PID Controller Synthesis for MIMO Plants with Delays*

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Abstract

Conditions are presented for closed-loop stabilizability of linear time-invariant (LTI) multi-input, multi-output (MIMO) plants with input delays using PID (Proportional + Integral + Derivative) controllers. We show that systems with at most two unstable poles can be stabilized by PID controllers provided a small gain condition is satisfied. For systems with only one unstable pole, this condition is equivalent to having sufficiently small delay-unstable pole product. Our method of synthesis of such controllers identify some free parameters that can be used to satisfy further design criteria than stability.

1 Introduction

While finite dimensional LTI systems are sufficiently accurate models for a wide range of dynamical phenomena, there are many cases in which delay effects cannot be ignored and have to be included in the model, [6]. A multi-output LTI system with r inputs $u_1, \ldots, u_r$ and delays in the input channels can be represented in time domain as follows:

$$\dot{x}(t) = Ax(t) + B[u_1(t-T_1)\ldots u_r(t-T_r)]^{\mathrm{T}}, \;\; y(t) = Cx(t) + D[u_1(t-T_1)\ldots u_r(t-T_r)]^{\mathrm{T}}$$

where x is the state of the finite dimensional part, y is the output vector, A, B, C, D are real matrices and the j^{th} input channel delay T_j is a non-negative real number, $j = 1, \ldots, r$. The transfer matrix of such a system is given by $P(s) = G(s)\Lambda(s)$, where $G(s) = C(sI - A)^{-1}B + D$ is the finite dimensional part, and $\Lambda(s) = \mathrm{diag}\left[e^{-T_1 s}, \cdots, e^{-T_r s}\right]$ is the delay matrix. This paper considers closed-loop stabilization (see Figure 1) of such systems using *proper PID-controllers* [5]:

$$C_{pid} = K_p + \frac{K_i}{s} + \frac{K_d\ s}{\tau_d s + 1}\ , \qquad (1)$$

where K_p, K_i, and K_d are real matrices.

Stability of delay systems of retarded type, or even neutral type, is extensively investigated and many delay-independent and delay-dependent stability results are available (see [6], [9]). The feedback stabilization of delay systems is also well investigated. Being a subclass of infinite dimensional systems,

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delay systems inherit the results on robust control of infinite dimensional systems, [4]. Also, since delay element is an integral part of process control systems, most of the tuning and internal model control techniques used in process control systems apply to delay systems, [1]. The more special, but practically very relevant (see [5]), problem of existence of stabilizing PID-controllers is unfortunately not easy to solve even for the delay-free case. One way of gaining insight into the difficulty of the problem is to note that, in the delay-free case, the existence of a stabilizing PID-controller is equivalent to that of a *constant stabilizing output feedback* for a transformed MIMO plant. Alternatively, the problem can be posed as determining conditions of existence of a fixed order, stable, and minimum-phase controller for a suitable plant, which is again well-known to be a difficult problem, [2, 17].

It should be mentioned that there *are* some computational PID-stabilization methods, which consist of "efficient search" in the parameter space, recently developed for single-input single-output (SISO) delay-free systems, see [11] and references therein. Some of these techniques have been extended to cover first-order, scalar, single-delay systems, [13]. A parameter-space approach for finding stability regions of a class of quasi-polynomials is proposed in, [7]. Moreover, this technique can be used for finding stability regions in the PID controller parameter space for delay systems. For example, in [12], the method developed by [7] is applied to PID controller tuning for active queue management problem.

For a plant consisting of a chain of integrators, stabilization using multiple delays is studied in [10, 8]. Although the motivation of [10, 8] is to stabilize non-delayed plants using delayed output with static gains, clearly, their problem includes proportional control design for an integrator (and oscillator in the case of [8]) with delay. This is one of the special cases we will study as well.

In this paper, making a novel use of the small gain theorem, we arrive at two main results: First, for MIMO plants with input delays, we obtain some sufficient conditions on the existence of stabilizing PID controllers, and second, we explicitly construct PID controllers for plants having only one unstable pole (under the condition that the product of unstable pole with delay is sufficiently small). This construction is extended to the case of two unstable poles, either on the real-axis or the imaginary-axis. As our goal is to establish existence of stabilizing PID-controllers at this point, we do not consider performance issues but propose freedom in the design parameters that can be used towards satisfaction of performance criteria.

Notation: As usual, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}_-$, $\mathbb{C}_+$ denote real, complex, open left-half plane complex and open right-half plane complex numbers. Throughout the paper, $\mathcal{U}$ denotes the extended closed right-half plane, i.e., $\mathcal{U} = \{ s \in \mathbb{C} \mid \mathcal{R}e(s) \geq 0 \} \cup \{\infty\}$; $\mathbf{R_p}$ denotes proper rational functions; $\mathbf{S}$ denotes stable proper real rational functions of s. We define $\mathcal{M}(\mathbf{S})$ as the set of matrices whose entries are in $\mathbf{S}$. The space H_∞ is the set of all bounded analytic functions in $\mathbb{C}_+$. For $h \in H_\infty$, the norm is defined as $\|h\|_\infty = \text{ess sup}_{s\in\mathbb{C}_+} |h(s)|$, where ess sup denotes the essential supremum. A matrix valued function H is in $\mathcal{M}(\mathcal{H}_\infty)$ if all its entries are in H_∞, and in this case $\|H\|_\infty = \text{ess sup}_{s\in\mathbb{C}_+} \bar{\sigma}(H(s))$, where $\bar{\sigma}$ denotes the maximum singular value. From the induced L^2 gain point of view, a system whose transfer matrix is H is stable if and only if $H \in \mathcal{M}(\mathcal{H}_\infty)$. Moreover, for a square transfer matrix $H \in \mathcal{M}(\mathcal{H}_\infty)$ we say that H is unimodular if $H^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$.

For simplicity, we drop (s) in transfer matrices such as $G(s)$ where this causes no confusion. Also, since all the norms we are interested in are H_∞ norms, we will drop the norm subscript, i.e. $\|\cdot\|_\infty \equiv \|\cdot\|$ whenever this is clear from the context.

2 Problem Description

Consider the standard unity-feedback system shown in Fig. 1, where $G \in \mathbf{R_p}^{r\times r}$ and $C \in \mathbf{R_p}^{r\times r}$ denote the plant without the time delay term (non-delayed plant, for short) and the controller transfer-functions. The delay term is $\Lambda = \text{diag}\left[e^{-sT_1}, e^{-sT_2}, \ldots, e^{-sT_r}\right]$, where, for $1 \leq j \leq r$, $T_j \in \Theta_j = [0\ ,\ T^j_{\max}) \subset \mathbb{R}_+$. We assume that the delay upper bound $T^j_{\max}$ is known for all input channels $j = 1, \ldots, r$. Define $\mathcal{T} := (T_1, \ldots, T_r)$ and $\Theta := (\Theta_1, \ldots, \Theta_r)$. As a shorthand notation we will write $\mathcal{T} \in \Theta$ to represent all possibilities $T_j \in \Theta_j$, $1 \leq j \leq r$. It is assumed that the feedback system is well-posed and that the non-delayed plant and controller have no unstable hidden-modes. It is also assumed that $G \in \mathbf{R_p}^{r\times r}$ is full normal rank.

