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A General Boundedness Result for Interconnected Nonlinear Systems

M. G. Kabuli and A. N. Gündeş

***Abstract*—A general boundedness result is given for the feedback interconnection of two nonlinear stable systems. Using the same input–output approach as in the standard small-gain theorem, the sufficient condition given here relaxes the finite-gain stability assumption and does not require a boundedness result for all possible bounded exogenous inputs. This condition has a simple graphical interpretation that utilizes graphs of bounding functions.**

***Index Terms*— Boundedness, nonlinear systems, small-gain theorem, stability.**

I. INTRODUCTION

The standard small-gain theorem establishes a sufficient condition to ensure stability of the feedback interconnection of two stable nonlinear systems [1], [5]; this theorem is central in many stability robustness results in the literature. The finite-gain setting of the theorem allows a natural extension of results on stable linear systems to the nonlinear case and associates a "gain" with each of the nonlinear stable systems in the feedback interconnection. The strong result of the small-gain theorem requires no existence, or uniqueness, or continuity assumptions. Stability analysis is reduced to the following simple scalar inequality condition: if the product of the gains is less than one, then the closed-loop feedback interconnection is finite-gain stable. This condition requires only two ingredients: 1) bounded-input/bounded-output stability of each subsystem and 2) a crucial pair of inequality constraints resulting from the property of seminorms.

In this paper, we characterize bounded-input/bounded-output stability of systems in a general form using nondecreasing bounding functions. For a given bound on exogenous inputs, we state a sufficient condition to guarantee that all resulting signals in the feedback interconnection are bounded. The proposed condition is solely based on a crucial pair of inequality constraints. It has a simple graphical interpretation using the graphs of two bounding functions and their translations due to the bounds on the exogenous signals. The level of conservatism in the standard finite-gain small-gain theorem is reduced in the present setting due to adopting a bounding function more general than an affine one, and due to not requiring a boundedness result for *any* bound on exogenous inputs.

This paper was motivated by a generalization of the small-gain theorem in [3], which also allows general monotone output-bounding functions (see also [4] and references therein). However, the conditions in [3] require additional assumptions in order to guarantee a bounded output for any bound on exogenous inputs. The condition in this paper extracts the boundedness result directly from a pair of inequalities instead of reducing them to a single inequality, which is the step where the additional requirements would be introduced.

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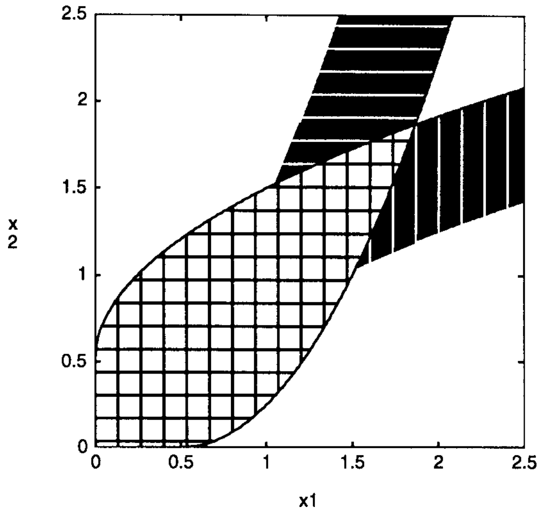

Fig. 1. $\varphi_1(\cdot) = \varphi_2(\cdot) = \sqrt{(\cdot)}$ and $\alpha = [0.5\ 0.5]^T$: below $(x_1, \alpha_2 + \varphi_1(x_1))$ and above $(\alpha_1 + \varphi_2(x_2), x_2)$.

