Proposition 3.1.

Proposition 3.1: Let $H \in \mathbf{S}^{n_j \times n_j}$. Let rank$H(0) = n_j$. Choose any $\hat{K}_p, \hat{K}_d \in \mathbb{R}^{n_j \times n_j}$, $\tau > 0$. Then, for any $\beta \in \mathbb{R}_+$ satisfying (4), a PID-controller that stabilizes H is given by (3); for $\hat{K}_d = 0$, (3) is a PI-controller; for $\hat{K}_d = \hat{K}_p = 0$, (3) is an I-controller:

$$C_{pid} = \beta\, \hat{K}_p + \frac{\beta\, H(0)^{-1}}{s} + \frac{\beta\, \hat{K}_d\, s}{\tau s + 1}\,, \qquad (3)$$

$$\beta < \| H(s)(\hat{K}_p + \frac{\hat{K}_d s}{\tau s+1}) + \frac{H(s)H^I(0) - I}{s} \|^{-1}. \blacksquare \quad (4)$$

We propose PID-controllers for three classes of unstable strongly stabilizable plants with restrictions on the blocking-zeros in $\mathcal{U}$ and complete freedom in the $\mathcal{U}$-poles.

1) Unstable plants with no zeros in the unstable region: Consider unstable G_2 with no zeros in the unstable region $\mathcal{U}$, including infinity. For these plants, $G_2^{-1} \in \mathbf{S}^{n_2 \times n_2}$. In the SISO case, G_2 has relative-degree equal to zero. A PID-controller synthesis is given in Proposition 3.2.

Proposition 3.2: Let $G_2 \in \mathbf{R_p}^{n_2 \times n_2}$, rank$G_2(s) = n_2$. Let G_2 have no transmission-zeros in $\mathcal{U}$ (including infinity). Choose $K_{p2}, K_{d2} \in \mathbb{R}^{n_2 \times n_2}$, $\tau_2 > 0$ such that det $R(\infty) = $ det $[\, G_2(\infty)^{-1} + K_{p2} + \tau_2^{-1} K_{d2}] \neq 0$, where $R := G_2^{-1}(s) + K_{p2} + \frac{K_{d2}s}{\tau_2 s+1}$. Then, for any $\gamma \in \mathbb{R}_+$ satisfying (6), a PID-controller that stabilizes G_2 is given by (5); if $K_{d2} = 0$, (5) is a PI-controller; if $K_{p2} = K_{d2} = 0$, (5) is an I-controller:

$$C_2 = K_{p2} + \frac{\gamma[G_2(\infty)^{-1} + K_{p2} + \tau_2^{-1} K_{d2}]}{s} + \frac{K_{d2}s}{\tau_2 s + 1}, \quad (5)$$

$$\gamma > \| s\, [\, R\, (G_2(\infty)^{-1} + K_{p2} + \tau_2^{-1} K_{d2}\,)^{-1} - I\,] \|. \blacksquare \quad (6)$$

2) Unstable plants with one positive real-axis zero including infinity: Consider unstable G_2 with one real-axis blocking-zero in $\mathcal{U}$; G_2 may have other transmission-zeros in the stable region. In the SISO case, G_2 has relative-degree equal to 0 or 1. We consider two cases where the real-axis $\mathcal{U}$-zero is either 1) "large", including infinity, or 2) "small". *Case 1)* Let $G_2 = [\frac{(1-s/z)}{s+a} G_2^{-1}]^{-1} \frac{(1-s/z)}{s+a} I =: \tilde{Y}^{-1}\tilde{X}$, $z \in \mathbb{R}$, $0 < z \le \infty$, $a \in \mathbb{R}_+$. Proposition 3.3 considers plants with one "large" real-axis zero satisfying $z > \|\Psi\|$ defined in (7). A zero at infinity satisfies this bound. In the SISO case, if G_2 is strictly-proper, this class corresponds to minimum-phase unstable plants with relative-degree one.

Proposition 3.3: Let $G_2 \in \mathbf{R_p}^{n_2 \times n_2}$, rank$G_2(s) = n_2$. Let G_2 have no transmission-zeros at $s = 0$. Let $z \in \mathbb{R}$, $0 < z \le \infty$. Let $\frac{(1-s/z)}{s+a} G_2^{-1} \in \mathcal{M}(\mathbf{S})$, for $a \in \mathbb{R}$, $a > 0$. Let $\tilde{Y}(\infty)^{-1} = (1-s/z)^{-1} s\, G_2(s)|_{s\to\infty}$. If $z > 0$ is finite, let $K_{d2} = 0$. If the zero is at infinity, choose any $K_{d2} \in \mathbb{R}^{n_2 \times n_2}$, $\tau_2 > 0$. Define

$$\Psi := (1-s/z)\ [\, G_2^{-1}(s) + \frac{K_{d2}\, s}{\tau_2 s+1}\ ]\tilde{Y}(\infty)^{-1} - s\, I\,. \quad (7)$$