The closed-loop map H_{cl} from (r, v) to (u, y) is given by

$$H_{cl} = \begin{bmatrix} C(I+G\Lambda C)^{-1} & -C(I+G\Lambda C)^{-1}G\Lambda \\ G\Lambda C(I+G\Lambda C)^{-1} & (I+G\Lambda C)^{-1}G\Lambda \end{bmatrix}. \tag{2}$$

We consider the proper form of PID-controllers in (1), where the real matrices K_p, K_i, K_d are called the proportional constant, the integral constant, and the derivative constant, respectively. Due to implementation issues of the derivative action, a pole is typically added to the derivative term (with $\tau \in \mathbb{R}$, $\tau_d > 0$ when $K_d \neq 0$) so that the transfer-function C_{pid} in (1) is proper. If one or more of the three terms K_p, K_i, K_d is zero, then the corresponding subscript is omitted from C_{pid}; i.e., $C_{pi} = K_p + \dfrac{K_i}{s}$ denotes a proportional+integral (PI), $C_{pd} = K_p + \dfrac{K_d\, s}{\tau_d s + 1}$ denotes a proportional+derivative (PD), $C_{id} = \dfrac{K_i}{s} + \dfrac{K_d\, s}{\tau_d s + 1}$ denotes an integral+derivative (ID) controller; C_p, C_i, C_d correspond to pure proportional (P), integral (I), derivative (D) controllers, respectively.

Definition 2.1 *a) The unity-feedback system $Sys(G\Lambda, C)$ is said to be stable iff the closed-loop map H_{cl} is in $\mathcal{M}(\mathcal{H}_\infty)$. b) A delayed plant $G\Lambda$, where $G \in \mathbf{R_p}^{r\times r}$, is said to admit a PID-controller iff there exists a PID-controller $C = C_{pid}$ as in (1) such that the system $Sys(G\Lambda, C)$ is stable. We say that $G\Lambda$ is stabilizable by a PID-controller, and C_{pid} is a stabilizing PID-controller.* ∎

Let $G = Y^{-1}X$ be any left coprime factorization (LCF) of the plant, $C = N_c D_c^{-1}$ be any right coprime factorization (RCF) of the controller, where we use coprime factorizations over $\mathbf{S}$; i.e., for $G \in \mathbf{R_p}^{r\times r}$, $X, Y \in \mathcal{M}(\mathbf{S})$ and $\det Y(\infty) \neq 0$, and similarly for $C \in \mathbf{R_p}^{r\times r}$, $N_c, D_c \in \mathcal{M}(\mathbf{S})$ and $\det D_c(\infty) \neq 0$. The controller C stabilizes G if and only if $M := YD_c + XN_c \in \mathcal{M}(\mathbf{S})$ is unimodular, i.e., $M^{-1} \in \mathcal{M}(\mathbf{S})$, [17]. When dealing with systems with time delays, $\mathcal{M}(\mathbf{S})$ is replaced with $\mathcal{M}(\mathcal{H}_\infty)$ in the above definitions. More precisely, the controller C stabilizes $G\Lambda$ if and only if $M_\Lambda := YD_c + X\Lambda N_c \in \mathcal{M}(\mathcal{H}_\infty)$ is unimodular, i.e., $M_\Lambda^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$, [15].

Lemma 2.1 (Two-step controller synthesis): *Let $G \in \mathbf{R_p}^{r\times r}$. Suppose that C_g is a controller that stabilizes $G\Lambda$, and C_h is a controller that stabilizes the stable system $H := H_{yv} = G\Lambda(I + C_g G\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$. Then*

$$C = C_g + C_h \tag{3}$$

is also a controller that stabilizes $G\Lambda$. ∎

Proof of Lemma 2.1: Let $G = Y^{-1}X$ be an LCF; let $C_g = N_g D_g^{-1}$ be an RCF. The controller $C_g = N_g D_g^{-1}$ stabilizes $G\Lambda = Y^{-1}X\Lambda$ if and only if $M_\Lambda := YD_g + X\Lambda N_g$ is unimodular. Since C_g stabilizes G, the transfer-functions $H = G\Lambda(I + C_g G\Lambda)^{-1}$ and $I - C_g H = (I + C_g G\Lambda)^{-1}$ are stable. Now C_h stabilizes $H \in \mathcal{M}(\mathcal{H}_\infty)$ if and only if $C_h(I + HC_h)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$, and $(I + HC_h)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$. Write (3) as $C = C_g + C_h = N_g D_g^{-1}(I + HC_h) + (I - C_g H)C_h(I + HC_h)^{-1}D_g D_g^{-1}(I + HC_h) = [N_g + (I - C_g H)C_h(I + HC_h)^{-1}D_g]\,[(I + HC_h)^{-1}D_g]^{-1}$. Define $N_c := [N_g + (I - C_g H)C_h(I + HC_h)^{-1}D_g] \in \mathcal{M}(\mathcal{H}_\infty)$, $D_c := (I + HC_h)^{-1}D_g \in \mathcal{M}(\mathcal{H}_\infty)$. Then $YD_c + X\Lambda N_c = Y(I + HC_h)^{-1}D_g + X\Lambda N_g + YY^{-1}X\Lambda(I - C_g H)C_h(I + HC_h)^{-1}D_g = Y[(I + HC_h)^{-1} + G\Lambda(I + C_g G\Lambda)^{-1}C_h(I + HC_h)^{-1}]D_g + X\Lambda N_g = Y[(I + HC_h)^{-1} + HC_h(I + HC_h)^{-1}]D_g + X\Lambda N_g = M_\Lambda$ is unimodular. Therefore, $C = N_c D_c^{-1}$ is an RCF of a stabilizing controller for $G\Lambda$. ∎

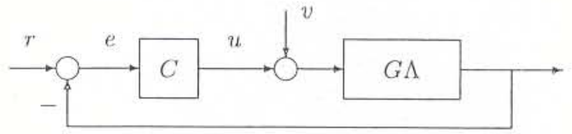

Figure 1: Unity-Feedback System $Sys(G\Lambda, C)$ with input delays.

3 Main Results

Although it is obvious that stable plants admit PID-controllers, the freedom in the stabilizing controller parameters is still worth investigating. We propose a PID-controller synthesis for stable plants in the following Proposition 3.1, which will be frequently referred to in the sequel.