II. Notation and Preliminaries

All nonlinear maps in this paper are causal, multi-input/multi-output, and defined over appropriate products of causal extension of the set $\mathcal{L}$ of bounded signals. The time-set $\mathcal{T}$ typically denotes nonnegative reals or integers. For $T \in \mathcal{T}$, let Π_T denote the usual truncation operator and $\|\cdot\|$ denote the associated norm used in describing the bounded signals in $\mathcal{L}$. The causal extension of $\mathcal{L}$ is denoted by $\mathcal{L}_e$. With a slight abuse of notation, $\|\cdot\|$ is also used in describing the norm in the product set of bounded signals $\mathcal{L}^n$. For a thorough treatment of general extended spaces within the input–output approach to nonlinear systems, see [1]. The extended space $\mathcal{L}_e$ is a means of incorporating unbounded signals in the analysis; however, although $\mathcal{L} \subset \mathcal{L}_e$, the component $(\mathcal{L})^C \backslash \mathcal{L}_e \neq \emptyset$, where $(\cdot)^C$ denotes the complement with respect to the set of all functions on $\mathcal{T}$. The nonempty intersection arises due to discontinuities that are not jump discontinuities. Such signals that exhibit "finite escape time" are not covered within the scope of extended spaces; therefore, domains restricted to a strictly proper subset of the input extended space might be necessary in describing the nonlinear maps. Hence, $\mathcal{L}$ describes the set of bounded signals and $\mathcal{L}_e \backslash \mathcal{L}$ denotes the set of unbounded signals (unbounded at infinity). An n_1-input/n_2-output causal nonlinear map $\mathcal{P}$ is considered as $\mathcal{P} : \mathcal{U} \subset \mathcal{L}_e^{n_1} \rightarrow \mathcal{L}_e^{n_2}$, where $\mathcal{U}$ denotes the domain. With appropriate domain and range matchings, the map $\mathcal{FG}$ denotes the composition of two nonlinear causal maps $\mathcal{F}$ and $\mathcal{G}$.

In an input–output approach to analysis and design of nonlinear systems, the notions of boundedness, and stability are crucial for the subsequent results. Unlike the finite-dimensional linear time-invariant case, most of the properties depend on the particular framework used. The following definition sets up the framework used here [2].

Definition 1: A causal map $\mathcal{H} : \mathcal{L}_e^{n_i} \rightarrow \mathcal{L}_e^{n_o}$ is said to be stable if and only if there exists a continuous nondecreasing bounding function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\|\mathcal{H}(u)\| \leq \varphi(\|u\|)\ \forall u \in \mathcal{L}^{n_i}$.

The bounding function φ in Definition 1 need not be strictly increasing, or one-to-one, or onto, or subadditive ($\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be subadditive if and only if $\varphi(x_1 + x_2) \leq \varphi(x_1) + \varphi(x_2)$ for all $x_1, x_2 \in \mathbb{R}_+$).

III. Main Result

Theorem 1 states a sufficient boundedness condition extracted from a crucial pair of inequalities resulting from applying the property of seminorms on the summing-junction equations of the unity-feedback interconnection.

Theorem 1: Let $\mathcal{H}_1 : \mathcal{L}_e^{n_1} \rightarrow \mathcal{L}_e^{n_2}$ and $\mathcal{H}_2 : \mathcal{L}_e^{n_2} \rightarrow \mathcal{L}_e^{n_1}$ be stable maps with the associated bounding functions φ_1 and φ_2, respectively. Consider the unity-feedback interconnection, where

$$e_1 = u_1 - \mathcal{H}_2(e_2) \tag{1}$$

$$e_2 = u_2 + \mathcal{H}_1(e_1). \tag{2}$$

For a given $\alpha \in \mathbb{R}_+^2$, let $\|u_1\| \leq \alpha_1$ and $\|u_2\| \leq \alpha_2$. Let $\beta \in \mathbb{R}_+^2$ be such that

$$\left\{ x \in \mathbb{R}_+^2 \middle| \begin{matrix} x_1 \leq \alpha_1 + \varphi_2(x_2) \\ x_2 \leq \alpha_2 + \varphi_1(x_1) \end{matrix} \right\} \subset [0, \beta_1] \times [0, \beta_2]. \tag{3}$$

Under these assumptions, if $(e_1, e_2) \in \mathcal{L}_e^{n_1} \times \mathcal{L}_e^{n_2}$, then $\|e_1\| \leq \beta_1$ and $\|e_2\| \leq \beta_2$. $\square$

As in the standard finite-gain small-gain setting, no assumptions of existence, uniqueness, or continuity of solutions are made in Theorem 1. Although this paper emphasizes a map setting, the theorem is valid for relations as well as maps.