If $z > \|\Psi\|$ then, for any $\alpha \in \mathbb{R}_+$ satisfying (9), let C_{pd} be as in (8):

$$C_{pd} = \frac{\alpha}{1+\alpha/z}\, \tilde{Y}(\infty) + \frac{K_{d2}\, s}{\tau_2\, s+1}, \quad (8)$$

$$\alpha\ >\ \|\ \Psi\ \|\ (\ 1 - \|\ \Psi\ \|/z\ )^{-1}\ . \quad (9)$$

Define $H_{pd} := G_2(I + C_{pd}G_2\,)^{-1}$. Then, for any $\gamma \in \mathbb{R}_+$ satisfying (15), a PID-controller that stabilizes G_2 is given by (10); if $K_{d2} = 0$, (10) is a PI-controller:

$$C_{pid} = C_{pd} + \frac{\gamma}{s}\, [G_2^{-1}(0) + \frac{\alpha}{1+\alpha/z}\, \tilde{Y}(\infty)]. \blacksquare \quad (10)$$

Case 2) Let $G_2 = [\frac{(s-z)}{as+1} G_2^{-1}]^{-1} \frac{(s-z)}{as+1} I =: \tilde{Y}^{-1}\tilde{X}$, $z \in \mathbb{R}$, $z > 0$, $a \in \mathbb{R}_+$. Proposition 3.4 gives a PID-controller synthesis for plants with one "small" real-axis zero satisfying $0 < z < \|\Phi\|^{-1}$ defined in (11).

Proposition 3.4: Let $G_2 \in \mathbf{R_p}^{n_2 \times n_2}$, rank$G_2(s) = n_2$. Let G_2 have no transmission-zeros at $s = 0$. Let G_2 have no poles at $s = 0$. Let $z \in \mathbb{R}$, $z > 0$. Let $\frac{(s-z)}{as+1} G_2^{-1} \in \mathcal{M}(\mathbf{S})$, for $a \in \mathbb{R}_+$. Let $\tilde{Y}(0)^{-1} = -z^{-1}\, G_2(0)$. Choose any $K_{d2} \in \mathbb{R}^{n_2 \times n_2}$, $\tau_2 > 0$. Define

$$\Phi := \frac{(s-z)G_2^{-1}(s)\tilde{Y}(0)^{-1} - I}{s} + \frac{(s-z)K_{d2}\tilde{Y}(0)^{-1}}{\tau_2 s+1}. \quad (11)$$

If $0 < z < \|\Phi\|^{-1}$ then, for any $\alpha \in \mathbb{R}_+$ satisfying (13), let C_{pd} be as in (12):

$$C_{pd} = \frac{\alpha}{1+\alpha\, z}\, \tilde{Y}(0) + \frac{K_{d2}\, s}{\tau_2 s+1}\ , \quad (12)$$

$$\alpha > (\ \|\ \Phi\ \|^{-1} - z\ )^{-1}\ . \quad (13)$$

Define $H_{pd} := G_2(I + C_{pd}G_2\,)^{-1}$. Then, for any $\gamma \in \mathbb{R}_+$ satisfying (15), a PID-controller that stabilizes G_2 is given by (14); if $K_{d2} = 0$, (14) is a PI-controller:

$$C_{pid} = C_{pd} + \frac{\gamma}{s}\, [\ G_2^{-1}(0) + \alpha\, (1+\alpha z)^{-1}\tilde{Y}(0)\ ], \quad (14)$$

$$\gamma < \| \frac{H_{pd}H_{pd}(0) - I}{s} \|^{-1} = \|(I + G_2 C_{pd}\,)^{-1} \frac{[\, G_2 G_2(0)^{-1} - I]}{s} - \frac{H_{pd}\, K_{d2}}{\tau_2 s+1} \|^{-1}. \blacksquare \quad (15)$$

3) Unstable plants with two positive real-axis zeros including infinity: Consider unstable G_2 with two real-axis blocking-zeros in $\mathcal{U}$; G_2 may have other transmission-zeros in the stable region. In the SISO case, G_2 has relative-degree equal to 0 or 1. We consider two cases where $z_1, z_2 \ge 0$ are either 1) both "large", including infinity, or both"small".

Case 1) Let

$$G_2 = [\ \prod_{\ell=1}^{2} \frac{(1-s/z_\ell)}{(s+a_\ell)} G_2^{-1}\,]^{-1} \prod_{\ell=1}^{2} \frac{(1-s/z_\ell)}{(s+a_\ell)} I =: \tilde{Y}^{-1}\tilde{X}\,,$$

$z_\ell \in \mathbb{R}$, $0 < z_\ell \le \infty$, $a_\ell \in \mathbb{R}_+$, $\ell = 1, 2$.

