Plant Order Reduction for Controller Design

A. B. Özgüler and A. N. Gündeş[1]

Electrical and Electronics Engineering
Bilkent University, Ankara, Turkey 06800
ozguler@ee.bilkent.edu.tr

Electrical and Computer Engineering
University of California, Davis, CA 95616
gundes@ece.ucdavis.edu

Abstract

Two dual methods of plant order reduction for controller design are proposed for linear, time-invariant, multi-input multi-output systems. The model reduction methods are tailored towards closed-loop stability and performance and they yield estimates for the stability robustness and performance of the final design. They can be considered as formalizations of two classical heuristic model reduction techniques: One method neglects a plant-pole sufficiently far to the left of dominant poles and the other cancels a sufficiently small stable plant-zero with a pole at the origin.

1 Introduction

In spite of numerous simplifying assumptions and approximations already performed at the modelling stage, an acceptable controller design for a linear plant may necessitate further simplifications. Since the number of plant poles and zeros directly influence the complexity of design, the simplification required is almost always in the form of "order reduction", both of the plant model and of the controller (to be) designed. Hence, many approximation methods of order reduction were proposed for linear time-invariant (LTI) systems.

Some old and simple methods of order reduction such as those surveyed in [2] remain obscure either because they offer no guaranteed performance or because they do not provide closed-form solutions. Among rigorous model reduction methods that come with some kind of a performance criterion, three are notable and best known: The balanced realization method [13], the Hankel norm approximation method [1, 11, 9], and the q-covariance equivalent method [20]. Irrespective of various extensions that have resulted in frequency weighted approximations and a more detailed analysis of error bounds, all three methods essentially apply to stable plants. In the case of an unstable plant, the reduction is performed only on the stable part after writing the plant as the sum of a stable and an anti-stable plant.

The closed-loop performance of reduced order models when used for the purpose of control system design is not sufficiently investigated. An exception is [3], where a fractional representation based controller reduction method is proposed and the methods are examined from the viewpoint of controller reduction and the associated loss of performance. The main difficulty with closed-loop performance assessment is that a satisfactory model reduction for control system design requires knowledge of the controller in advance and *vice versa* [3]. This brings a logical circularity into the whole process. The situation is similar in model identification, where the end-use of the model to be identified makes a huge difference in the identification procedure.

Since the difference between the plant model and its approximation can be considered as a perturbation on the plant, stability and performance of approximate models in a closed-loop system can be studied by the existing robust controller design tools (e.g., [5, 3]). While it is possible to obtain order reduction based design methods by using such results, these usually cannot yield explicit error bounds for stability and performance.

The motivation for this paper comes from perhaps the oldest simple reduction techniques covered in classical control textbooks such as [12, 17]. The first heuristic method relies on identifying dominant *versus* insignificant poles. The basic rule is that, poles having at least 5 times as large real parts as poles which are nearest the $j\omega$-axis are considered insignificant [12], provided there are no zeros nearby [14]. Such poles can be deleted from a transfer function making sure the low-frequency gain is unchanged and design can be carried out on the reduced order plant, in hopes of resulting in an acceptable controller for the original plant. The dominant pole based approximation is widely used on a closed-loop transfer function for analysis purposes. Occasionally, such approximations are also used on the open-loop transfer functions ([12], p. 416). The second heuristic method is part of a specific proportional-integral-derivative (PID) controller design. The PI part of a PID controller is usually employed to improve the steady-state (low-frequency) performance of a system since it increases the open-loop system type. If there are additional transient performance specifications, then a controller zero is placed much closer to the origin than any other stable plant pole and the requirement is satisfied as if the controller is a proportional one. In other words, the cascade of the PI controller and the plant transfer functions is approximated by the original plant and any further design proceeds with a proportional controller ([12], p. 695). Since PID controllers can be designed by cascading consecutive PI and PD design stages, this method simplifies the second stage. These two seemingly contradictory heuristic

[1]Research supported by the NSF Grant ECS-9905729.

methods were justified and shown to be dual model reduction methods in [15]. The purpose of this paper is to further justify these approximation techniques from the viewpoint of performance and formalize them as systematic design methods.

The approximation-based design results here apply to multi-input multi-output (MIMO) systems. Theorem 1 shows that, if a stabilizing controller achieving a sufficiently quenched complementary sensitivity function at high-frequencies can be determined for the reduced plant obtained by deleting candidate insignificant left-half plane poles from a given plant, then the same controller stabilizes the original plant and achieves a complementary sensitivity with similar high-frequency characteristics. Theorem 2 shows that, if a stabilizing controller achieving a sufficiently quenched sensitivity function at low-frequencies can be determined for the reduced plant obtained by cancelling candidate insignificant left-half plane zeros with poles at zero, then the same controller stabilizes the original plant and achieves a sensitivity with similar low-frequency characteristics. An iterative application of each result starting with the left-most pole or the right-most zero yields a model reduction based design algorithm, ensuring a certain degree of stability robustness and performance for the closed-loop system at each stage.

The set of stable proper real rational functions of s (real-rational H_∞ functions) is denoted by $\mathbf{S}$; matrices whose entries are in $\mathbf{S}$ is denoted by $\mathcal{M}(\mathbf{S})$. The H_∞-norm of a matrix $M(s) \in \mathcal{M}(\mathbf{S})$ is denoted by $\|M(s)\|$ (i.e., for $M \in \mathcal{M}(\mathbf{S})$, the norm $\|\cdot\|$ is defined as $\|M\| = \sup_{s\in\partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ denotes the maximum singular value and $\partial\mathcal{U}$ denotes the boundary of the extended closed right-half-plane $\mathcal{U}$). We also denote the real, complex, and left-half plane complex numbers by $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{C}_-$. For simplicity, we drop (s) in transfer matrices such as $G(s)$.

2 Main Results

A set $E := \{\epsilon_i \in \mathbb{C}\ ,\ i = 1, ..., q\}$ is called *conjugate symmetric* if for every $\epsilon_i \notin \mathbb{R}$ in the set E, the complex-conjugate $\bar{\epsilon}_i$ is also in the set E. We assume ϵ_i and $\bar{\epsilon}_i$ are assigned consecutive indices for each $\epsilon_i \notin \mathbb{R}$.