Proposition 3.1 (PID-controller synthesis for stable plants): *Let* $H \in \mathbf{S}^{r\times r}$, *let (normal)* $\text{rank}H(s) = r$.
i) PD-design: Choose any $\hat{K}_p, \hat{K}_d \in \mathbb{R}^{r\times r}$, $\tau_d > 0$. *Then,*

$$C_{pd} = K_p + \frac{K_d\, s}{\tau_d s + 1} = \alpha\,\hat{K}_p + \frac{\alpha\,\hat{K}_d\, s}{\tau_d s + 1} \tag{4}$$

is a PD-controller that stabilizes $H\Lambda$ *for* $\mathcal{T} \in \Theta$, *where* $\alpha \in \mathbb{R}$ *satisfies*

$$0 < \alpha < \ \| \ H(s)\ (\ \hat{K}_p + \frac{\hat{K}_d\, s}{\tau_d s + 1}\ )\ \|^{-1}\ . \tag{5}$$

For $\hat{K}_d = 0$, *(4) is a P-controller* C_p *; for* $\hat{K}_p = 0$, *(4) is a D-controller* C_d .
ii) PID-design: Let $\text{rank}H(0) = r$. *Choose any* $\hat{K}_p, \hat{K}_d \in \mathbb{R}^{r\times r}$, $\tau_d > 0$. *Then,*

$$C_{pid} = \gamma\,\hat{K}_p + \frac{\gamma\,H(0)^{-1}}{s} + \frac{\gamma\,\hat{K}_d s}{\tau_d s + 1} \tag{6}$$

is a PID-controller that stabilizes $H\Lambda$ *for* $\mathcal{T} \in \Theta$, *where* $\gamma \in \mathbb{R}$ *satisfies*

$$0 < \gamma < \min_{\mathcal{T}\in\Theta}\ \| \ H(s)\Lambda(s)(\hat{K}_p + \frac{\hat{K}_d s}{\tau_d s + 1}) + \frac{H(s)\Lambda(s)H(0)^{-1} - I}{s}\ \|^{-1}. \tag{7}$$

For $\hat{K}_d = 0$, *(6) is a PI-controller* C_{pi} *; for* $\hat{K}_p = 0$, *(6) is an ID-controller* C_{id} *; for* $\hat{K}_d = \hat{K}_p = 0$, *(6) is an I-controller* C_i . ■

Proof of Proposition 3.1: *i)* Let C_{pd} be as in (4) for $\alpha \in \mathbb{R}$ satisfying (5). Then $M_{pd} := I + H\Lambda C_{pd} = I + \alpha H\Lambda(\hat{K}_p + \frac{\hat{K}_d s}{\tau_d s + 1})$ is unimodular since $\alpha \ \|\ H(s)\ \Lambda(s)\ (\ \hat{K}_p + \frac{\hat{K}_d\, s}{\tau_d s+1}\ )\ \| = \alpha\ \|\ H(s)\ (\ \hat{K}_p + \frac{\hat{K}_d\, s}{\tau_d s+1}\ )\ \| < 1$. Therefore, C_{pd} stabilizes $H\Lambda$. Since $\hat{K}_p, \hat{K}_d$ are arbitrary, they can be chosen as zero. *ii)* Let C_{pid} be as in (6) for $\gamma \in \mathbb{R}$ satisfying (7). Then $M_{pid} := \frac{s}{s+\gamma} I + H\Lambda \frac{s}{s+\gamma} C_{pid} = I + \frac{\gamma s}{s+\gamma}\,(\,H\Lambda\,(\hat{K}_p + \frac{\hat{K}_d s}{\tau_d s + 1}) + \frac{H(s)\Lambda(s)H(0)^{-1} - I}{s}\,)$ is unimodular. Therefore, C_{pid} stabilizes $H\Lambda$. Since $\hat{K}_p, \hat{K}_d$ are arbitrary, they can be chosen as zero. ■

In the following proposition, we state some general existence conditions for stabilizing PID controllers.

Proposition 3.2 (General existence conditions for stabilizing PID-controllers): *Let* $G \in \mathbf{R_p}^{r\times r}$. *Let (normal)* $\text{rank}G(s) = r$. **a)** *If* $G\Lambda$ *admits a PID-controller such that the integral constant* $K_i \in \mathbb{R}^{r\times r}$ *is nonzero, then* G *has no transmission-zeros at* $s = 0$ *and* $\text{rank}K_i = r$.
b) *If* $G\Lambda$ *admits a PID-controller such that any one of the three constants* K_p, K_d, K_i *is nonzero, then* $G\Lambda$ *admits a PID-controller such that any two of the three constants is nonzero, and* $G\Lambda$ *admits a PID-controller such that all of the three constants is nonzero.*
c) *If* $G\Lambda$ *admits a PID-controller such that two of the three constants* K_p, K_d, K_i *is nonzero, then* $G\Lambda$ *admits a PID-controller such that all of the three constants is nonzero.*
In **b)** *and* **c)**, *the integral constant* $K_i \neq 0$ *only if* G *has no transmission-zeros at* $s = 0$. ■

Proof of Proposition 3.2: **a)** Let $G = Y^{-1}X$ be an LCF of G. Let $C_{pid} = K_p + \frac{K_i}{s} + \frac{K_d\, s}{\tau_d s+1}$ be a PID-controller that stabilizes $G\Lambda$. For any positive $a \in \mathbb{R}$, an RCF $C_{pid} = N_c D_c^{-1}$ is

$$C_{pid} = N_c D_c^{-1} = \left[(\ K_p + \frac{K_d\, s}{\tau_d s + 1}\ ) \frac{s}{s+a} + \frac{K_i}{s+a} \right] \left[\frac{s}{s+a} I_r \right]^{-1} . \tag{8}$$

Since C_{pid} stabilizes $G\Lambda$, $M_\Lambda = Y D_c + X \Lambda N_c$ is unimodular, which implies rank$M_\Lambda(0) = r =$ rank$X(0) K_i$. Therefore, rank$X(0) = r$, equivalently, G has no transmission-zeros at $s = 0$, and rank$K_i = r$. **b)** Suppose that $G\Lambda$ is stabilized by C_p, equivalently $H_p = G\Lambda(I + C_p G\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$; or by C_d, equivalently $H_d = G\Lambda(I + C_d G\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$; or by C_i, which implies $H_i = G\Lambda(I + C_i G\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$. The (normal) rank of H_p, H_d, H_i are equal to rank$G = r$. By Proposition 3.1-(i), there exists a a P-controller for H_d, for H_i, and for H_{id}; there exists a D-controller for H_p, for H_i, and for H_{pi}. By Proposition 3.1-(ii), there exists an I-controller for H_p, for H_d, and for H_{pd}. Consider $H_p \in \mathcal{M}(\mathcal{H}_\infty)$: If G has no transmission-zeros at $s = 0$, then rank$H_p(0) =$ rank$(Y + X\Lambda C_p)^{-1}(0)X(0)\Lambda(0) =$ rank$X(0) = r$ (because $(Y + X\Lambda C_p)$ is unimodular). Let C_{dh} be a D-controller and C_{ih} be an I-controller for H_p. By Lemma 2.1, the PD-controller $C_{pd} = C_p + C_{dh}$ and the PI-controller $C_{pi} = C_p + C_{ih}$ stabilize $G\Lambda$. Similarly, consider $H_d \in \mathcal{M}(\mathcal{H}_\infty)$: Since $M_{d\Lambda} := (Y + X\Lambda C_d)$ is unimodular, rank$M_{d\Lambda}(0) =$ rank$Y(0) = r$; i.e., G has no poles at $s = 0$. If G has no transmission-zeros at $s = 0$, then rank$H_d(0) =$ rank$M_{d\Lambda}^{-1}(0)X(0)\Lambda(0) =$ rank$X(0) = r$. Let C_{ph} be a P-controller and C_{ih} be an I-controller for H_d. By Lemma 2.1, the PD-controller $C_{dp} = C_d + C_{ph}$ and the ID-controller $C_{di} = C_d + C_{ih}$ stabilize $G\Lambda$. Now consider $H_i \in \mathcal{M}(\mathcal{H}_\infty)$: Let C_{ph} be a P-controller and C_{dh} be a D-controller for H_i. By Lemma 2.1, the PI-controller $C_{ip} = C_i + C_{ph}$ and the ID-controller $C_{id} = C_i + C_{dh}$ stabilize $G\Lambda$. **c)** Suppose that $G\Lambda$ is stabilized by C_{pd}, equivalently $H_{pd} = G\Lambda(I + C_{pd}G\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$; or by C_{pi}, which implies $H_{pi} = G\Lambda(I + C_{pi}G\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$; or by C_{id}, which implies $H_{id} = G\Lambda(I + C_{id}G\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$. The (normal) rank of H_{pd}, H_{pi}, H_{id} are equal to rank$G = r$. Consider $H_{pd} \in \mathcal{M}(\mathcal{H}_\infty)$: If G has no transmission-zeros at $s = 0$, then rank$H_{pd}(0) =$ rank$(Y + X\Lambda C_{pd})^{-1}(0)X(0)\Lambda(0) =$ rank$X(0) = r$. Let C_{ih} be an I-controller for H_{pd}. Let C_{dh} be a D-controller for H_{pi}. Let C_{ph} be a P-controller for H_{id}. By Lemma 2.1, each of the PID-controllers $C_{pdi} = C_{pd} + C_{ih}$, $C_{pid} = C_{pi} + C_{dh}$, and $C_{idp} = C_{id} + C_{ph}$ stabilize $G\Lambda$. ■