Proposition 3.5: Let $G_2 \in \mathbf{R_p}^{n_2 \times n_2}$, rank$G_2(s) = n_2$. Let $z_\ell \in \mathbb{R}$, $0 < z_\ell \le \infty$, $\ell = 1, 2$. Let $\prod_{\ell=1}^{2} \frac{(1-s/z_\ell)}{(s+a_\ell)} G_2^{-1} \in \mathcal{M}(\mathbf{S})$, for $a_\ell \in \mathbb{R}_+$. Let $\tilde{Y}(\infty)^{-1} = (1-s/z_1)^{-1}(1-s/z_2)^{-1} s^2\, G_2(s)|_{s\to\infty}$. Choose any $\delta > 0$. Define

$$\Psi_1 := \frac{(1-s/z_1)(1-s/z_2)}{(s+\delta)} G_2^{-1}(s)\tilde{Y}(\infty)^{-1} - sI. \quad (16)$$

If $\|\Psi_1\| < z_1 \le \infty$, then choose any $\alpha \in \mathbb{R}_+$ satisfying

$$\alpha\ >\ \|\Psi_1\|\ (\ 1 - \|\Psi_1\|/z_1\ )^{-1}\ . \quad (17)$$

Define

$$\Psi_2 := \frac{(s+\delta)}{(1+\alpha/z_1)} [I + \frac{\alpha(s+\delta)G_2(s)\tilde{Y}(\infty)}{(1+\alpha/z_1)(1-s/z_2)}]^{-1} - sI. \quad (18)$$

If $(1+\alpha/z_1)^{-1}\|\Psi_2\| < z_2 \le \infty$, then choose any $\beta \in \mathbb{R}_+$ satisfying

$$\|\Psi_2\| \,(1 - \|\Psi_2\|/z_2)^{-1} < \beta < \alpha^{-1} z_1 z_2 \,. \quad (19)$$

With $\eta := (1 - \frac{\alpha\beta}{z_1 z_2})$, $\tau_2 = \eta[\beta(1+\frac{\alpha}{z_1}) + \delta(1+\frac{\beta}{z_2})]^{-1}$, $K_{p2} = \eta^{-1}\tau_2\delta\alpha\beta\tilde{Y}(\infty)$, $K_{d2} = (\frac{1}{\delta} - \tau_2)K_{p2}$, let

$$C_{pd} = K_{p2} + \frac{K_{d2}\, s}{\tau_2 s+1} = \frac{\eta^{-1}\tau_2\alpha\beta(s+\delta)}{\tau_2 s+1}\tilde{Y}(\infty). \quad (20)$$

Then, for any $\gamma \in \mathbb{R}_+$ satisfying (15), a PID-controller that stabilizes G_2 is given by (21):

$$C_{pid} = C_{pd} + \frac{\gamma}{s}\,[\,G_2^{-1}(0) + \eta^{-1}\tau_2\delta\,\alpha\,\beta\,\tilde{Y}(\infty)\,]. \,\blacksquare \quad (21)$$

Case 2) Let

$$G_2 = [\,\prod_{\ell=1}^{2}\frac{(s-z_\ell)}{(a_\ell s+1)}G_2^{-1}]^{-1}\prod_{\ell=1}^{2}\frac{(s-z_\ell)}{(a_\ell s+1)}I =: \tilde{Y}^{-1}\tilde{X}\,,$$

$z_\ell \in \mathbb{R}$, $z_\ell > 0$, $a_\ell \in \mathbb{R}_+$, $\ell = 1, 2$.

Proposition 3.6: Let $G_2 \in \mathbf{R_p}^{n_2\times n_2}$, rank$G_2(s) = n_2$. Let G_2 have no poles at $s = 0$. Let $z_\ell \in \mathbb{R}$, $z_\ell > 0$. Let $\prod_{\ell=1}^{2}\frac{(s-z_\ell)}{(a_\ell s+1)}G_2^{-1} \in \mathcal{M}(\mathbf{S})$, for $a, b \in \mathbb{R}$, $a, b > 0$. Let $\tilde{Y}(0)^{-1} = z_1^{-1}z_2^{-1}\,G_2(0)$. Choose any $\delta > 0$. Define

$$\Phi_1 := s^{-1}(\frac{(s-z_1)(s-z_2)}{(\delta s+1)}G_2^{-1}(s)\tilde{Y}(0)^{-1} - I). \quad (22)$$

If $0 < z_1 < \|\phi_1\|^{-1}$, then choose any $\alpha \in \mathbb{R}_+$ satisfying

$$\alpha \;>\; (\,\|\Phi_1\|^{-1} - z_1\,)^{-1}\,. \quad (23)$$

Define

$$\Phi_2 := s^{-1}(\frac{(\delta s+1)}{(1+\alpha z_1)}[I + \frac{\alpha(\delta s+1)G_2(s)\tilde{Y}(0)}{(1+\alpha z_1)(s-z_2)}]^{-1} - I). \quad (24)$$