2.1 Insignificant Poles

Consider the unity-feedback system shown in Figure 1. Let G be the plant's transfer matrix, C be the controller's transfer matrix. Let $G = ND^{-1}$ be a right-coprime-factorization (RCF), $C = D_c^{-1}N_c$ be a left-coprime-factorization (LCF) over $\mathbf{S}$. For $k \geq 1$, define

$$G_0 := G\ ,\ G_k := \frac{G_{k-1}}{(\epsilon_k s+1)} = \frac{G}{\prod_{i=1}^{k}(\epsilon_i s+1)}, \quad (1)$$

$$N_0 := N\ ,\ N_k = \frac{N_{k-1}}{(\epsilon_k s+1)} = \frac{N}{\prod_{i=1}^{k}(\epsilon_i s+1)}. \quad (2)$$

Then $G_k = N_k D^{-1}$ is an RCF of G_k. With G_k as the plant in the unity-feedback control system, let the sensitivity function S_k and the complementary sensitivity function $T_k = I - S_k$ be given by

$$S_k = (I + G_k C)^{-1},\ T_k = G_k C(I + G_k C)^{-1}. \quad (3)$$

The input-to-error and the input-to-output transfer-functions are $H_{er} = S_k$, $H_{yr} = I - H_{er} = T_k = I - S_k$.

The following lemma roughly states that if C is a stabilizing controller for a plant G, then we can add any number of poles in the stable region to G and it is still stabilized by the same controller as long as these poles are "sufficiently far from the imaginary axis". This was stated in [18] for scalar plants with stable controllers; it has also been independently used in [10] to establish a simultaneous stabilization result. This lemma can also be proved as a corollary of the result in [5].

In Lemma 1, it is assumed that GC is strictly-proper, equivalently $T_0 = GC(I + GC)^{-1}$ is strictly-proper, $S_0(\infty) = I$. This assumption is automatically satisfied if G or C is strictly-proper. Any stabilizing controller $C = D_c^{-1}N_c$ can be modified to be strictly-proper, for example as $C' = ((I + BN_cN)D_c)^{-1}(I - BD_cD)N_c$ where $B := (D_cD)(\infty)^{-1}$. Therefore, there is no loss of generality in assuming GC is strictly-proper, with the controller chosen as strictly-proper as necessary.

Lemma 1. *Let a plant G be stabilized by a controller C, where GC is strictly-proper.* **a)** *For $\epsilon_k \in \mathbb{R}$, $\epsilon_k > 0$, if*

$$\epsilon_k < \|sT_{k-1}\|^{-1}, \quad (4)$$

then the same C also stabilizes the higher-order plant $G_k = \frac{G_{k-1}}{(\epsilon_k s+1)}$. **b)** *For $\epsilon_k \notin \mathbb{R}$, $-\epsilon_k \in \mathbb{C}_-$, let $\epsilon_{k+1} = \bar{\epsilon}_k$ and define $r_k := \frac{1}{\mathcal{R}e(1/\epsilon_k)}$; if*

$$2r_k < \|sT_{k-1}\|^{-1}, \quad (5)$$

then the same C also stabilizes the higher-order plant $G_{k+1} = \frac{G_{k-1}}{(\epsilon_k s+1)(\bar{\epsilon}_k s+1)}$. **c)** *For any conjugate symmetric set $\{-\epsilon_i \in \mathbb{C}_-, i = 1, ..., q\}$, where $\epsilon_i \in \mathbb{R}$ satisfies (4) and $\epsilon_i \notin \mathbb{R}$ satisfies (5), the controller C also stabilizes the higher-order plant $G_q = \frac{G}{\prod_{i=1}^{q}(\epsilon_i s+1)}$.*

Proof. Let $G = ND^{-1}$ be an RCF and let $C = D_c^{-1}N_c$ be an LCF. For $k \geq 0$, define $U_k := D_cD + N_cN_k$. The controller C stabilizes G_k if and only if U_k is unimodular, i.e., $U_k^{-1} \in \mathcal{M}(\mathbf{S})$. By assumption, $U_0 = D_cD + N_cN$ is unimodular since C stabilizes G. We show that C also stabilizes G_k by induction: For $k \geq 1$, $U_k = U_{k-1} - \frac{\epsilon_k s}{\epsilon_k s+1} N_c N_{k-1}$. If C stabilizes G_{k-1}, then U_{k-1} is unimodular. Since GC is strictly proper, so is $T_0 = GC(I + GC)^{-1} = NU_0^{-1}N_c$; hence, for $k \geq 1$, $sT_{k-1} = (sN_{k-1}U_{k-1}^{-1}N_c) \in \mathcal{M}(\mathbf{S})$. **a)** Define $\frac{1}{\epsilon_k s+1} =: x_k$. For $\epsilon_k \in \mathbb{R}$, $U_{k-1}^{-1}U_k = I - \frac{\epsilon_k s}{\epsilon_k s+1} U_{k-1}^{-1} N_c N_{k-1}$ is unimodular if and only if $I - \epsilon_k x_k (sN_{k-1}U_{k-1}^{-1}N_c) = I - \epsilon_k x_k sT_{k-1}$ is unimodular. By (4), $\|x_k\| = 1$ implies $\|\epsilon_k x_k sT_{k-1}\| \leq \|\epsilon_k x_k\| \|sT_{k-1}\| = \epsilon_k \|sT_{k-1}\| < 1$. Therefore, $U_{k-1}^{-1}U_k$ is unimodular, equivalently, U_k is unimodular, and hence, C stabilizes G_k. **b)** For $\epsilon_k \notin \mathbb{R}$, define $a + jb := (1/\epsilon_k)$, where $a, b \in \mathbb{R}$, $a > 0$ since $-\epsilon_k \in \mathbb{C}_-$, and without loss of generality, $b > 0$ since