Proposition 3.2 states that if a stabilizing P, I, or D-controller exists, then it can be extended to a stabilizing PI, ID, PD, PID-controller. However, Proposition 3.2 does not explicitly define the plant classes that admit P, I, or D-controllers. We investigate specific classes of plants that admit such controllers and propose stabilizing PID-controller design methods in Section 3.1.

3.1 Delayed plant classes that admit PID-controllers

Plants that admit PID-controllers are a subset of plants that are strongly stabilizable, i.e., that can be stabilized using stable controllers. This condition is formalized next in Lemma 3.1.

Lemma 3.1 (Strong stabilizability as a necessary condition for PID stabilization): *Let $G \in \mathbf{R_p}^{r\times r}$. Let (normal)* rank$G(s) = r$. *If $G\Lambda$ admits a PID-controller for any $T \in \Theta$, then G is strongly stabilizable.*

Proof of Lemma 3.1: Let $G = Y^{-1}X$ be an LCF of G. Let C_{pid} be a PID-controller that stabilizes $G\Lambda$. An RCF $C_{pid} = N_c D_c^{-1}$ is given by (8). Then det $D_c(z_i) =$ det $\dfrac{z_i}{z_i + a} I_r > 0$ for all $z_i > 0$. If C_{pid} stabilizes $G\Lambda$, then $M_\Lambda = Y D_c + X \Lambda N_c$ is unimodular, which implies det $M_\Lambda(z_i) =$ det$Y(z_i)$ det $D_c(z_i)$ has the same sign for all $z_i \in \mathcal{U}$ such that $X(z_i) = 0$; equivalently, det$Y(z_i)$ has the same sign at all blocking-zeros of G. Therefore, G has the parity-interlacing-property; hence, it is strongly stabilizable, [17]. ■

By Lemma 3.1, plants that admit PID-controllers are necessarily strongly stabilizable. In Proposition 3.1, we gave a PID-controller synthesis method for stable plants and showed that if $G \in \mathbf{S}^{r\times r}$, then there exist P, D, PD-controllers for $G\Lambda$. If rank$G(0) = r$, then there also exist I, PI, ID, PID-controllers for $G\Lambda$. When we consider unstable plants, it is clear that the location of the $\mathcal{U}$-poles of G is important in determining if $G\Lambda$ admits PID-controllers. But even when the real-axis $\mathcal{U}$-poles and zeros satisfy the parity-interlacing-property, $G\Lambda$ may not admit PID-controllers since the strong stabilizability condition is far from being sufficient. For example, PID controllers are at most second order (SISO case); on the other hand, there are bounds on the order of strongly stabilizing controllers for a given plant, [14]. If the right half plane pole-zero pattern of the plant is "close to violating parity interlacing property," then the minimal order of the strongly stabilizing controller is high, [14]; moreover, in this case, the smallest H_∞ norm achievable using a strongly stabilizing controller is large, [18].

We now consider plants with a limited number of $\mathcal{U}$-poles on the positive real-axis, including the origin. Such limitations on the number of $\mathcal{U}$-poles are not surprising; for example, as it can be easily shown in the case of $(s-p)^{-3}$ for $p \geq 0$, plants that have more than two poles in $\mathcal{U}$ do not necessarily admit PID-controllers.

3.1.1 Plants with only one $\mathcal{U}$-pole at $s=0$ or on the positive real-axis

Let $G \in \mathbf{R_p}^{r\times r}$ have full (normal) rank. Let G have no transmission-zeros at $s=0$. Let G have any number of poles in the stable region. Other than a $\mathcal{U}$-pole at $s=p \in \mathbb{R}$, $p \geq 0$, let G have no poles in the unstable region $\mathcal{U}$. The pole at $s=p\geq 0$ may appear in some or all entries of G. Therefore, G has an LCF $G = Y^{-1}X = [\ \frac{(s-p)}{as+1} I\ ]^{-1}\frac{(s-p)}{as+1}G$, $p \in \mathbb{R}$, $p \geq 0$ and $a \in \mathbb{R}$, $a > 0$, where $\text{rank}X(p) = \text{rank}(s-p)G(s)|_{s=p} = r$. Furthermore, since G has no transmission-zeros at $s=0$, $\text{rank}X(0) = \text{rank}(s-p)G(s)|_{s=0} = r$. Under certain assumptions on the pole at $s=p\geq 0$, the delayed plant $G\Lambda$ admits P, PD, PI, PID-controllers. If $p=0$, then plants in this class do not admit D-controllers of the form $C_d = K_d s/(\tau_d s+1)$ since the plant pole at $s=0$ would then cancel the controller's zero. Furthermore, some plants may not admit I-controllers; for example, let $G = \frac{1}{s(s+e)}$ where $\epsilon \geq 0$; then G does not admit any controllers of the form $C_i = K_i/s$. Stabilizing D-controllers or I-controllers may not exist when $p > 0$. For example, let $G = \frac{s-z}{s-p}$ where $z, p > 0$; then obviously G does not admit D-controllers $C_d = K_d s/(\tau_d s+1)$ for any $\tau_d > 0$ and it does not admit I-controllers $C_i = K_i/s$.

Proposition 3.3 develops a PID-controller synthesis for plants with one "small" real-axis pole at $p \geq 0$ subject to the norm-bound $0 \leq p < \|\Phi_\Lambda\|^{-1}$ as defined in (9). Obviously, $p=0$ satisfies this condition. Let $0 < z_{min} \in \mathbb{R}$ be the smallest positive real-axis blocking-zero of G; then $X(z_{min}) = 0$. Consider pure proportional controller synthesis ($K_d = 0$ in C_{pd}): Then $\|\Phi_\Lambda\| \geq 1/z_{min}$ implies $p < \|\Phi_\Lambda\|^{-1} \leq z_{min}$, i.e., p must be closer to the origin than the smallest positive zero.