If $0 < z_2 < (1+\alpha z_1)^{-1}\|\Phi_2\|^{-1}$, then choose any $\beta \in \mathbb{R}_+$ satisfying

$$(\,\|\Phi_2\|^{-1} - z_2\,)^{-1} \;<\; \beta \;<\; (\,\alpha\, z_1\, z_2\,)^{-1}\,. \quad (25)$$

With $\eta := (1 - \alpha\,\beta\, z_1\, z_2)$, $\tau_2 = \eta^{-1}\,[\,\beta\,(1+\alpha z_1) + \delta(1+\beta z_2)\,]$, $K_{p2} = \eta^{-1}\,\alpha\,\beta\,\tilde{Y}(0)$, $K_{d2} = (\,\delta - \tau_2\,)\,K_{p2}$, let

$$C_{pd} = K_{p2} + \frac{K_{d2}\, s}{\tau_2 s+1} = \frac{\eta^{-1}\,\alpha\,\beta\,(\,\delta\, s+1\,)}{\tau_2\, s+1}\,\tilde{Y}(0)\,. \quad (26)$$

If $\beta = (\,\alpha\, z_1\, z_2\,)^{-1}$, then (26) is a PI-controller $C_{pi} = \alpha\,\beta\,[\,\beta\,(1+\alpha z_1) + \delta(1+\beta z_2)\,]^{-1}\,(\,\delta\, s+1\,)/s$. Then, for any $\gamma \in \mathbb{R}_+$ satisfying (15), a PID-controller that stabilizes G_2 is given by (27):

$$C_{pid} \;=\; C_{pd} \;+\; \frac{\gamma}{s}\,[\;G_2^{-1}(0) + \eta^{-1}\,\alpha\,\beta\,\tilde{Y}(0)\;]. \,\blacksquare \quad (27)$$

IV. Reliable decentralized design

In this section, for partially reliable controller design, assume rank$G = n$, rank$G_2 = n_2$ and G and G_2 have no transmission-zeros at $s = 0$. Based on Theorem 2.1, first design a PID-controller C_2 that stabilizes G_2 by using the synthesis methods in Section III. Then design a PID-controller C_1 that stabilizes the stable $W = G_1 - G_{12}C_2(I + G_2C_2)^{-1}G_{21}$ following the synthesis in Proposition 3.1.

For stable G, we can design partially reliable decentralized PID-controllers if and only if rank$G(0) = n$, rank$G_2(0) = n_2$. We follow the synthesis method of Proposition 3.1 to design a PID-controller C_2 that stabilizes G_2 and then a PID-controller C_1 that stabilizes the stable transfer-function W. By Lemma 3.2, fully reliable decentralized PID-controller design requires rank$G(0) = n$, rank$G_1(0) = n_1$, rank$G_2(0) = n_2$. We further assume the sufficient condition that $W(0)G_1^{-1}(0) > 0$. A fully reliable decentralized PID-controller synthesis is given in Proposition 4.1:

Proposition 4.1: Let $G \in \mathbf{S}^{n\times n}$. Let rank$G(0) = n$, rank$G_j(0) = n_j$, $j = 1, 2$. Let $W(0)G_1^{-1}(0) := I - G_{12}(0)G_2^{-1}(0)G_{21}(0)G_1^{-1}(0)$ be symmetric, positive-definite. Choose any $\hat{K}_{p2}, \hat{K}_{d2} \in \mathbb{R}^{n_2\times n_2}$, $\tau_2 > 0$. For any $\beta_2 \in \mathbb{R}_+$ satisfying (29), let C_2 be given by (28):

$$C_2 = \beta_2\,\hat{K}_{p2} + \frac{\beta_2\,G_2(0)^{-1}}{s} + \frac{\beta_2\,\hat{K}_{d2}\,s}{\tau_2\,s+1}, \quad (28)$$

$$\beta_2 < \|G_2(s)(\hat{K}_{p2} + \frac{\hat{K}_{d2}\,s}{\tau_2 s+1}) + \frac{G_2(s)G_2^{-1}(0) - I}{s}\,\|^{-1}. \quad (29)$$

Let $W := G_1 - G_{12}C_2(I + G_2C_2)^{-1}G_{21}$. Choose any $\hat{K}_{p1}, \hat{K}_{d1} \in \mathbb{R}^{n_1\times n_1}$, $\tau_1 > 0$. For any $\beta_1 \in \mathbb{R}_+$ satisfying (31), let C_1 be as in (30):

$$C_1 = \beta_1\,\hat{K}_{p1} + \frac{\beta_1\,G_1(0)^{-1}}{s} + \frac{\beta_1\,\hat{K}_{d1}\,s}{\tau_1 s+1}, \quad (30)$$

$$\beta_1 < \min\big\{\|G_1(s)(\hat{K}_{p1} + \frac{\hat{K}_{d1}\,s}{\tau_1 s+1}) + \frac{G_1(s)G_1^{-1}(0) - I}{s}\|^{-1},$$
$$\|W(\hat{K}_{p1} + \frac{\hat{K}_{d1}\,s}{\tau_1 s+1}) + \frac{[W(s) - W(0)]G_1^{-1}(0)}{s}\|^{-1}\big\}. \quad (31)$$

Then $C_D = \mathrm{diag}\,[\,C_1, C_2\,]$ is a fully reliable decentralized PID-controller. For $\hat{K}_{dj} = 0$, (28) and (30) are PI-controllers; for $\hat{K}_{dj} = \hat{K}_{pj} = 0$, (28) and (30) are I-controllers. $\blacksquare$

Example 4.1 illustrates partially reliable decentralized PID design for a linearized model of a sugar mill process, [4].