Proceedings of the American Control Conference
Denver, Colorado June 4-6, 2003

$\epsilon_{k+1} = \bar{\epsilon}_k$. We find an upper-bound on $\|\frac{1-x_k\bar{x}_k}{s}\| = \|\frac{s+2a}{s^2+2as+a^2+b^2}\| \in \mathbf{S}$ as follows: Consider $\omega \geq 0$. Let $d^2 := |a^2+b^2-\omega^2+2aj\omega|^2 = ((b^2-\omega^2)^2+a^4+2a^2(b^2+\omega^2))$; then $d \geq |b^2-\omega^2|$. Consider two cases: (i) If $a \leq b$, then $|2a+j\omega| \leq \sqrt{4b^2+4\omega^2} \leq 2(b+\omega)$. Therefore, $\frac{|2a+j\omega|}{d} \leq \frac{2(b+\omega)}{(b^2-\omega^2)} = \frac{2}{|b-\omega|} \leq 2/b \leq 2/a$. (ii) If $a > b$, then $|2a+j\omega| \leq \sqrt{4a^2+4\omega^2} \leq 2(a+\omega)$. Also, $(4a^2-2b^2) \geq 0$ implies $d^2 = ((a^2-\omega^2)^2 + b^4 + 2a^2b^2 + (4a^2-2b^2)\omega^2)) \geq (a^2-\omega^2)^2$. Therefore, $\frac{|2a+j\omega|}{d} \leq \frac{2(a+\omega)}{(a^2-\omega^2)} = \frac{2}{|a-\omega|} \leq 2/a$. We conclude that $\|\frac{1-x_k\bar{x}_k}{s}\| \leq \frac{2}{a} = \frac{2}{\mathcal{R}e(1/\epsilon_k)} = 2r_k$. With $\epsilon_{k+1} = \bar{\epsilon}_k$, $U_{k-1}^{-1}U_{k+1} = I - (1-x_k\bar{x}_k)U_{k-1}^{-1}N_cN_{k-1}$ is unimodular if and only if $I - \frac{(1-x_k\bar{x}_k)}{s}sT_{k-1}$ is unimodular. By (5), $\|\frac{(1-x_k\bar{x}_k)}{s}sT_{k-1}\| \leq \|\frac{1-x_k\bar{x}_k}{s}\|\|sT_{k-1}\| \leq 2r_k\|sT_{k-1}\| < 1$. Therefore, $U_{k-1}^{-1}U_{k+1}$ is unimodular, equivalently, U_{k+1} is unimodular; hence, C stabilizes G_{k+1}. **c)** It follows by induction from (a) and (b) that the plant G_k for $k = q$ is also stabilized by the same C. ∎

Lemma 1 justifies and generalizes to the MIMO case methods in which a stabilizing controller is determined by neglecting the insignificant poles in a loop-gain transfer function and performing the design on the lower order approximation G. The terms that are discarded are such that the low-frequency gain $G(0)$ of G and of (1) are the same. A real pole $-\frac{1}{\epsilon_i}$ of (1) is *insignificant* if $-\frac{1}{\epsilon_i} < -\alpha_i$, where $\alpha_i := \|sT_{i-1}\|$, i.e., if it is sufficiently far on the left-half plane. Based on condition (5), a complex-conjugate pair of insignificant poles lies to the left of a line at $-2\alpha_i$.

The definition of α_i's obviously depends on the controller choice, making the definition of an insignificant pole circular. Theorem 1 removes this circularity by an iterative procedure resulting in a design algorithm.

Theorem 1. *Let C be a stabilizing controller for the plant G_k for some $k \in \{0, \ldots, q-1\}$, where $\{-\epsilon_i \in \mathbf{C}_-, i = 1, ..., k\}$ is a conjugate symmetric set. Let C be such that GC is strictly-proper. For $k+1 \leq i \leq q$, let $\alpha_i := \|sT_{i-1}\|$. Suppose that $\{-\epsilon_j \in \mathbf{C}_-,\ j = k+1, ..., i,\ k+1 \leq i \leq q\}$ is a conjugate symmetric set and that there exists a real $\delta > 0$ such that*

$$\|sT_k\| \leq (\delta + \sum_{j=k+1}^{q} r_j)^{-1}\ . \tag{6}$$

Under these assumptions: **a)** *When $\epsilon_i \in \mathbb{R}$, the same controller C also stabilizes $G_i = \frac{G_{i-1}}{\epsilon_i s+1}$. Furthermore,*

$$\|sT_i\| \leq (\ \delta + \sum_{j=i+1}^{q} r_j\ )^{-1} \tag{7}$$

and the following sensitivity and complementary sensitivity bounds are achieved:

$$\begin{array}{l}\frac{1}{1+\epsilon_i\alpha_i}\|S_{i-1}\| \leq \|S_i\| \leq \frac{1}{1-\epsilon_i\alpha_i}\|S_{i-1}\|,\\ \frac{1}{1+\epsilon_i\alpha_i}\|T_{i-1}\| - \frac{\epsilon_i\alpha_i}{1+\epsilon_i\alpha_i} \leq \|T_i\| \leq \frac{1}{1-\epsilon_i\alpha_i}\|T_{i-1}\|.\end{array} \tag{8}$$

b) *When $\epsilon_i \notin \mathbb{R}$, $\epsilon_{i+1} = \bar{\epsilon}_i$ (where $a_i + jb_i := (1/\epsilon_i)$, $a_i, b_i \in \mathbb{R}$, $a_i, b_i > 0$), the same controller C also stabilizes $G_{i+1} = \frac{G_{i-1}}{(\epsilon_i s+1)(\bar{\epsilon}_i s+1)}$. Furthermore,*

$$\|sT_{i+1}\| \leq \frac{(a_i^2+b_i^2)}{2a_ib_i}(\ \delta + \sum_{j=i+2}^{q} r_j\ )^{-1}, \tag{9}$$

and the following sensitivity and complementary sensitivity bounds are achieved:

$$\begin{array}{l}\frac{1}{1+2r_i\alpha_i}\|S_{i-1}\| \leq \|S_{i+1}\| \leq \frac{1}{1-2r_i\alpha_i}\|S_{i-1}\|,\\ \frac{1}{1+2r_i\alpha_i}\|T_{i-1}\| - \frac{2r_i\alpha_i}{1+2r_i\alpha_i} \leq \|T_{i+1}\|\\ \qquad \leq \frac{1}{1-2r_i\alpha_i}\|T_{i-1}\| + \frac{2r_i\alpha_i}{1-2r_i\alpha_i}.\end{array} \tag{10}$$

