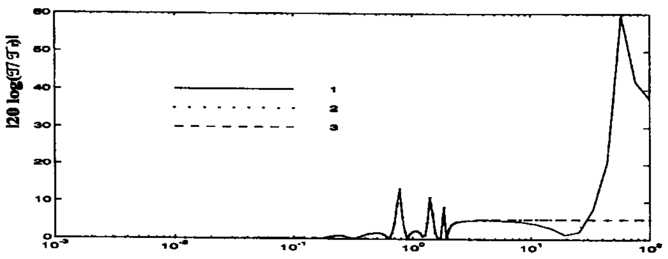

Fig. 6. Errors of approximation of the full-order closed loop by the closed loops with the reduced second order controllers.

The results show that when fast-sampling is used during the reduction process, a superior result is obtained. It is sufficient in this case to use $N = 3$ as the fast sampling rate. This corresponds to an angular frequency of approximately 190 rad/sec; the improvement in matching of $\mathcal{T}$ and $\mathcal{T}_r$ is evident starting at about 10 rad/sec.

Needless to say, whatever the sampling frequency is, it makes sense, especially if there is a problem with stability of the sampled-data closed loop, to use a more sophisticated scheme for obtaining the original (high order) discrete controller transfer function [9], [10].

V. Conclusion

The proposed method allows one to reduce a discrete-time controller which is used in a closed loop with a continuous-time plant, sampler, zero-order hold, and antialiasing filter. This reduction is based on information describing the system's behavior not only at the sampling instants, but in intersample periods as well, and aims to preserve the closed-loop behavior of the sampled-data loop. To get information about the intersample behavior of the system, fast-sampling has been applied, followed by a lifting operation, which gives a time-invariant system. Obviously, the fast sampling procedure incurs an approximation error.

In the whole reduction process, there are actually three different types of error:

i) The error due to replacing a hybrid system by a multirate sampled-data system—this error can be made as small as desired by choosing a fast enough sampling rate for the faster of the two rates.

ii) The error involved in replacing the problem of matching closed-loop transfer functions by the problem of matching (with weights) the open-loop responses of the controller—this error arises from neglecting second-order terms and has the potential to lead to a (mildly) less than optimal result in terms of closed-loop matching.

iii) The error associated with approximating a high-order transfer function by a low-order one—this is obviously unavoidable.

The feasibility, efficiency, and advantage of the proposed method have been confirmed by a practical numerical example.

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Simultaneous Stabilization of Linear Systems Under Stable Additive or Feedback Perturbations

A. N. Gündeş and M. G. Kabuli

***Abstract*— In the standard linear, time-invariant, multi-input multi-output unity-feedback system, it is shown that a given plant and one obtained by a known stable additive (or feedback) perturbation of this plant can be simultaneously stabilized by a common controller. The plant is not necessarily stable. No small-gain restrictions are imposed on the stable perturbations. A set of simultaneously stabilizing controllers is explicitly derived for any such pairs of plants. The results extend the standard single connected set of plants description in robust control design methods to two (possibly disjoint) sets of plants.**

I. Introduction

In the standard linear time-invariant (LTI), multi-input multi-output (MIMO) unity-feedback system, we consider simultaneous stabilization of a pair of plants, a nominal plant P, and an additively-perturbed plant $(P + \mathbf{G}_A)$ (similarly, P and a feedback-perturbed

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plant $P(I+\mathbf{G}_F P)^{-1}$), where $\mathbf{G}_A(\mathbf{G}_F)$ is a known stable perturbation. The plant is not necessarily stable. No small-gain restrictions are imposed on the stable perturbation. We show that there exists a common controller that stabilizes any such plant pairs and derive a set of simultaneously stabilizing controllers explicitly.