The proof of Theorem 1 is a simple exercise using causality, the truncation operator Π_T, and the property of seminorms on (1) and (2): let $(e_1, e_2) \in \mathcal{L}_e^{n_1} \times \mathcal{L}_e^{n_2}$. For any $T \in \mathcal{T}$

$$\|\Pi_T e_1\| \leq \|\Pi_T u_1\| + \|\Pi_T \mathcal{H}_2(e_2)\| \leq \alpha_1 + \varphi_2(\|\Pi_T e_2\|)$$
$$\|\Pi_T e_2\| \leq \|\Pi_T u_2\| + \|\Pi_T \mathcal{H}_1(e_1)\| \leq \alpha_2 + \varphi_1(\|\Pi_T e_1\|).$$

Since (3) holds, $\|\Pi_T e_1\| \leq \beta_1$ and $\|\Pi_T e_2\| \leq \beta_2$, for all $T \in \mathcal{T}$, and the conclusion follows.

IV. Application

A simple two-dimensional graphics environment is all that is required in order to apply the result in Theorem 1. Optimization is not required unless the least upper-bounding β is sought. Two graphs in $\mathbb{R}_+^2$, i.e., $\{(x_1, \alpha_2 + \varphi_1(x_1)), x_1 \in \mathbb{R}_+\}$ and $\{(\alpha_1 + \varphi_2(x_2), x_2) \mid x_2 \in \mathbb{R}_+\}$, are drawn. The desired intersection is the union of possibly disjoint sets formed by intersecting the region *below* the first graph and *above* the second graph. No conditions are imposed on the bounding functions or their compositions. In the case of affine bounding functions with slopes k_1 and k_2, as in the finite-gain small-gain theorem, a bounded intersection in the nonnegative quadrant is possible if and only if $k_1 k_2 \in [0, 1)$. Changing $\alpha \in \mathbb{R}_+^2$ corresponds to translating the two graphs appropriately. Since bounded intersections may not exist for all α in general (as in the case of Fig. 2 explained below), the user can easily see the effect of exogenous signal bounds and extract a tight bound such that the sufficient condition still holds. Theorem 1 also yields a one-dimensional graphical interpretation using, for example, only $x_1 \leq \alpha_1 + \varphi_2(\alpha_2 + \varphi_1(x_1))$. However, this approach would not have the same simple interpretation in terms of the graphs of φ_1 and φ_2 since for each α_2, the graph of a new function $\varphi_2(\alpha_2 + \varphi_1(\cdot))$ would need to be computed. The two-dimensional approach above uses translations of the same pair of graphs for all α. The one-dimensional approach can be further simplified at the expense of additional assumptions, such as subadditivity of φ_1 or φ_2. Regardless of which graphical interpretation of Theorem 1 is used, when at least one of the bounding functions φ_1 or φ_2 is uniformly bounded, there exists a bounded β for any bounded α.

Example 1: Consider Fig. 1, where the sufficient condition of Theorem 1 is satisfied. For this example, the finite-gain stability setting of the standard small-gain theorem would have required affine bounds on φ_1 and φ_2, which would bring conservatism in bounding the intersection region.