Proof. For a proof by induction, suppose G_{i-1} is stabilized by C and that (6) holds for $(i-1)$, i.e., $\delta + \sum_{j=i}^{q} r_j \leq \|sT_{i-1}\|^{-1}$. **a)** Let $\epsilon_i \in \mathbb{R}$. By Lemma 1, $\epsilon_i = r_i < \|sT_{i-1}\|^{-1}$ implies G_i is also stabilized by C. With $x_i := \frac{1}{\epsilon_i s+1}$, $T_i = N_iU_i^{-1}N_c = N_iU_i^{-1}U_{i-1}U_{i-1}^{-1}N_c = N_iU_i^{-1}(U_i + \epsilon_i x_i s N_c N_{i-1})U_{i-1}^{-1}N_c = x_i[N_{i-1}U_{i-1}^{-1}N_c + \epsilon_i N_i U_i N_c s N_{i-1} U_{i-1}^{-1} N_c]$ implies

$$T_i = x_i\ (\ T_{i-1} + \epsilon_i T_i s T_{i-1}\ )\ . \tag{11}$$

By (11), $\|sT_i\| = \|x_i(sT_{i-1} + \epsilon_i sT_i sT_{i-1})\| \leq \|(1+\epsilon_i\|sT_i\|)\|\|sT_{i-1}\| \leq (1+\epsilon_i\|sT_i\|)(\delta + \sum_{j=i}^{q} r_j)^{-1}$. Therefore, $[1 - \epsilon_i(\delta + \sum_{j=i}^{q} r_j)^{-1}]\|sT_i\| \leq (\delta + \sum_{j=i}^{q} r_j)^{-1}$ implies that (7) holds. The bounds in (8) follow from (11); $\|T_i\| \leq \|T_{i-1}\| + \epsilon_i\|T_i\|\|sT_{i-1}\|$ implies $(1-\epsilon_i\alpha_i)\|T_i\| \leq \|T_{i-1}\|$, which establishes the upper-bound on $\|T_i\|$; $T_i = (1-\epsilon_i x_i s)T_{i-1} + \epsilon_i x_i T_i s T_{i-1}$ implies $T_{i-1} = (1-\epsilon_i x_i s T_{i-1})T_i + \epsilon_i x_i s T_{i-1}$; therefore $\|T_{i-1}\| \leq (1+\epsilon_i\alpha_i)\|T_i\| + \epsilon_i\alpha_i$ establishes the lower-bound on $\|T_i\|$. The bounds on $\|S_i\|$ follow by writing (11) as $I - S_i = (1-\epsilon_i x_i s)T_{i-1} + \epsilon_i x_i(I - S_i)sT_{i-1} = I - S_{i-1} - \epsilon_i x_i S_i s T_{i-1}$; therefore, $\|S_{i-1}\| - \epsilon_i\alpha_i\|S_i\| \leq \|S_i\| \leq \|S_{i-1}\| + \epsilon_i\alpha_i\|S_i\|$. **b)** Let $\epsilon_i \notin \mathbb{R}$. Since $\epsilon_{i+1} = \bar{\epsilon}_i$ implies $r_i = r_{i+1}$, (6) implies $\delta + 2r_i + \sum_{j=i+2}^{q} r_j \leq \|sT_{i-1}\|^{-1}$. By Lemma 1, G_{i+1} is also stabilized by C since $2r_i < \|sT_{i-1}\|^{-1}$. By $T_{i+1} = N_{i+1}U_{i+1}^{-1}N_c = N_{i+1}U_{i+1}^{-1}U_{i-1}U_{i-1}^{-1}N_c = N_{i+1}U_{i+1}^{-1}(U_{i+1} + (1 - x_i\bar{x}_i)N_cN_{i-1})U_{i-1}^{-1}N_c = x_i\bar{x}_iN_{i-1}U_{i-1}^{-1}N_c + N_{i+1}U_{i+1}^{-1}N_c(1-x_i\bar{x}_i)N_{i-1}U_{i-1}^{-1}N_c$,

$$T_{i+1} = x_i\bar{x}_iT_{i-1} + \frac{(1-x_i\bar{x}_i)}{s}T_{i+1}sT_{i-1}\ . \tag{12}$$

We find an upper-bound on $\|x_i\bar{x}_i\| = \frac{a_i^2+b_i^2}{s^2+2a_is+a_i^2+b_i^2}$: For $a_i \leq b_i$, $\|x_i\bar{x}_i\| = (a_i^2+b_i^2)/2a_ib_i$; for $a_i > b_i$, $\|x_i\bar{x}_i\| = 1$. But $(a_i - b_i)^2 > 0$ implies $(a_i^2+b_i^2)/2a_ib_i > 1$; hence, $\|x_i\bar{x}_i\| \leq (a_i^2+b_i^2)/2a_ib_i$. Therefore, (12) implies $\|sT_{i+1}\| \leq \frac{a_i^2+b_i^2}{2a_ib_i}\|sT_{i-1}\| + 2r_i\|sT_{i+1}\|\|sT_{i-1})\| \leq \left(\frac{a_i^2+b_i^2}{2a_ib_i} + 2r_i\|sT_{i+1}\|\right)(\delta + \sum_{j=i}^{q} r_j)^{-1}$; condition (9) follows from $(1 - 2r_i(\delta + 2r_i + \sum_{j=i+2}^{q} r_j)^{-1})\|sT_{i+1}\| \leq \frac{a_i^2+b_i^2}{2a_ib_i}(\delta + \sum_{j=i}^{q} r_j)^{-1}$. The bounds in (10) follow from (12); $\|T_{i+1}\| \leq \|T_{i-1}\| + 2r_i(1 + \|T_{i+1}\|)\|sT_{i-1}\|$ implies $(1 - 2r_i\alpha_i)\|T_{i+1}\| \leq \|T_{i-1}\| + 2r_i\alpha_i$, which establishes