Example 4.1: Let $G = \begin{bmatrix} \frac{-5}{25s+1} & \frac{s^2-0.005(s+1)}{s(s+1)} \\ \frac{1}{25s+1} & \frac{-0.0023}{s} \end{bmatrix} = \begin{bmatrix} G_1 & \frac{s^2-0.005(s+1)}{(s+a)(s+1)} \\ G_{21} & \frac{-0.0023}{(s+a)} \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & \frac{s}{(s+a)} \end{bmatrix}^{-1}$; With $Y = \frac{s}{s+a}$, $a > 0$, G satisfies the *plant assumptions*: $G_1, G_{21} \in \mathcal{M}(\mathbf{S})$, $G_{12}Y \in \mathcal{M}(\mathbf{S})$. The only $\mathcal{U}$-pole of G_2 is at $s = 0$, which also appears as a pole of G_{12}. The plant G and G_2 have no transmission-zeros at $s = 0$; G has a transmission-zero at $s = 0.137 \in \mathcal{U}$ (and another at $s = -0.1205$). The only zero of G_2 is at infinity. Following Proposition 3.3, design C_2: Choose $K_{d2} = -1$, $\tau_2 = 0.1$. Then (9) holds for $\alpha > 0.023$; take $\alpha = 0.06$. Then (15) holds for $\gamma < 0.06$; take $\gamma = 0.04$. The PID-controller $C_2 = -26.0870 - \frac{s}{0.1s+1} - \frac{1.0435}{s}$.

Following Proposition 3.1, design C_1 stabilizing $W = G_1 - G_{12}C_2(I+G_2C_2)^{-1}G_{21} \in \mathbf{S}$: Choose $\hat{K}_{d1} = 0$, $\hat{K}_{p1} = -4$. Then $\beta < \|W\hat{K}_{p1} + \frac{W(s)W(0)-I}{s}\|^{-1} = 0.1168$; take $\beta = 0.1$. The PI-controller $C_1 = -0.4 - \frac{0.0139}{s}$. Fig. 2 shows the step responses for the two outputs y_1, y_2, with unit-steps applied at both references r_1, r_2. The controller $C_D = [\,C_1, C_2\,]$ is active. Fig. 3 shows the step responses when C_1 fails, i.e., $C_D = [\,0, C_2\,]$, with only the second channel operational. The partially reliable design guarantees closed-loop stability when $C_1 = 0$ but asymptotic tracking with zero steady-state error is achieved only in the second channel with integral-action. ∎

To illustrate the fully reliable decentralized PID-controller design approach, the synthesis procedure in Proposition 4.1 is applied in Example 4.2 to design a control system that manipulates the flow rate of two drugs, dopamine and sodium nitroprusside, to a critical care patient. We use the simplified model in [2] without input delays. The anesthesiologist infuses several drugs to the patient during surgery to maintain the outputs, the main arterial pressure and cardiac output, close to their desired setpoints.

Example 4.2: Let $G = \begin{bmatrix} \frac{-6}{0.67s+1} & \frac{3}{2s+1} \\ \frac{12}{0.67s+1} & \frac{5}{5s+1} \end{bmatrix} \in \mathcal{M}(\mathbf{S})$; then $\text{rank}G(0) = 2$, $G_1(0) \neq 0, G_2(0) \neq 0$, $I - G_{12}(0)G_2^{-1}G_{21}(0)G_1^{-1}(0) = 2.2 > 0$. Design C_2: Choose $\hat{K}_{p2} = 1.05$, $\hat{K}_{d2} = 0$. With $\beta_2 = 1.5 < 4$ satisfying (29), we obtain $K_{p2} = 1.575$, $K_{i2} = 0.3$. The PI-controller $C_2 = \frac{1.575s+0.3}{s}$ as in (28). Design C_1 that simultaneously stabilizes G_1 and $W = \frac{-8.955s^3-62.69s^2-27.16s-2.955}{s^4+3.768s^3+4.583s^2+1.922s+0.2239}$: Choose $\hat{K}_{p1} = -0.1$, $\hat{K}_{d1} = -0.1$, $\tau_1 = 0.01$. With $\beta_1 = 0.25 < \min\{1.1346, 0.3779\}$ satisfying (31), we obtain $K_{p1} = -0.025$, $K_{d1} = -0.025$, $K_{i1} = -0.0417$. The PID-controller $C_1 = \frac{-0.02525s^2-0.02542s-0.04167}{s\,(0.01s+1)}$ as in (30). When $C_D = \text{diag}[\,C_1, C_2\,]$, the closed-loop poles are $\{-121.81, -3.0562, -0.21554, -0.18956, -0.55344 \pm j0.70938\}$. Fig. 4 shows the step responses of $S(G, C_D)$ for the two outputs y_1 (dashed), y_2 (solid), with unit-steps applied at both references r_1, r_2, with $C_D = [\,C_1, C_2\,]$ having both channels active. Both channels achieve asymptotic tracking with zero steady-state error. Fig. 5 shows the step responses when C_1 is taken out, i.e., $C_D = [\,0, C_2\,]$. The output y_1 does not track the step reference due to the lack of integral action in the first channel. Fig. 6 shows the step responses when C_2 is turned off, i.e., $C_D = [\,C_1, 0\,]$.