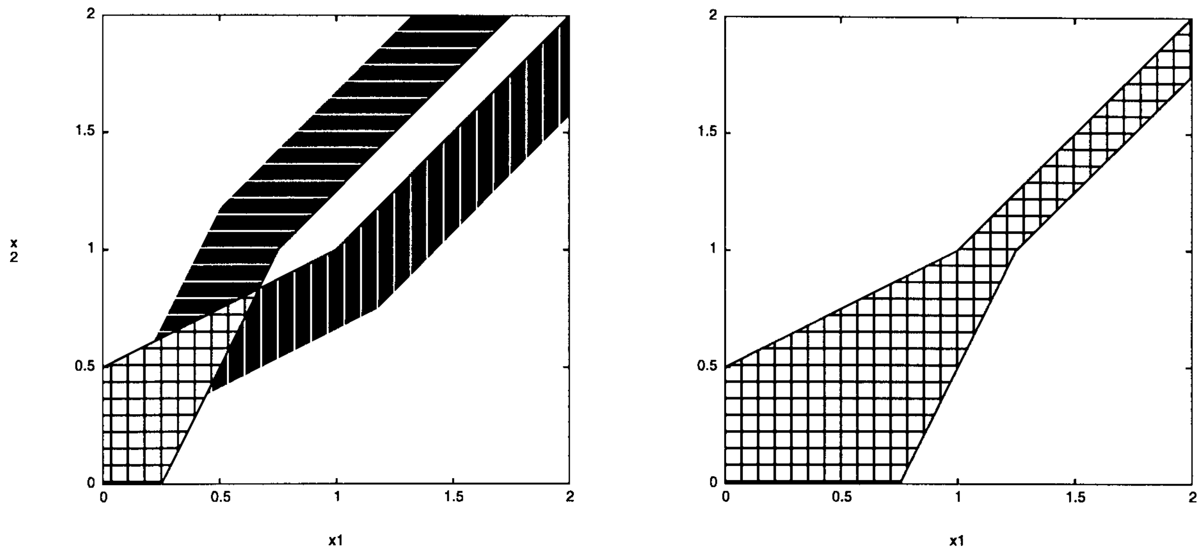

Fig. 2. Feasible regions for $\alpha = [0.25 \;\; 0.5]^T$ and $\alpha = [0.75 \;\; 0.5]^T$ for two identical piecewise-linear bounding functions: below $(x_1, \alpha_2 + \varphi_1(x_1))$ and above $(\alpha_1 + \varphi_2(x_2), x_2)$. Changing α_1 corresponds to horizontal translation of $(\varphi_2(x_2), x_2)$.

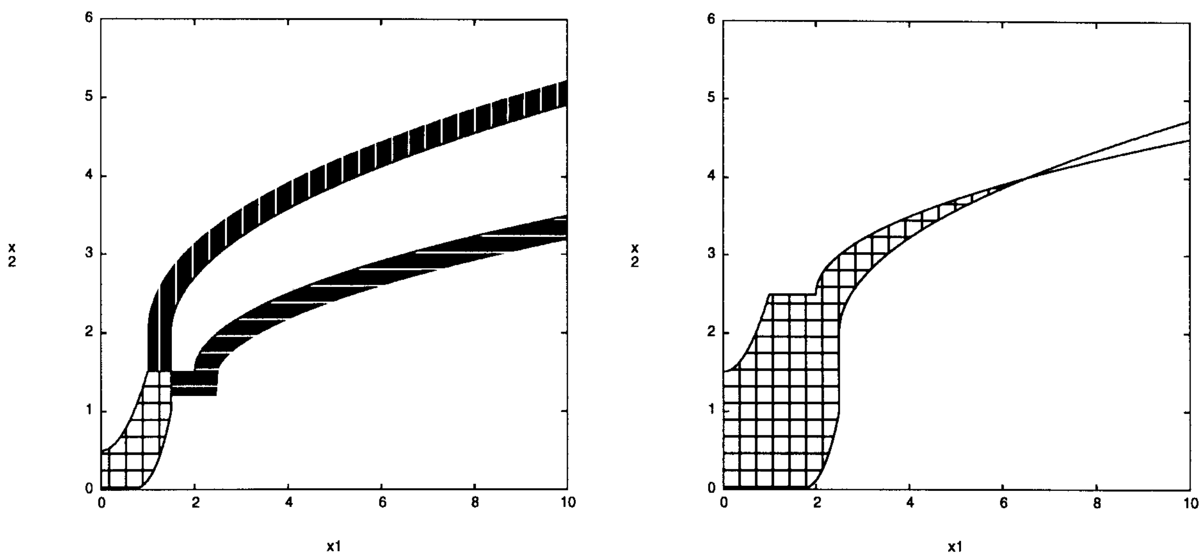

Fig. 3. The feasible regions for the bounding functions in (4) and (5) for $\hat{\alpha} = 0.5$ and $\hat{\alpha} = 1.5$: below $(x_1, \hat{\alpha} + \varphi_1(x_1))$ and above $(\hat{\alpha} + \varphi_2(x_2), x_2)$.