Proceedings of the American Control Conference
Denver, Colorado June 4-6, 2003

the upper-bound on $\|T_{i+1}\|$; $T_{i-1} = T_{i+1} + \frac{(1-x_i\bar{x}_i)}{s}(1 - T_{i+1})sT_{i-1}$ implies $\|T_{i-1}\| \leq (1+2r_i\alpha_i)\|T_{i+1}\| + 2r_i\alpha_i$, which establishes the lower-bound on $\|T_{i+1}\|$. The bounds on $\|S_{i+1}\|$ follow by writing (12) as $I - S_{i+1} = I - S_{i-1} - (1-x_i\bar{x}_i)T_{i-1} + (1-x_i\bar{x}_i)(I - S_{i+1})T_{i-1}$, i.e., $S_{i+1} = S_{i-1} + \frac{(1-x_i\bar{x}_i)}{s}S_{i+1}sT_{i-1}$; therefore, $\|S_{i-1}\| - 2r_i\alpha_i\|S_{i+1}\| \leq \|S_{i+1}\| \leq \|S_{i-1}\| + 2r_i\alpha_i\|S_{i+1}\|$. ∎

Remarks: 1) Condition (6) is a high-frequency performance requirement on the plant G_k. In the scalar case, this condition is equivalent to $\sup_{\omega\geq 0} |\omega|\,|T_k(j\omega)| \leq (\delta + \sum_{j=k+1}^{q} r_j)^{-1}$, which implies

$$|T_k(j\omega)| \leq (\ \omega\,(\delta + \sum_{j=k+1}^{q} r_j)\ )^{-1}\ ,\ \forall\ \omega \geq 0.$$

This means in particular that $|T_k(j\omega)| < 1$ for all $\omega \geq (\sum_{j=k+1}^{q} r_j)^{-1}$. By Theorem 1, a similar performance holds true for each plant G_i, $i \in [k+1, q]$, stabilized by the same controller. If G_i has a pole in the open-right-half plane and its associated complementary sensitivity function T_i has small magnitude over some frequency range, then its H_∞-norm must necessarily get large ([8], section V). The bounds in (8) show that the $\|T_i\|$'s (and $\|S_i\|$'s) nevertheless remain bounded by a multiple of $\|T_k\|$ ($\|S_k\|$, repectively). In the MIMO case, (6) implies

$$\bar{\sigma}(T_k(j\omega)) \leq (\ \omega\,(\delta + \sum_{j=k+1}^{q} r_j)\ )^{-1}\ ,\ \forall\ \omega \geq 0.$$

△

2) The high-frequency requirement (6) can be represented in terms of the plant G_k and a nominal stabilizing controller C_o for G_k. Let $C_o = D_{co}^{-1}N_{co}$ be an LCF such that $U_k = D_{co}D + N_{co}N_k = I$. All stabilizing controllers for G_k are expressed as $(D_{co} - Q\tilde{N}_k)^{-1}(N_{co} + Q\tilde{D}_k)$, where $G_k = \tilde{D}^{-1}\tilde{N}_k$ is any LCF of G_k, and $Q \in \mathcal{M}(\mathbf{S})$. Suppose that for some $\delta > 0$,

$$\min_Q \|sN_k(N_{co} + Q\tilde{D}_k)\| \leq (\delta + \sum_{j=k+1}^{q} r_j)^{-1}; \quad (13)$$

the minimum is taken over all $Q \in \mathcal{M}(\mathbf{S})$ *such that* $N_k(N_{co} + Q\tilde{D}_k)$ is strictly-proper. If Q_* denotes the argument minimum of the left hand side, then the controller $D_c^{-1}N_c := (D_{co} - Q_*\tilde{N}_k)^{-1}(N_{co} + Q_*\tilde{D}_k)$ satisfies $D_cD + N_cN_k = I = U_k$ and $\|sN_kU_k^{-1}N_c\| = \|sN_k(N_{co} + Q\tilde{D}_k)\| \leq (\delta + \sum_{j=k+1}^{q} r_j)^{-1}$, so (6) holds. Checking if (13) holds requires the solution of a well-known H_∞-problem [7, 4], with *weights* $U_k = I$. △

3) Using the consequence (7) of (6), $\alpha_i \leq (\delta + \sum_{j=i}^{q} r_j)^{-1}$ for $i \in [1, q]$. Conditions (8) hence remain valid when $(\delta + \sum_{j=i}^{q} r_j)^{-1}$ replaces α_i everywhere it occurs. This gives sensitivity and complementary sensitivity bounds in terms of insignificant poles and the positive constant δ. The resulting bounds, however, are looser than the bounds in terms of α_i. △

4) Theorem 1 provides an iterative reduction procedure, *which normally starts out without any of the left-half plane poles* $\{-1/\epsilon_i, i = 1, ..., q\}$ and checks if (6) can be satisfied by a stabilizing controller for G. If not, then the pole(s) $-1/\epsilon_i$ are appended one at a time to G, starting with the one "closest" to the imaginary axis. In the case of real poles, if $\epsilon_i < \epsilon_j$ for some $i, j \in [1, q]$, then the pole $-1/\epsilon_j$ is closer to the imaginary axis, i.e., $-1/\epsilon_j > -1/\epsilon_i$. To see why it is reasonable to start the reduction algorithm by appending the right-most real pole to increase the order, consider two possibilities, $G_1^\ell = \frac{1}{\epsilon_1^\ell s+1}G$, $G_1^m = \frac{1}{\epsilon_1^m s+1}G$, with $\epsilon_1^\ell > \epsilon_1^m$. Since $(\ \delta + \epsilon_1^\ell + \sum_{j=2}^{q} r_j\ )^{-1} \leq (\ \delta + \epsilon_1^m + \sum_{j=2}^{q} r_j\ )^{-1}$, the upper-bound given in (6) on $\|sT_1^\ell\|$ is larger than the one on $\|sT_1^m\|$ (for a controller which achieves close values for these norms); i.e., for G_1^ℓ and G_1^m having similar high frequency performances, the inequality (6) is easier to satisfy with G_1^ℓ than with G_1^m. Although this simple reasoning justifies increasing the order by including the right-most real pole, a similar easy rule cannot be stated in the case of complex-conjugate pairs of candidate insignificant poles. △

The following *single-step order reduction in* Corollary 1 states an easier interpretation of condition (6):