The problem considered here is a special case of the simultaneous stabilization of two plants (see, for example, [2], [3], [6], [8], [9]). It is well known that the existence of controllers that simultaneously stabilize two plants is equivalent to the existence of one stable controller that stabilizes an associated "pseudoplant" [7], [8]. We show that this strong stabilizability condition is always satisfied for the problem considered here by using the parity-interlacing property of the pseudoplant. After the parity-interlacing property is verified, constructing a controller is nontrivial; the only known method for single-input single-output (SISO) plants relies on interpolation conditions and does not provide a controller explicitly for MIMO plants [8]. On the other hand, the results of this note guarantee existence of simultaneously stabilizing controllers without obtaining the associated pseudoplant and checking the appropriate parity-interlacing property for each given perturbation. Furthermore, simultaneously stabilizing controllers are constructed, and a subclass of all such controllers is developed explicitly for the general MIMO case.

The systems under consideration are described in Section II. Conditions for closed-loop stability are stated in Section III. The main result is Theorem 3.2; it states that the nominal and the additively-perturbed plant (or the nominal and the feedback-perturbed plant) can always be simultaneously stabilized by a common controller and proposes a class of simultaneously stabilizing controllers. In Remark 3.3, the result is compared with other approaches, and it is shown that the finite-dimensional controller parameters proposed in Theorem 3.2 are applicable to the MIMO case, whereas the interpolation method, applicable only to the SISO case, would not give finite-dimensional controller parameters. In Comment 3.4, it is shown that it is possible to design a simultaneously stabilizing controller to ensure robust stability for sufficiently small uncertainties around the nominal plant and the known perturbation. Example 3.5 illustrates the comparison with the small-gain approach. Conclusions are given in Section IV.

Notation: Let $\mathcal{U}$ contain the extended closed right-half plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). Let $\mathbb{R}$ denote the set of real numbers. Let $\mathcal{R}$ denote proper rational functions which do not have any poles in the region of instability $\mathcal{U}$; let $R_p(R_{\rm sp})$ denote proper (strictly proper) rational functions with real coefficients. Let $\mathcal{M}(\mathcal{R})$ denote the set of matrices whose entries are in $\mathcal{R}$. A matrix M is called $\mathcal{R}$-stable iff $M \in \mathcal{M}(\mathcal{R})$; an $\mathcal{R}$-stable M is called $\mathcal{R}$-unimodular iff M^{-1} is $\mathcal{R}$-stable. For an $\mathcal{R}$-stable M, the norm $\|\cdot\|$ is defined as $\| M \| = \sup_{s\in\partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ denotes the maximum singular value and $\partial\mathcal{U}$ denotes the boundary of $\mathcal{U}$. A right-coprime factorization (RCF) and a left-coprime factorization (LCF) of $P \in R_p^{n_o\times n_i}$ are denoted by (N,D) and $(\tilde{D},\tilde{N})$, where $N,D,\tilde{N},\tilde{D} \in \mathcal{M}(\mathcal{R})$, $D^{-1},\tilde{D}^{-1}$ are proper and $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$. If (N,D) is an RCF and $(\tilde{D},\tilde{N})$ is an LCF of the (nominal) plant P, then $(N+\mathbf{G}_A D, D)$ is an RCF and $(\tilde{D},\tilde{N}+\tilde{D}\mathbf{G}_A)$ is an LCF of the additively-perturbed plant $(P+\mathbf{G}_A)$; similarly, $(N, D+\mathbf{G}_F N)$ is an RCF and $(\tilde{D}+\tilde{N}\mathbf{G}_F,\tilde{N})$ is an LCF of the feedback-perturbed plant $P(I+\mathbf{G}_F P)^{-1}$. The identity map (of appropriate dimension) is denoted by I. $a := b$ means a is defined as b.