Example 2: Consider the first plot of Fig. 2, where the sufficient condition of Theorem 1 is satisfied for certain exogenous signal bounds, although the finite-gain approximation would have been inconclusive due to the maximum slopes of the bounding functions. Changing the bound α_1 on the exogenous input u_1 from 0.25 to 0.75 would simply correspond to dragging the graph $\{(\varphi_2(x_2), x_2) \mid x_2 \in \mathbb{R}_+\}$ suitably to the right as seen in the second plot of Fig. 2. Consequently, the sufficient condition no longer holds. Such a graphical interface allows the designer to visually extract tight bounds without any use of optimization or further analysis requiring compositions of functions.

Example 3: Let φ_1 and φ_2 be described by

$$\varphi_1(x) = \begin{cases} x^2, & x \in [0,1] \\ 1, & x \in [1,2] \\ 1 + \sqrt{\frac{(x-2)}{2}}, & x \in [2,\infty) \end{cases} \tag{4}$$

$$\varphi_2(x) = \begin{cases} x^{\frac{1}{3}}, & x \in [0,1] \\ 1, & x \in [1,2] \\ 1 + (x-2)^2, & x \in [2,\infty). \end{cases} \tag{5}$$

Note that φ_1 and φ_2 are not subadditive; furthermore, φ_1 and φ_2 are not one-to-one, hence their inverses constitute relations. Due to the quadratic term in φ_2, the finite-gain setting would be inapplicable since an affine bound cannot be obtained on φ_2. Even if these functions are modified to be bijections by coalescing the breakpoints at $x = 1$ and $x = 2$, their inverses will not be subadditive due to the quadratic and cubic segments.

Let $\hat{\alpha} := \max\{\alpha_1, \alpha_2\}$. As illustrated in Fig. 3, given any $\hat{\alpha}$, there exists a bounded β as defined in the notation of Theorem 1. This is due to the fact that the final segments in bounding functions will always intersect for $\hat{\alpha} \geq 1$. This is a simple visual observation, and one can easily generate admissible $\hat{\alpha}$, β values by simply dragging the curves and clicking near the intersections. In fact, the least upper-bounding β can be derived in terms of $\hat{\alpha}$ as follows.

Let $\delta(\hat{\alpha}) = (\hat{\alpha} - 1) + \sqrt{(2\hat{\alpha} - 1)(\hat{\alpha} - 1)}$ for $\hat{\alpha} \geq 1$. Then

$$\beta_1 = \begin{cases} \hat{\alpha} + 1, & \hat{\alpha} \in [0,1] \\ 2(1 + \delta(\hat{\alpha})^2), & \hat{\alpha} \in [1,\infty) \end{cases}$$

$$\beta_2 = \begin{cases} \hat{\alpha} + 1, & \hat{\alpha} \in [0,1] \\ \hat{\alpha} + 1 + \delta(\hat{\alpha}), & \hat{\alpha} \in [1,\infty). \end{cases}$$

Hence the feedback interconnection of two stable subsystems with bounding functions φ_1 and φ_2 is stable since the internal signals can be bounded for any bounded exogenous input.

V. Conclusion

A boundedness result for nonlinear unity-feedback interconnections with stable subsystems is introduced in Theorem 1. Each subsystem is associated with a nondecreasing bounding function. The result reduces to translating two curves that denote the boundaries of feasible regions and seeking bounded intersections. No further analysis or construction of functions other than the original pair of bounding functions is required. The application of this result has a simple two-dimensional graphical interpretation.