Corollary 1. *Under the assumptions of Theorem 1, with* $i := k+1$, *if there exists a real* $\delta > 0$ *such that*

$$\|sT_k\| \leq \begin{cases} (\epsilon_i + \delta)^{-1}, & \epsilon_i \in \mathbb{R}, \\ (2r_i + \delta)^{-1}, & \epsilon_i \notin \mathbb{R}\ ,\ \epsilon_{i+1} = \bar{\epsilon}_i \end{cases} \quad (14)$$

then **a)** *when* $\epsilon_i \in \mathbb{R}$, *C stabilizes* G_i *and satisfies* $\|sT_i\| \leq \frac{1}{\delta}$; **b)** *when* $\epsilon_i \notin \mathbb{R}$, $\epsilon_{i+1} = \bar{\epsilon}_i$, *C stabilizes* G_{i+1} *and satisfies* $\|sT_{i+1}\| \leq \frac{a_i^2 + b_i^2}{2a_ib_i\delta}$. △

Based on (14), a real pole $-1/\epsilon_i$ that lies to the left of $-\alpha_i = -\|sT_{i-1}\|$ *can be considered insignificant for* order reduction. For $-1/\epsilon_i \notin \mathbb{R}$ to be insignificant, $-r_i = -\mathcal{R}e(1/\epsilon_i) < -2\alpha_i$, i.e., the complex-conjugate pair of poles $-1/\epsilon_i$, $-1/\bar{\epsilon}_i$ should lie to the left of the line at $-2\alpha_i$. As $\|sT_{i-1}\|$ gets smaller, this line moves closer to the imaginary axis, enlarging the region for insignificant poles.

2.2 Insignificant Zeros

Consider the unity-feedback system, with P and $\tilde{C}$ as the plant's and the controller's transfer matrix. Let $P = \tilde{D}^{-1}\tilde{N}$ be an LCF, $\tilde{C} = \tilde{N}_c\tilde{D}_c^{-1}$ be an RCF over $\mathbf{S}$. Let P be full row-rank and have no transmission-zeros at $s = 0$, equivalently, let $\tilde{N}(0)$ be full row-rank. For $k \geq 1$, define

$$P_0 := P\ ,\quad P_k := P_{k-1}\frac{(s+z_k)}{s} = P\prod_{i=1}^{k}\frac{(s+z_i)}{s}, \quad (15)$$

$$\tilde{D}_0 := \tilde{D}\ ,\quad \tilde{D}_k = \frac{\tilde{D}_{k-1}s}{(s+z_k)} = \tilde{D}\prod_{i=1}^{k}\frac{s}{(s+z_i)}. \quad (16)$$

Then $P_k = \tilde{D}_k^{-1}\tilde{N}$ is an LCF of P_k . With P_k as the plant in the unity-feedback control system (replacing G_k in Section 2.1), let the sensitivity function S_k and the complementary sensitivity function T_k be $S_k = (I + P_k\tilde{C})^{-1}$, $T_k = P_k\tilde{C}(I + P_k\tilde{C})^{-1}$ as in (3).

We now give a dual of Lemma 1, where it was assumed that $GC(\infty) = 0$, equivalently, $T_0(\infty) = 0 = I - S_0(\infty)$. In the dual Lemma 2, we assume $S_0(0) = 0 = I - T_0(0)$, which implies $P\tilde{C}(0)$ has poles at $s = 0$. A transfer matrix $P\tilde{C}$ is said to be *of type-1 or greater* if $S_0(0) = 0$. This assumption is automatically satisfied if $\tilde{D}(0) = 0$ in $P = \tilde{D}^{-1}\tilde{N}$ or if $\tilde{D}_c(0) = 0$ in $\tilde{C} = \tilde{N}_c\tilde{D}_c^{-1}$, in which case the stabilizing controller *has integral-action*. Any stabilizing controller $\tilde{C} = \tilde{N}_c\tilde{D}_c^{-1}$ can be modified to be one with integral action, for example as $\tilde{C}' = \tilde{N}_c(I + \tilde{D}\tilde{D}_c\tilde{B})(\tilde{D}_c(I - \tilde{N}\tilde{N}_c\tilde{B}))^{-1}$ where $\tilde{B} = (\tilde{N}\tilde{N}_c)(0)^{-1}$. Therefore, there is no loss of generality in assuming $P\tilde{C}$ is of type-1 or greater, with the controller chosen to have integral action as necessary.

Lemma 2. *Let a plant P be stabilized by a controller $\tilde{C}$, where $P\tilde{C}$ is of type-1 or greater.* **a)** *For $z_k \in \mathbb{R}$, $z_k > 0$, if*

$$z_k < \|s^{-1}S_{k-1}\|^{-1}, \tag{17}$$

then the same $\tilde{C}$ also stabilizes the higher-order plant $P_k = P_{k-1}\frac{(s+z_k)}{s}$. **b)** *For $z_k \notin \mathbb{R}$, $-z_k \in \mathbb{C}_-$, let $z_{k+1} = \bar{z}_k$ and define $r_k := \frac{1}{Re(1/z_k)}$; if*

$$2r_k < \|s^{-1}S_{k-1}\|^{-1}, \tag{18}$$

then the same $\tilde{C}$ also stabilizes the higher-order plant $P_{k+1} = P_{k-1}\frac{(s+z_k)(s+\bar{z}_k)}{s^2}$. **c)** *For any conjugate symmetric set $\{-z_i \in \mathbb{C}_-, i = 1, ..., q\}$, where $z_i \in \mathbb{R}$ satisfies (17) and $z_i \notin \mathbb{R}$ satisfies (18), the controller $\tilde{C}$ also stabilizes the higher-order plant $P_q = P\prod_{i=1}^{q}\frac{(s+z_i)}{s}$.*