V. Conclusions

We proposed systematic synthesis of decentralized PID-controllers that achieve closed-loop stability and asymptotic tracking of step-input references at each output channel when both channels are operational, and maintain closed-loop stability even when one of the controllers is turned off. Although we considered the two-channel decentralized case here, the results may be extended to more channels.

VI. Appendix: Proofs

Proof of Proposition 3.1: Let C_{pid} be as in (3). Then $M_{pid} := \frac{s}{s+\beta}I + H\frac{s}{s+\beta}C_{pid} = I + \frac{\beta\, s}{s+\beta}[\,H\,(\hat{K}_p + \frac{\hat{K}_d s}{\tau s+1}) + \frac{HH^I(0)-I}{s}\,]$ is unimodular. Therefore, C_{pid} stabilizes H. ∎

Proof of Proposition 3.2: By assumption, K_{p2}, K_{d2}, τ_2 are such that $R(\infty)^{-1}$ exists. By (6), $M_{pid} := \frac{s}{s+\gamma}G^{-1} + \frac{s}{s+\gamma}C_2 = \frac{s}{s+\gamma}R + \frac{\gamma}{s+\gamma}R(\infty) = [\,I + \frac{1}{s+\gamma}s\,(\,R(s)R^{-1}(\infty) - I\,)\,]\,R(\infty)$ is unimodular. Therefore, C_2 stabilizes G_2. ∎

Proof of Proposition 3.3: By (9), $M_{pd} := \tilde{Y} + \tilde{X}C_{pd} = \frac{(1-s/z)}{s+a}[G_2^{-1} + C_{pd}] = [I + \frac{(1+\alpha/z)}{s+\alpha}\Psi]\tilde{Y}(\infty)\frac{(s+\alpha)}{(1+\alpha/z)(s+a)}$ is unimodular . Therefore, C_{pd} stabilizes G_2 and $H_{pd} \in \mathcal{M}(\mathbf{S})$; $H_{pd}(0)^{-1} = G_2^{-1}(0) + K_{p2}$. Since K_i/s stabilizes H_{pd}, $C_{pid} = C_{pd} + K_{i2}/s$ stabilizes G_2. ∎

Proof of Proposition 3.4: *i)* By (13), $M_{pd} := \tilde{Y} + \tilde{X}C_{pd} = \frac{(s-z)}{as+1}[G_2^{-1} + C_{pd}] = [\frac{\alpha(s-z)}{as+1}I + (1+\alpha z)\frac{(s-z)}{as+1}(G_2^{-1} + \frac{K_{d2}\,s}{\tau_2 s+1})\tilde{Y}(0)^{-1}]\frac{\tilde{Y}(0)}{(1+\alpha\,z)} = [\,I + \frac{(1+\alpha\,z)s}{\alpha\,s+1}\Phi]\tilde{Y}(0)\frac{(\alpha s+1)}{(1+\alpha z)(as+1)}$ is unimodular. Therefore, C_{pd} stabilizes G_2 and $H_{pd} := M_{pd}^{-1}\tilde{X} = G_2(I + C_{pd}G_2)^{-1} \in \mathcal{M}(\mathbf{S})$; $H_{pd}(0)^{-1} = \alpha(1+\alpha z)^{-1}\tilde{Y}(0) + G_2^{-1}(0)$. Since K_i/s stabilizes H_{pd}, $C_{pid} = C_{pd} + K_i/s$ stabilizes G_2. ∎