Proposition 3.3 *Let $G \in \mathbf{R_p}^{r\times r}$, $\text{rank}G(s) = r$. Let $p \in \mathbb{R}$, $p \geq 0$. Let $X = \frac{(s-p)}{as+1}G \in \mathcal{M}(\mathbf{S})$, for $a \in \mathbb{R}$, $a > 0$. Let $\text{rank}X(p) = \text{rank}(s-p)G(s)|_{s=p} = r$. Let $X(0) = (s-p)G(s)|_{s=0}$ be nonsingular, $G^{-1}(0) = -p\,X(0)^{-1}$.*
i) PD-design: Choose any $\hat{K}_d \in \mathbb{R}^{r\times r}$, $\tau_d > 0$. Define

$$\Phi_\Lambda := \frac{(s-p)\,G(s)\Lambda(s)\,X(0)^{-1} - I}{s} + (s-p)\,G(s)\Lambda(s)\,\frac{\hat{K}_d}{\tau_d s+1}\,. \tag{9}$$

If $0 \leq p < \min_{T\in\Theta} \|\ \Phi_\Lambda\ \|^{-1}$, then

$$C_{pd} = (\,\alpha+p\,)\,X(0)^{-1} + \frac{(\,\alpha+p\,)\,\hat{K}_d\ s}{\tau_d s+1} \tag{10}$$

is a PD-controller that stabilizes $G\Lambda$ for $T \in \Theta$, where $\alpha \in \mathbb{R}$ satisfies

$$0 < \alpha < \min_{T\in\Theta} \|\ \Phi_\Lambda\ \|^{-1} - p\,. \tag{11}$$

If $\hat{K}_d = 0$, (10) is a P-controller.
ii) PID-design: Let $0 \leq p < \min_{T\in\Theta} \|\ \Phi_\Lambda\ \|^{-1}$. Let C_{pd} be as in (10). Let $H_{pd} := G\Lambda(I + C_{pd}G\Lambda)^{-1}$. Then

$$C_{pid} = (\,\alpha+p\,)\,X(0)^{-1} + \frac{\gamma\,\alpha\,X(0)^{-1}}{s} + \frac{(\,\alpha+p\,)\,\hat{K}_d\,s}{\tau_d s+1} \tag{12}$$

is a PID-controller that stabilizes $G\Lambda$ for $T \in \Theta$, where $\gamma \in \mathbb{R}$ satisfies

$$0 < \gamma < \min_{T\in\Theta} \|\ \frac{H_{pd}(s)H_{pd}(0)^{-1} - I}{s} \|^{-1}$$

$$= \min_{T\in\Theta} \|\ (I + G\Lambda C_{pd})^{-1}\ \frac{G(s)\Lambda G^{-1}(0) - I}{s} - H_{pd}(s)\frac{K_d}{\tau_d s+1}\ \|^{-1}\,, \tag{13}$$

and $H_{pd}(0)^{-1} = \alpha\,X(0)^{-1}$. If $\hat{K}_d = 0$, (12) is a PI-controller. ∎

Proof of Proposition 3.3: i) By (11), $M_{pd} := Y + X\Lambda C_{pd} = \frac{(s-p)}{as+1}[I + (\alpha+p)G\Lambda(X(0)^{-1} + \frac{\hat{K}_d s}{\tau_d s+1})] = [I + \frac{(\alpha+p)s}{s+\alpha}\Phi_\Lambda]\frac{(s+\alpha)}{(as+1)}$ is unimodular since $a, \alpha > 0$. Therefore, C_{pd} stabilizes $G\Lambda$. Since $\hat{K}_d$ is arbitrary, it can be chosen as zero. *ii)* Since C_{pd} in (10) stabilizes $G\Lambda$, $H_{pd} := M_{pd}^{-1}X\Lambda = G\Lambda(I + C_{pd}G\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$, where $H_{pd}(0)^{-1} = G^{-1}(0) + K_p = X(0)^{-1}Y(0) + (\alpha+p)X(0)^{-1} = \alpha X(0)^{-1}$. For any $\gamma \in \mathbb{R}$ satisfying (13), the I-controller $K_i/s = \gamma H_{pd}(0)^{-1}/s$ stabilizes H_{pd}. By Lemma 2.1, $C_{pid} = C_{pd} + K_i/s$ in (12) stabilizes $G\Lambda$. ■

Example 3.1 *Consider the plant* $G\Lambda = \frac{s-z}{s}e^{-sT}$, *where* $z > 0$. *Then for* $a > 0$, $X := (s-z)/(as+1)$, $X(0) = -z$. *Choose any* $\hat{K}_d \in \mathbb{R}$ *and* $\tau_d > 0$. *Following Proposition 3.3, for any positive* $\alpha < \min_{T\in\Theta} \| \frac{-1}{z}e^{-sT} + \frac{e^{-sT}-1}{s} + e^{-sT}\frac{\hat{K}_d(s-z)}{\tau_d s+1} \|^{-1}$, $C_{pd} = \frac{-\alpha}{z} + \frac{\alpha\hat{K}_d s}{\tau_d s+1}$ *is a stabilizing PD-controller for* $G\Lambda$. *Now let* $H_{pd}(s) := G(s)\Lambda(s)(I + C_{pd}(s)G(s)\Lambda(s))^{-1} = e^{-sT}(s-z)(s + C_{pd}(s)(s-z)e^{-sT})^{-1}$. *Choose any positive* $\gamma \in \mathbb{R}$ *satisfying* $\gamma < \min_{T\in\Theta} \| \frac{-\alpha z^{-1}H_{pd}(s) - I}{s} \|$. *Then* $C_{pid}(s) = \frac{-\alpha}{z} + \frac{-\gamma\alpha}{zs} + \frac{\alpha\hat{K}_d s}{\tau_d s+1}$ *is a PID-controller that stabilizes* $\frac{s-z}{s}e^{-sT}$ *for all* $T \in \Theta$.

Clearly, for a fixed time delay T, *and the right half plane zero* z, *there is an upper bound on the controller gain* α. *For example, when* $\hat{K}_d = 0$, $T\alpha < \| \frac{-1}{Tz}e^{-sT} + \frac{e^{-sT}-1}{Ts} \|^{-1}$, *and this relation is shown in Figure 2.* ■

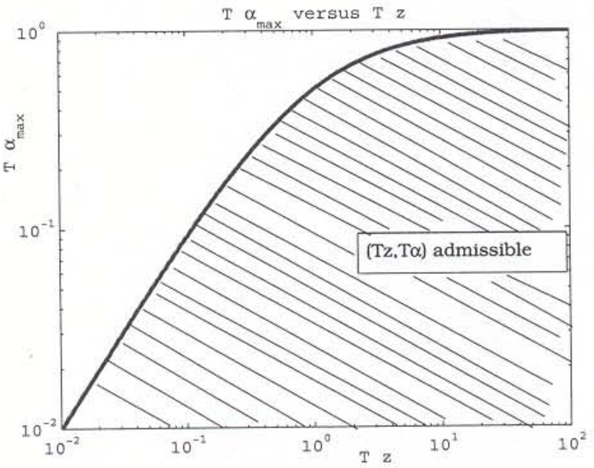

Figure 2: Maximum α versus z for $\hat{K}_d = 0$.