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A New Necessary and Sufficient Condition for Static Output Feedback Stabilizability and Comments on "Stabilization via Static Output Feedback"

Yong-Yan Cao, You-Xian Sun, and Wei-Jie Mao

***Abstract*— In this paper, a counterexample of the above-mentioned paper[1] is reported. It is pointed out that one of the conditions for a linear system to be stabilizable via static output feedback is not correct. A modified necessary and sufficient condition for this problem is also presented.**

***Index Terms*—Stabilization, static output feedback.**

I. Introduction

Stabilization of linear systems by static output feedback is a problem that is practically important and theoretically appealing. Recently, Trofino-Neto and Kucera presented two necessary and sufficient conditions for the existence of a stabilizing static output feedback gain matrix,[1] but one of them is incorrect.

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[1] A. Trofino-Neto and V. Kucera, *IEEE Trans. Automat. Contr.*, vol. 38, pp. 764–765, 1993.

Consider the system described by the equations

$$\dot{x} = Ax + Bu; \qquad y = Cx \tag{1}$$

where x is the state vector, u is the control vector, y is the measurement vector, and A, B, and C are constant matrices. Let E_i and E_k denote the orthogonal projection matrices on the $\operatorname{Im} C^T$ and $\operatorname{Ker} C$, respectively, i.e.,

$$E_i = C^+ C, \qquad E_k = I - E_i \tag{2}$$

and the Hamiltonian matrix be defined as

$$H(S) = \begin{bmatrix} A - BR^{-1}S & BR^{-1}B^T \\ Q - S^T R^{-1} S & -(A - BR^{-1}S)^T \end{bmatrix}. \tag{3}$$

Theorem 3.1 of the paper is stated as follows.

Let E_i and E_k be the projection matrices defined in (2) and $H(s)$ the Hamiltonian matrix in (3). Then, the following statements are equivalent.

i) System (1) is stabilizable via static output feedback.

ii) There exist matrices $Q > 0$, $R > 0$, and L of compatible dimensions such that the following algebraic matrix equation:

$$A^T P + PA - E_i(PB + L^T)R^{-1}(B^T P + L)E_i + Q = 0 \tag{4}$$

has a unique solution $P > 0$ and $K = R^{-1}(L + B^T P)E_i$ is stabilizing.

iii) The pair (A, B) is stabilizable (by state feedback) and there exist $P > 0$, $Q > 0$, $R > 0$, and a matrix L in (4) such that the Hamiltonian matrix in (3) has no pure imaginary eigenvalues for $S = LE_i - B^T PE_k$.

First, let us see a counterexample. Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = [1 \quad -a]$$

which is not stable since its two eigenvalues are 1 and -1. It is not difficult to find that this system cannot be stabilized via static output feedback when $a > 0$, while it can be when $a < 0$. Let $a = 0.5$ and

$$R = 1, \qquad L = [0.2 \quad 1.7], \qquad Q = \begin{bmatrix} 5.96 & -4.88 \\ -4.88 & 4.04 \end{bmatrix}.$$

The algebraic matrix equation (4) has a unique solution

$$P = \begin{bmatrix} 1.8 & -1.7 \\ -1.7 & 1.8 \end{bmatrix} > 0$$

and then $S = [-0.2 \;\; 0.9]$. So the output feedback gain is $F = 1.6$. But the eigenvalues of the closed-loop system $A - BFC$ are -2.362 and 0.762. Obviously, it is not stable. So the statements i) and iii) in Theorem 3.1 of the paper are not equivalent.

II. A Corrected Stabilizability Condition

In this section, we give a new necessary and sufficient condition for the existence of a stabilizing static output feedback gain matrix. The following lemma is well known [1], [2].

Lemma 1: Let the linear time-invariant system (1) be given. Then, the following statements are equivalent.

1) System (1) is stabilizable via state feedback.

2) There exist matrices $Q > 0$ and $R > 0$ of compatible dimensions such that the following algebraic Riccati equation (ARE):

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

has a unique solution $P > 0$.