Proof of Proposition 3.5: *i)* By (19), $1 - \alpha\beta(z_1 z_2)^{-1} = \eta > 0$. By (17), $U_d := \frac{(s+a_2)}{s+\delta}\tilde{Y} + \frac{\alpha(1-s/z_1)\tilde{Y}(\infty)}{(1+\alpha/z_1)(s+a_1)} = \frac{(s+\alpha)}{(1+\alpha/z_1)(s+a_1)}[\frac{(1+\alpha/z_1)(1-s/z_1)(1-s/z_2)}{(s+\alpha)(s+\delta)}G_2^{-1}\tilde{Y}(\infty)^{-1} + \frac{\alpha(1-s/z_1)}{s+\alpha}I]\tilde{Y}(\infty) = \frac{(s+\alpha)}{(1+\alpha/z_1)(s+a_1)}[\frac{(1+\alpha/z_1)}{s+\alpha}\Psi_1 + I]\tilde{Y}(\infty)$ is unimodular. Since $z_2 > \|\,\Psi_2\,\|$, by (19), $M_{pd} := \tilde{Y} + \tilde{X}C_{pd} = \frac{\tau_2\,(s+\delta)}{\eta(\tau_2 s+1)}[(1 + \beta/z_2)\tilde{Y} + \frac{\beta(1+\alpha/z_1)(1-s/z_2)}{(s+\delta)}\tilde{Y} + \tilde{X}\alpha\beta\tilde{Y}(\infty)] = \frac{\tau_2(s+\delta)}{\eta(\tau_2 s+1)}[(1 + \beta/z_2)\tilde{Y} + U_d\frac{\beta(1+\alpha/z_1)(1-s/z_2)}{(s+a_2)}] = \frac{(s+\beta)(1+\alpha/z_1)\tau_2(s+\delta)}{(s+a_2)\eta(\tau_2 s+1)}U_d[\frac{(1+\beta/z_2)(s+a_2)}{(s+\beta)(1+\alpha/z_1)}U_d^{-1}\tilde{Y} + \frac{\beta\,(1-s/z_2)}{s+\beta}I] = \frac{(s+\beta)(1+\alpha/z_1)\tau_2(s+\delta)}{(s+a_2)\eta(\tau_2 s+1)}U_d\;[\frac{(1+\beta/z_2)}{s+\beta}\Psi_2 + I]$ is unimodular. Therefore, C_{pd} stabilizes G_2 and $H_{pd} := M_{pd}^{-1}\tilde{X} = G_2(I + C_{pd}G_2)^{-1} \in \mathcal{M}(\mathbf{S})$; $H_{pd}(0)^{-1} = \tau_2\delta\alpha\beta\tilde{Y}(\infty)/\eta + G_2^{-1}(0)$. Since K_{i2}/s stabilizes H_{pd}, $C_{pid} = C_{pd} + K_{i2}/s$ stabilizes G_2. ∎

Proof of Proposition 3.6: By (25), $\eta \geq 0$. By (23), $U_d := \frac{\alpha(s-z_1)\tilde{Y}(0)}{(1+\alpha z_1)(a_1 s+1)} + \frac{(a_2 s+1)}{\delta s+1}\tilde{Y} = \frac{(\alpha s+1)}{(1+\alpha z_1)(a_1 s+1)}[\frac{\alpha(s-z_1)}{\alpha s+1}I +$

$\frac{(1+\alpha z_1)(s-z_1)(s-z_2)}{(\alpha s+1)(\delta s+1)} G_2^{-1}\tilde{Y}(0)^{-1}]\tilde{Y}(0) = \frac{(\alpha s+1)}{(1+\alpha z_1)(a_1 s+1)}[I + \frac{(1+\alpha z_1)\,s}{\alpha s+1}\Phi_1]\tilde{Y}(0)$ is unimodular. By (25), $M_{pd} := \tilde{Y}+\tilde{X}C_{pd} = \frac{(\delta s+1)}{\eta(\tau_2 s+1)}[(1+\beta z_2)\tilde{Y} + \frac{\beta(1+\alpha z_1)(s-z_2)}{(\delta s+1)}\tilde{Y} + \tilde{X}\alpha\beta\tilde{Y}(0)] = \frac{(\delta s+1)}{\eta(\tau_2 s+1)}[(1+\beta z_2)\tilde{Y} + U_d\frac{\beta(1+\alpha z_1)(s-z_2)}{(a_2 s+1)}] = \frac{(\beta s+1)(1+\alpha z_1)(\delta s+1)}{\eta(a_2 s+1)(\tau_2 s+1)}U_d[\frac{(1+\beta z_2)s}{\beta s+1}\Phi_2 + I]$ is unimodular. Therefore, C_{pd} stabilizes G_2 and $H_{pd} := M_{pd}^{-1}\tilde{X} \in \mathcal{M}(\mathbf{S})$; $H_{pd}(0)^{-1} = \alpha\beta\tilde{Y}(0)/\eta + G_2^{-1}(0)$. Since K_{i2}/s stabilizes H_{pd}, $C_{pid} = C_{pd} + K_{i2}/s$ stabilizes G_2. ∎