Example 3.2 *Consider the plant* $G(s)\Lambda(s) = \frac{e^{-sT}}{s-p}$, *where* $p > 0$. *Then for* $a > 0$, $X := 1/(as+1)$, $X(0) = 1$. *Choose any* $\hat{K}_d \in \mathbb{R}$ *and* $\tau_d > 0$. *Following Proposition 3.3, if* $p < \min_{T\in\Theta} \| \Phi_\Lambda \|^{-1} = \min_{T\in\Theta} \| \frac{e^{-sT}-1}{s} + e^{-sT}\frac{\hat{K}_d}{\tau_d s+1} \|^{-1}$, *then for any positive* $\alpha < \min_{T\in\Theta} \| \Phi_\Lambda \|^{-1} - p$ *as in (11),* $C_{pd}(s) = (p+\alpha) + \frac{(p+\alpha)\hat{K}_d s}{\tau_d s+1}$ *is a stabilizing PD-controller for* $G\Lambda$. *Now let* $H_{pd}(s) := G(s)\Lambda(s)(I + C_{pd}(s)G(s)\Lambda(s))^{-1} = e^{-sT}(s - p + C_{pd}(s)e^{-sT})^{-1}$. *Choose any positive* $\gamma \in \mathbb{R}$ *satisfying* $\gamma < \min_{T\in\Theta} \| \frac{\alpha H_{pd}(s) - I}{s} \|$. *Then* $C_{pid}(s) = (p+\alpha) + \frac{\gamma\alpha}{s} + \frac{(p+\alpha)\hat{K}_d s}{\tau_d s+1}$ *is a PID-controller that stabilizes* $\frac{e^{-sT}}{s-p}$ *for all* $T \in \Theta$.

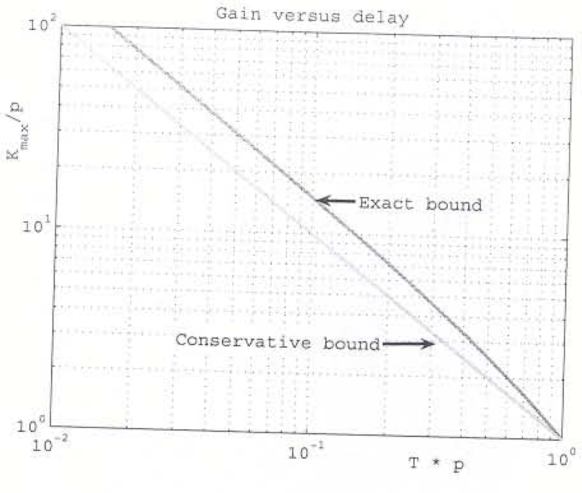

Figure 3: Maximum proportional gain versus pole-delay product pT.

Let us now consider proportional controller design for a fixed T and p in this example. It is easy to show that a stabilizing proportional controller exists if and only if $pT < 1$. Moreover, for any fixed $pT < 1$, there is a maximum allowable gain $K_{\max}$ for the proportional controller; this is shown in Figure 3 as the exact bound. On the other hand, our approach uses the small gain argument and leads to $C_p = (p+\alpha)$ as the controller gain. With $\|\Phi_\Lambda\| = \| T\,[\frac{e^{-sT}-1}{sT}]\,\| = T$, the condition $p < \|\Phi_\Lambda\|^{-1}$ is the same as $pT < 1$. From the bound given in (11), $\alpha < T^{-1} - p$; we see that the largest controller gain we can use in our case is $1/T$. This bound is also shown in Figure 3, which illustrates that the approach used here is not too conservative. The figure also demonstrates the difficulty of controlling this plant using a proportional controller when the product of the unstable pole with delay is relatively large. Other fundamental performance limitations can also be quantified, in terms of smallest achievable sensitivity level, [16], or mixed sensitivity H^∞ cost, [4, 3]. See [16] for further discussion and motivations for studying this benchmark problem.

It is also clear that by using the derivative term we can improve the bound on largest allowable pT. The largest pole delay product for which we can find a PD-controller is $1.38 = 1/0.725$, and that corresponds to $\tau_d \to 0$ and $\hat{K}_d/T = 0.31$, see Figure 4.

■

3.1.2 Plants with two $\mathcal{U}$-poles at $s=0$ or on the positive real-axis

Let $G \in \mathbf{R_p}^{r\times r}$ have full (normal) rank. Let G have no transmission-zeros at $s=0$. Let G have any number of poles in the stable region. Other than $\mathcal{U}$-poles at $s=p_1 \in \mathbb{R}$, $s=p_2 \in \mathbb{R}$, $p_1 \geq 0$, $p_2 \geq 0$, let G have no poles in the unstable region $\mathcal{U}$. Some or all of the entries of G contain one or both of the poles at p_1 and p_2. Therefore, G has an LCF $G = Y^{-1}X = [\,\frac{(s-p_1)(s-p_2)}{(bs+1)(as+1)}\,I\,]^{-1}\frac{(s-p_1)(s-p_2)}{(bs+1)(as+1)}\,G$, $p_1, p_2 \in \mathbb{R}$, $p_1 \geq 0$, $p_2 \geq 0$, and $a, b \in \mathbb{R}$, $a, b > 0$, where $\mathrm{rank}X(p_1) = \mathrm{rank}(s-p_1)(s-p_2)G(s)|_{s=p_1} = r$ and $\mathrm{rank}X(p_1) = \mathrm{rank}(s-p_1)(s-p_2)G(s)|_{s=p_2} = r$. Furthermore, since G has no transmission-zeros at $s=0$, $\mathrm{rank}X(0) = \mathrm{rank}(s-p_1)(s-p_2)G(s)|_{s=0} = r$. Proposition 3.4 shows that under certain assumptions on the poles at $p_1, p_2 \geq 0$, the delayed plant $G\Lambda$ admits PD and PID-controllers. Stabilizing P-controllers, D-controllers or I-controllers may not exist. For example, for $G = \frac{1}{(s-p_1)(s-p_2)}$, where $p_1 \geq 0$, $p_2 \geq 0$, $G\Lambda$ does not admit P-controllers $C_p = K_p$, or D-controllers $C_d = K_d s/(\tau_d s+1)$ for any $\tau_d > 0$, or I-controllers $C_i = K_i/s$.