Proof. Let $P = \tilde{D}^{-1}\tilde{N}$ be an LCF and let $\tilde{C} = \tilde{N}_c\tilde{D}_c^{-1}$ be an RCF. For $k \geq 0$, define $V_k := \tilde{D}_k\tilde{D}_c + \tilde{N}\tilde{N}_c$. The controller $\tilde{C}$ stabilizes P_k if and only if V_k is unimodular, i.e., $V_k^{-1} \in \mathcal{M}(\mathbf{S})$. By assumption, $V_0 = \tilde{D}\tilde{D}_c + \tilde{N}\tilde{N}_c$ is unimodular since $\tilde{C}$ stabilizes P. We show that $\tilde{C}$ also stabilizes P_k by induction: For $k \geq 1$, $V_k = V_{k-1} - \frac{z_k}{z_k s+1}\tilde{D}_k\tilde{D}_c$. If $\tilde{C}$ stabilizes P_{k-1}, then V_{k-1} is unimodular. Since $P\tilde{C}$ is of type-1 or greater, $s^{-1}S_0 = s^{-1}\tilde{D}_cV_0^{-1}\tilde{D} \in \mathcal{M}(\mathbf{S})$. Note that $M(s) \in \mathcal{M}(\mathbf{S})$ if and only if $\hat{M} := M(1/s) \in \mathcal{M}(\mathbf{S})$; i.e., stability is preserved under the transformation $s \to s^{-1}$. Let $\hat{S}_{k-1} := S_{k-1}(1/s)$; then since $s^{-1}S_{k-1} \in \mathcal{M}(\mathbf{S})$ for $k \geq 1$, it follows that $s\hat{S}_{k-1} \in \mathcal{M}(\mathbf{S})$. Now for $k \geq 1$, $V_{k-1}^{-1}V_k = I - \frac{z_k}{s+z_k}V_{k-1}^{-1}\tilde{D}_{k-1}\tilde{D}_c$ is unimodular if and only if $M_k := I - \frac{z_k}{s+z_k}\tilde{D}_cV_{k-1}^{-1}\tilde{D}_{k-1} = I - \frac{z_k s}{s+z_k}(s^{-1}S_{k-1})$ is unimodular. Applying the transformation $s \to s^{-1}$, M_k is unimodular if and only if $\hat{M}_k := M_k(1/s) = I - \frac{z_k}{z_k s+1}(s\hat{S}_{k-1})$ is unimodular. Therefore, we now have the problem cast in the setting of Lemma 1, replacing (sT_{k-1}) by $(s\hat{S}_{k-1})$; hence, the proof follows as in the proof of Lemma 1, by finally using the transformation $s^{-1} \to s$. ∎

Lemma 2 justifies methods of design where a loop-gain transfer function (15) is approximated by a function of type-1 or greater, in designing a stabilizing controller. The terms that are discarded are such that the high-frequency gain of P and that of P_k are the same, i.e., each insignificant zero is cancelled with exactly one pole at the origin. A real zero $-z_i$ is *insignificant*, or, *cancellable with a pole at the origin*, if $-z_i$ is in the interval $(-1/\beta_i, 0)$, where $\beta_i := \|s^{-1}S_{i-1}\|$, i.e., it is sufficiently close to the origin. Based on condition (18), a complex-conjugate pair of cancellable zeros lies inside the circle of radius $(4\beta_i)^{-1}$ centered at $-(4\beta_i)^{-1}$.

The proof given for Lemma 2, based on the transformation $s \to s^{-1}$, clarifies the relationship between the two design methods: An insignificant denominator-term $(\epsilon s + 1)$ under the transformation $s \to s^{-1}$ gives a PI controller $\frac{s+\epsilon}{s}$ having an insignificant zero.

We now give a dual of Theorem 1: If for some $k < q$, we can determine a stabilizing controller, which achieves a certain amount of closed-loop performance for P_k, then the same controller stabilizes every P_i for $i \geq k$ and has, to some degree, a guaranteed closed-loop performance.

Theorem 2. *Let $\tilde{C}$ be a stabilizing controller for the plant P_k for some $k \in \{0, \dots, q-1\}$, where $\{-z_i \in \mathbf{C}_-, i = 1, ..., k\}$ is a conjugate symmetric set. Let $\tilde{C}$ be such that $P\tilde{C}$ is of type-1 or greater. For $k+1 \leq i \leq q$, let $\beta_i := \|s^{-1}S_{i-1}\|$. Suppose that $\{-z_j \in \mathbf{C}_-, j = k+1, ..., i, k+1 \leq i \leq q\}$ is a conjugate symmetric set and that there exists a real $\delta > 0$ such that*

$$\|s^{-1}S_k\| \leq (\delta + \sum_{j=k+1}^{q} r_j)^{-1}. \tag{19}$$

Under these assumptions: **a)** *When $z_i \in \mathbb{R}$, the same controller $\tilde{C}$ also stabilizes $P_i = P_{i-1}\frac{(s+z_i)}{s}$. Furthermore,*

$$\|s^{-1}S_i\| \leq (\delta + \sum_{j=i+1}^{q} r_j)^{-1} \tag{20}$$

and the following sensitivity and complementary sensitivity bounds are achieved:

$$\begin{aligned} &\tfrac{1}{1+z_i\beta_i}\|S_{i-1}\| - \tfrac{z_i\beta_i}{1+z_i\beta_i} \leq \|S_i\| \leq \tfrac{1}{1-z_i\beta_i}\|S_{i-1}\|, \\ &\tfrac{1}{1+z_i\beta_i}\|T_{i-1}\| \leq \|T_i\| \leq \tfrac{1}{1-z_i\beta_i}\|T_{i-1}\|. \end{aligned} \tag{21}$$

b) *When $z_i \notin \mathbb{R}$, $z_{i+1} = \bar{z}_i$ (where $a_i + jb_i := (1/z_i)$, $a_i, b_i \in \mathbb{R}$, $a_i, b_i > 0$), the same controller $\tilde{C}$ also stabilizes $P_{i+1} = G_{i-1}\frac{(s+z_i)(s+\bar{z}_i)}{s^2}$. Furthermore,*

$$\|s^{-1}S_{i+1}\| \leq \frac{(a_i^2 + b_i^2)}{2a_ib_i}(\delta + \sum_{j=i+2}^{q} r_j)^{-1}, \tag{22}$$

and the following sensitivity and complementary sensitivity bounds are achieved:

$$\begin{aligned} &\tfrac{1}{1+2r_i\beta_i}\|S_{i-1}\| - \tfrac{2r_i\beta_i}{1+2r_i\beta_i} \leq \|S_{i+1}\| \\ &\qquad \leq \tfrac{1}{1-2r_i\beta_i}\|S_{i-1}\| + \tfrac{2r_i\beta_i}{1-2r_i\beta_i}, \\ &\tfrac{1}{1+2r_i\beta_i}\|T_{i-1}\| \leq \|T_{i+1}\| \leq \tfrac{1}{1-2r_i\beta_i}\|T_{i-1}\|. \end{aligned} \tag{23}$$

Proof. The result can be obtained from Theorem 1 by the transformation $s \to s^{-1}$ and by appropriate