Proof of Proposition 4.1: By Proposition 3.1, C_2 in (28) stabilizes $G_2 \in \mathcal{M}(\mathbf{S})$ and C_1 in (28) stabilizes $G_1 \in \mathcal{M}(\mathbf{S})$. By Theorem 2.1-(b), we must show that C_1 stabilizes W. Now $C_2(I+G_2C_2)^{-1} \in \mathcal{M}(\mathbf{S})$ since C_2 stabilizes G_2, and hence, $W \in \mathcal{M}(\mathbf{S})$. Furthermore, $C_2(I+G_2C_2)^{-1}(0) = G_2^{-1}(0)$ implies $W(0) = G_1(0) - G_{12}(0)G_2^{-1}(0)G_{21}(0)$. By assumption, $\Theta := W(0)G_1^{-1}(0) > 0$ implies $\|sI(sI+\beta_1\Theta)^{-1}\| = 1$ for $\beta_1 > 0$. Then $M_w := sI(sI+\beta_1\Theta)^{-1} + WC_1 sI(sI+\beta_1\Theta)^{-1} = sI(sI+\beta_1\Theta)^{-1} + \beta_1 W(\hat{K}_{p1} + \frac{\hat{K}_{d1}s}{\tau_1 s+1} + \frac{G_1^{-1}(0)}{s})\, sI\,(sI+\beta_1\Theta)^{-1} = I + \beta_1[W(\hat{K}_{p1} + \frac{\hat{K}_{d1}s}{\tau_1 s+1}) + \frac{W(s)G_1^{-1}(0)-\Theta}{s}]\, sI\,(sI+\beta_1\Theta)^{-1}$ is unimodular. Therefore, C_1 stabilizes W. ∎

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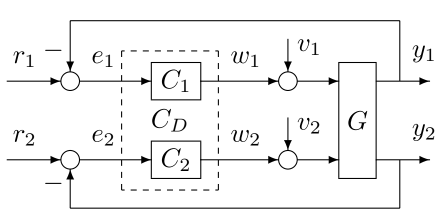

Fig. 1: The two-channel decentralized system $S(G, C_D)$.

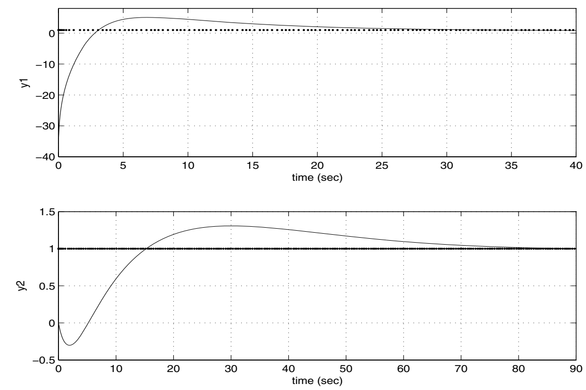

Fig. 2: Example 4.1 step-responses with $C_D = [\,C_1, C_2\,]$.

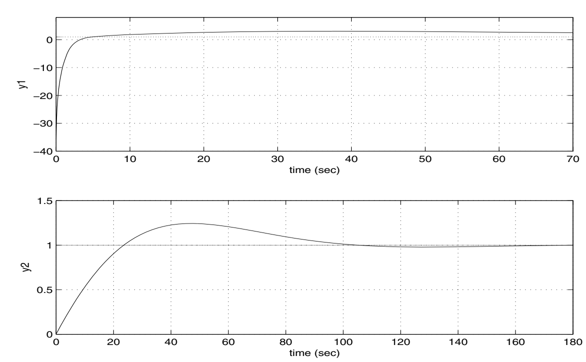

Fig. 3: Example 4.1 step-responses with $C_D = [\,0, C_2\,]$.

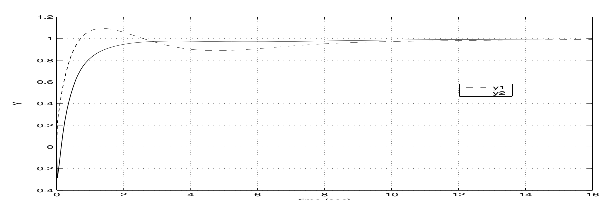

Fig. 4: Example 4.2 step-responses with $C_D = [\,C_1, C_2\,]$.

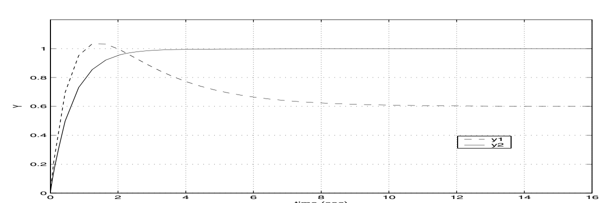

Fig. 5: Example 4.2 step-responses with $C_D = [\,0, C_2\,]$.

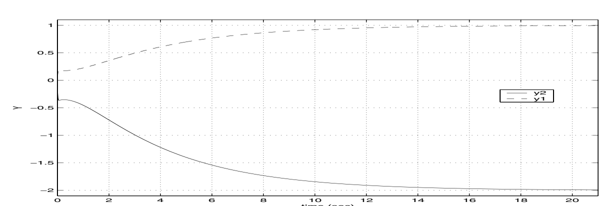

Fig. 6: Example 4.2 step-responses with $C_D = [\,C_1, 0\,]$.