Proposition 3.4 *Let $G \in \mathbf{R_p}^{r\times r}$, $\mathrm{rank}G(s) = r$. Let $p_1, p_2 \in \mathbb{R}$, $p_1 \geq 0$, $p_2 \geq 0$. Let $X = \frac{(s-p_1)(s-p_2)}{(as+1)(bs+1)}G \in \mathbf{S}^{r\times r}$, for $a, b \in \mathbb{R}, a, b > 0$. Let $X(0) = (s-p_1)(s-p_2)G(s)|_{s=0}$ be nonsingular, where $G^{-1}(0) = p_1 p_2 X(0)^{-1}$.*

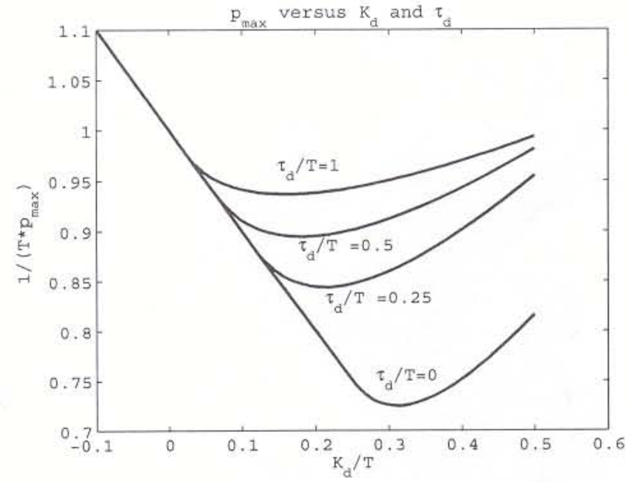

Figure 4: Maximum pole-delay product pT versus $\hat{K}_d$ and τ_d.

i) PD-design: Choose any $\tau_d > 0$. Define

$$\Phi_{1\Lambda} \;:=\; \frac{(\tau_d s+1)^{-1}(s-p_1)(s-p_2)\; G(s)\Lambda(s)\,X(0)^{-1} - I}{s} \;. \tag{14}$$

If $0 \le p_1 < \min_{\mathcal{T}\in\Theta} \;\|\, \Phi_{1\Lambda} \,\|^{-1}$, then choose any positive $\alpha \in \mathbb{R}$ satisfying

$$0 < \alpha < \min_{\mathcal{T}\in\Theta} \;\|\, \Phi_{1\Lambda} \,\|^{-1} - p_1 \;. \tag{15}$$

Define

$$\Phi_{2\Lambda} \;:=\; \frac{\alpha\,[\,I + \dfrac{(\alpha+p_1)}{\tau_d s+1}\,(s-p_2)\,G(s)\Lambda(s)\,X(0)^{-1}\,]^{-1}(s-p_2)\,G(s)\Lambda(s)\,X(0)^{-1} - I}{s} \;. \tag{16}$$

If $0 \le p_2 < \min_{\mathcal{T}\in\Theta} \;\|\, \Phi_{2\Lambda} \,\|^{-1}$, then choose any positive $\beta \in \mathbb{R}$ satisfying

$$0 < \beta < \min_{\mathcal{T}\in\Theta} \;\|\, \Phi_{2\Lambda} \,\|^{-1} - p_2 \;. \tag{17}$$

Let $K_p = (\alpha\,\beta - p_1\,p_2)X(0)^{-1}$, $K_d = (\alpha + p_1\,)\,(1+\tau_d\,p_2)X(0)^{-1}$; then

$$C_{pd} = (\alpha\,\beta - p_1\,p_2)\,X(0)^{-1} + \frac{(\alpha+p_1\,)\,(1+\tau_d\,p_2)\,X(0)^{-1}\,s}{\tau_d s+1} \tag{18}$$

is a PD-controller that stabilizes $G\Lambda$ for $\mathcal{T} \in \Theta$.
ii) PID design: Let C_{pd} be as in (18). Then

$$C_{pid} = (\alpha\,\beta - p_1\,p_2)\,X(0)^{-1} + \frac{\gamma\,\alpha\,\beta\,X(0)^{-1}}{s} + \frac{(\alpha+p_1\,)\,(1+\tau_d\,p_2)\,X(0)^{-1}\,s}{\tau_d s+1} \tag{19}$$

is a PID-controller that stabilizes $G\Lambda$ for $\mathcal{T} \in \Theta$, where $\gamma \in \mathbb{R}$ satisfies (13) with $H_{pd}(0)^{-1} = \alpha\,\beta\,X(0)^{-1}$.

■

Proof of Proposition 3.4: *i)* By (15), $W_d \;:=\; \dfrac{(s-p_1)}{as+1}I \;+\; (\alpha+p_1)\,\dfrac{(bs+1)}{\tau_d s+1}X(s)\Lambda(s)X(0)^{-1} \;=\; \dfrac{s-p_1}{as+1}I + \dfrac{(s-p_1)}{as+1}(\alpha+p_1)G(s)\Lambda(s)X(0)^{-1}\dfrac{(s-p_2)}{\tau_d s+1} = [\,I + \dfrac{(\alpha+p_1)\,s}{s+\alpha}\;\Phi_{1\Lambda}\,]\dfrac{(s+\alpha)}{as+1}$ is unimodular. Define $\hat{H} := (bs+1)W_d^{-1}\,X\Lambda$; then $\hat{H}(0)^{-1} = \alpha X(0)^{-1}$. By (18), $C_{pd} = (\alpha+p_1)\dfrac{(s-p_2)}{\tau_d s+1}X(0)^{-1} + \alpha(\beta +$

$p_2)X(0)^{-1}$. By (17), $M_{pd} := Y + X\Lambda C_{pd} = \frac{(s-p_2)}{bs+1}[\frac{(s-p_1)}{as+1}I + (\alpha+p_1)\frac{(bs+1)}{\tau_d s+1}X(s)\Lambda(s)X(0)^{-1}] + \alpha(\beta+p_2)X(s)\Lambda(s)X(0)^{-1} = W_d[\frac{s-p_2}{bs+1}I + W_d^{-1}\alpha(\beta+p_2)X(s)\Lambda(s)X(0)^{-1}] = W_d[\frac{s-p_2}{bs+1}I + \frac{\alpha\beta\hat{H}X(0)^{-1}}{bs+1}] = W_d[I + \frac{(\beta+p_2)s}{s+\beta}\Phi_{2\Lambda}]\frac{(s+\beta)}{bs+1}$ is unimodular since $b,\beta>0$ and W_d is unimodular. Therefore, C_{pd} stabilizes $G\Lambda$. *ii)* Since C_{pd} in (18) stabilizes G, $H_{pd} := M_{pd}^{-1}X\Lambda = G\Lambda(I+C_{pd}G\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$; $H_{pd}(0)^{-1} = K_p + X(0)^{-1}Y(0) = \alpha\beta X(0)^{-1}$. For any $\gamma\in\mathbb{R}$ satisfying (13), the I-controller $K_i/s = \gamma H_{pd}(0)^{-1}/s$ stabilizes H_{pd}. By Lemma 2.1, $C_{pid} = C_{pd} + K_i/s$ in (19) stabilizes $G\Lambda$. ■

3.1.3 Plants with one pair of imaginary poles

Let $G \in \mathbf{R_p}^{r\times r}$ have full (normal) rank. Let G have no transmission-zeros at $s=0$. Let G have any number of poles in the stable region. Other than a complex conjugate pair of $j\omega$-axis poles at $s=\pm j\rho\in\mathbb{C}$, $\rho>0$, let G have no poles in the unstable region $\mathcal{U}$. The poles at $s=\pm j\rho$ may appear in some or all of the entries of G. Therefore, G has an LCF $G = Y^{-1}X = [\frac{(s^2+\rho^2)}{(as+1)(bs+1)}I]^{-1}\frac{(s^2+\rho^2)}{(as+1)(bs+1)}G$, $\rho\in\mathbb{R}$, $\rho>0$ and $a,b\in\mathbb{R}$, $a,b>0$, where $\text{rank}X(j\rho) = \text{rank}(s^2+\rho^2)G(s)|_{s=j\rho} = r$. Furthermore, since G has no transmission-zeros at $s=0$, $X(0)=\rho^2G(0)$ is nonsingular. Under certain assumptions on the magnitude ρ of the poles at $s=\pm j\rho$, the delayed plant $G\Lambda$ admits D, PD, ID, PID-controllers. P-controllers or I-controllers may not exist for plants in this class. For example, $G=\frac{1}{s^2+\rho^2}$ does not admit P-controllers or I-controllers for any $\rho\geq 0$.