Proceedings of the American Control Conference
Denver, Colorado June 4-6, 2003

changes in the notation. Alternately, it can be proved directly using Lemma 2 following similar steps as in the proof of Theorem 1. ∎

Remarks: **5)** Condition (19) is a low-frequency performance requirement on the plant P_k. In the scalar case, it is equivalent to $\sup_{\omega\geq 0}|\omega|^{-1}|S_k(j\omega)| \leq (\delta + \sum_{j=k+1}^{q} r_j)^{-1}$, which implies $|S_k(j\omega)| \leq |\omega|(\delta + \sum_{j=k+1}^{q} r_j)^{-1}$, $\forall\omega \geq 0$. This means in particular that $|S_k(j\omega)| < 1$ for all $\omega \leq \sum_{j=k+1}^{q} r_j$. By Theorem 2, a similar performance holds true for each plant P_k, $i \in [k+1, q]$, stabilized by the same controller. Again by [8], if P_k has a strict right-half plane zero and its associated sensitivity function gets small in magnitude in a frequency range, then its H_∞-norm necessarily gets large. The bounds in (21) show that the $\|S_k\|$'s nevertheless remain bounded by a multiple of $\|S_k\|$. △
6) As a counterpart for Corollary 1, a single step order reduction condition can easily be written from Theorem 2. A real zero $-z_i$ is cancellable if $z_i < 1/\beta_i$. A complex-conjugate pair $\{-z_i, -\bar{z}_i\} \in \mathbb{C}_-$ is cancellable if the zeros lie strictly in the circle of diameter $1/2\beta_i$. As $\|s^{-1}S_{i-1}(s)\|$ gets smaller, this region gets larger.

3 Conclusions

In Theorems 1 and 2, we provided dual model reduction methods from the viewpoint of closed-loop stability and performance. The iterative design algorithms hinge on the existence of a controller having a certain performance as quantified by conditions (6) and (19). The most important merit of the methods presented is that they directly focus on closed-loop performance and provide estimates in terms of eliminated poles or zeros for achievable performance and stability robustness. The design methods provide an MIMO generalization of the scalar design approximation methods. It should be noted that the candidate insignificant poles and zeros are "blocking" poles and zeros in the sense that they appear in every entry of the transfer matrix. These methods do not restrict the approximated plant to be stable or minimum-phase; the only requirement is that the discarded poles and zeros are in the open left-half plane. Unlike most other reduction methods, these do not require any additive decomposition of the plant into stable and anti-stable parts.

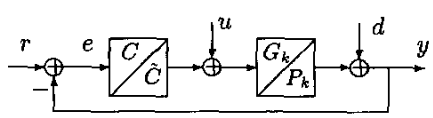

Figure 1: Unity-Feedback Control System

References

[1] V. M. Adamjan, D. Z. Arov, M. G. Krein, "Analytic properties of Schmidt pairs for Hankel operator and the generalized Schur-Takagi problem," *Math. USSR Sbornik*, Vol. 15, No. 1, pp. 31-73, 1971.

[2] U. M. Al-Saggaf, G. F. Franklin, "Model reduction via balanced realizations: An extension and frequency weighting techniques," *IEEE Trans. Auto. Contr.*, AC-33: 7, pp. 687-692, 1988.

[3] B. D. O. Anderson, Yi Liu, "Controller reduction: Concepts and approaches," *IEEE Trans. Auto. Contr.*, AC-34: 8, pp. 802-812, 1989.

[4] J. C. Doyle, K. Glover, P. P. Khargonekar, B. A. Francis, "State-space solutions to standard H_2 and H_∞ control problems," *IEEE Trans. Auto. Contr.*, AC-34: 8, pp. 831-847, 1989.

[5] J. C. Doyle, G. Stein, "Multivariable feedback design: Concepts for a classical/modern design," *IEEE Trans. Auto. Contr.*, AC-26: 1, pp. 4-16, 1981.

[6] D. F. Enns, "Model reduction with balanced realizations: An error bound and a frequency weighted generalization," *Proc. 23rd Conf. Decision Contr.*, pp. 127-132, 1984.

[7] B. A. Francis, *A Course in H_∞ Control Theory*, New York, Springer-Verlag 1987.

[8] B. A. Francis, G. Zames, "On H^∞-optimal sensitivity theory for SISO feedback systems," *IEEE Trans. Auto. Contr.*, AC-29: 1, pp. 9-16, 1984.

[9] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L_∞-error bounds," *International Journal of Control*, 39: 6, pp. 1115-1193, 1984.

[10] A. N. Gündeş, M. G. Kabuli, "Simultaneously stabilizing controller design for a class of MIMO systems," *Automatica*, 37, pp. 1989-1996, 2001.

[11] S. Y. Kung, D. W. Lin, "Optimal Hankel-norm model reductions: Multivariable systems," *IEEE Trans. Auto. Contr.*, AC-26: 4, pp. 832-852, 1981.

[12] B. C. Kuo, *Automatic Control Systems, 7th edition*, Prentice Hall, New Jersey, 1995.

[13] B. C. Moore, "Principal component analysis in linear systems: Controllability, observability, and model reduction," *IEEE Trans. Auto. Contr.*, AC-26: 1, pp. 17-32, 1981.

[14] K. Ogata, *Modern Control Engineering, 3rd edition*, Prentice Hall, New Jersey, 1997.

[15] A. B. Özgüler, A. N. Gündeş, "Approximations in compensator design: A duality," *Electronics Letters*, 38: 10, pp. 489-490, 2002.

[16] L. Pernebo, L. M. Silverman, "Model reduction via state space representations," *IEEE Trans. Auto. Contr.*, AC-27: 2, pp. 382-387, 1982.

[17] C. E. Rohrs, J. L. Melsa, D. G. Schultz, *Linear Control Systems*, McGraw Hill, New Jersey, 1993.

[18] M. C. Smith, K. P. Sondergeld, "On the order of stable compensators," *Automatica*, 22: 1, pp. 127-129, 1986.

[19] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, Massachusetts, USA, 1985.

[20] A. Yousuff, D. A. Wagie, R. Skelton, "Linear system approximation via covariance equivalent realizations," *J. Math. Anal. Appl.*, 106: 1, pp. 91-115, 1985.