Proposition 3.5 develops a PID-controller synthesis for plants with one complex conjugate pair of $j\omega$-axis poles with "small" magnitude, subject to the norm-bound $\rho < 0.5\|\Psi_\Lambda\|^{-1}$ as defined in (20). Note that $\|\Psi_\Lambda\|\geq 1/z_{min}$ implies $\rho<0.5\|\Psi_\Lambda\|^{-1}\leq 0.5z_{min}$, i.e., the magnitude ρ is smaller than half of the smallest positive zero z_{min}.

Proposition 3.5 *Let* $G\in\mathbf{R_p}^{r\times r}$, $\text{rank}G(s)=r$. *Let* $\rho\in\mathbb{R}$, $\rho>0$. *Let* $X = \frac{(s^2+\rho^2)}{(as+1)(bs+1)}G \in \mathcal{M}(\mathbf{S})$, *for* $a,b\in\mathbb{R}$, $a,b>0$. *Let* $\text{rank}X(p) = \text{rank}(s^2+\rho^2)G(s)|_{s=j\rho} = r$. *Let* $X(0)=\rho^2G(0)$ *be nonsingular.*
i) PD-design: Choose any $\tau_d>0$. *Define*

$$\Psi_\Lambda := \frac{(\tau_d s+1)^{-1}(s^2+\rho^2)G(s)\Lambda(s)\frac{G(0)^{-1}}{\rho^2} - I}{s}. \tag{20}$$

If $0<\rho<\frac{1}{2}\min_{\mathcal{T}\in\Theta}\|\Psi_\Lambda\|^{-1}$, *then choose* $\alpha,\beta\in\mathbb{R}$, $\alpha,\beta\geq 0$, *satisfying*

$$0\leq\alpha+\beta<\min_{\mathcal{T}\in\Theta}\|\Psi_\Lambda\|^{-1}-2\rho. \tag{21}$$

Let $K_p = [\alpha\beta+\rho(\alpha+\beta)]X(0)^{-1}$, $K_d = (\alpha+\beta+2\rho)X(0)^{-1} - \tau_d K_p$; *then*

$$C_{pd} = K_p + \frac{K_d s}{\tau_d s+1} = [\frac{(\alpha+\beta+2\rho)s+\alpha\beta+\rho(\alpha+\beta)}{\tau_d s+1}]\frac{G(0)^{-1}}{\rho^2} \tag{22}$$

is a PD-controller that stabilizes $G\Lambda$ *for* $\mathcal{T}\in\Theta$. *If* $\alpha=\beta=0$, *(22) is a D-controller.*
ii) PID-design: Let $0<\rho<\frac{1}{2}\min_{\mathcal{T}\in\Theta}\|\Psi_\Lambda\|^{-1}$. *Let* C_{pd} *be as in (22). Then*

$$C_{pid} = [\frac{(\alpha+\beta+2\rho)s+\alpha\beta+\rho(\alpha+\beta)}{\tau_d s+1} + \frac{\rho^2+\alpha\beta+\rho(\alpha+\beta)}{s}]\frac{G(0)^{-1}}{\rho^2} \tag{23}$$

is a PID-controller that stabilizes $G\Lambda$ *for* $\mathcal{T}\in\Theta$, *where* $\gamma\in\mathbb{R}$ *satisfies (13) with* $H_{pd}(0)^{-1} = [\rho^2+\alpha\beta+\rho(\alpha+\beta)]X(0)^{-1}$. *If* $\alpha=\beta=0$, *(23) is an ID-controller.*
■

Proof of Proposition 3.5: *i)* Define $\frac{n(s)}{d(s)} := \frac{s\,[\,(\alpha+\beta+2\rho)\,s+\alpha\beta+\rho(\alpha+\beta)\,]}{(s+\alpha+\rho)(s+\beta+\rho)}$. By (21), $\|\frac{n(s)}{d(s)}\Psi_\Lambda\| \le (\alpha+\beta+2\rho)\Psi_\Lambda < 1$ implies $M_{pd} := Y + X\Lambda C_{pd} = \frac{(s^2+\rho^2)}{(as+1)(bs+1)}[I + G\Lambda C_{pd}] = [I + \frac{n(s)}{d(s)}\Psi_\Lambda]\frac{d(s)}{(as+1)(bs+1)}$ is unimodular since $a, b > 0, \alpha, \beta \ge 0$. Therefore, C_{pd} stabilizes $G\Lambda$. *ii)* Since C_{pd} in (22) stabilizes $G\Lambda$, $H_{pd} := M_{pd}^{-1}X\Lambda = G\Lambda(I + C_{pd}G\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$; $H_{pd}(0)^{-1} = G^{-1}(0) + K_p$. For any $\gamma \in \mathbb{R}$ satisfying (13), the I-controller $K_i/s = \gamma H_{pd}(0)^{-1}/s$ stabilizes H_{pd}. By Lemma 2.1, $C_{pid} = C_{pd} + K_i/s$ in (23) stabilizes $G\Lambda$. ∎

4 Conclusions

We showed existence of stabilizing PID-controllers for a class of LTI, MIMO plants with delays in the input channels. Moreover, for plants with only one or two unstable poles we have given explicit formulae for PID controller parameters. These results are obtained from a small gain based argument. Therefore, they are conservative. We were able to quantify the level of conservatism on an SISO benchmark example.

Plants with more than two poles on the positive real-axis do not necessarily admit PID-controllers even if they are strongly stabilizable. Further assumptions are needed on such plants, which would impose restrictions on the transmission-zeros. For example, as it can be easily shown in the case of $\frac{1}{(s-p)^3}$ or $\frac{1}{(s-p)(s^2+p^2)}$ for $p \ge 0$, many plants that have more than two poles in the unstable region cannot be stabilized using PID-controllers even when the magnitudes of these poles are "small".

For MIMO plants with output delays, if Y^{-1} commutes with the delay matrix Λ (recall that the finite dimensional part of the plant was factored as $G = Y^{-1}X$), then the proofs remain valid. Indeed, all explicit controller design problems studied here had diagonal Y^{-1}, hence the results are valid for output delay case as well (of course with necessary minor modifications). Similarly, we can handle the case of both input and output delays, again by making minor modifications in the proofs under the assumption that Y^{-1} commutes with the output channel delay matrix. Details will be included in the full version of the paper which is in preparation.

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