II. System Description

Consider the LTI, MIMO systems $\mathcal{S}(P+\mathbf{G}_A, C)$ (Fig. 1) and $\mathcal{S}(P(I+\mathbf{G}_F P)^{-1}, C)$ (Fig. 2) where $P \in R_p^{n_o\times n_i}$ and $C \in R_p^{n_i\times n_o}$ are the transfer-functions of the plant and the controller. The LTI $\mathcal{R}$-stable transfer-functions $\mathbf{G}_A$ and $\mathbf{G}_F$ represent additive-perturbations and feedback-perturbations. If $\mathbf{G}_A = 0$ in $\mathcal{S}(P+\mathbf{G}_A, C)$ or if $\mathbf{G}_F = 0$ in $\mathcal{S}(P(I+\mathbf{G}_F P)^{-1}, C)$, these systems become the standard unity-feedback system $\mathcal{S}(P,C)$, called the nominal system. It is assumed that P and C do not have any hidden modes associated with eigenvalues in $\mathcal{U}$ and that the systems $\mathcal{S}(P+\mathbf{G}_A, C)$ and $\mathcal{S}(P(I+\mathbf{G}_F P)^{-1}, C)$ are well posed. The additively-perturbed plant and the feedback-perturbed plant are denoted by $(P+\mathbf{G}_A)$ and $P(I+\mathbf{G}_F P)^{-1}$. The (nominal) plant P is not necessarily $\mathcal{R}$-stable; $(P+\mathbf{G}_A)$ is $\mathcal{R}$-stable if and only if P is $\mathcal{R}$-stable; $P(I+\mathbf{G}_F P)^{-1}$ may not be $\mathcal{R}$-stable even if P is $\mathcal{R}$-stable, and it may be $\mathcal{R}$-stable when P is not $\mathcal{R}$-stable. The additively-perturbed plant $(P+\mathbf{G}_A)$ has the same $\mathcal{U}$-poles as P; the feedback-perturbed plant $P(I+\mathbf{G}_F P)^{-1}$ has the same $\mathcal{U}$-zeros as P.

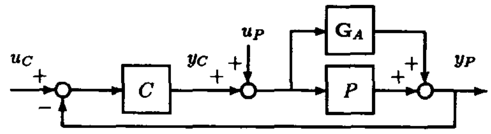

Fig. 1. The system $\mathcal{S}(P+\mathbf{G}_A, C)$.

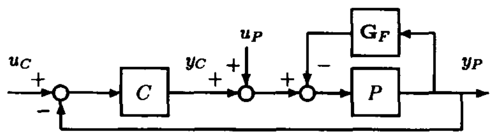

Fig. 2. The system $\mathcal{S}(P(I+\mathbf{G}_F P)^{-1}, C)$.

Using an RCF (N,D) of P and an RCF (N_C, D_C) of C, with $D\xi_P = u_P + y_C$ and $D_C\xi_C = u_C - y_P$, the system $\mathcal{S}(P+\mathbf{G}_A, C)$ is described as follows

$$[\tilde{D}_C D + \tilde{N}_C(N+\mathbf{G}_A D)]\xi_P = [\tilde{D}_C \quad \tilde{N}_C]\begin{bmatrix} u_P \\ u_C\end{bmatrix},$$

$$\begin{bmatrix} N+\mathbf{G}_A D \\ D\end{bmatrix}\xi_P + \begin{bmatrix} 0 \\ -u_P\end{bmatrix} = \begin{bmatrix} y_P \\ y_C\end{bmatrix}. \tag{1}$$

Equation (1) is of the form $\mathbf{D}_A\xi = N_L u$, $y = N_R\xi + Eu$; the closed-loop transfer-function $H : u \mapsto y$ is $H = N_R\mathbf{D}_A^{-1}N_L + E$. The system $\mathcal{S}(P(I+\mathbf{G}_F P)^{-1}, C)$ is described similarly, where $\mathbf{D}_A$, N_R in (1) are replaced by

$$\mathbf{D}_F := [\tilde{D}_C(D+\mathbf{G}_F N) + \tilde{N}_C N]$$

$$N_R := \begin{bmatrix} N \\ D+\mathbf{G}_F N\end{bmatrix}.$$

The system $\mathcal{S}(P+\mathbf{G}_A, C)$ is well posed if and only if $\mathbf{D}_A$ is biproper, equivalently, the transfer-function $H : u \mapsto y$ is proper; similarly, the system $\mathcal{S}(P(I+\mathbf{G}_F P)^{-1}, C)$ is well posed if and only if $\mathbf{D}_F$ is biproper. If P or C is strictly proper, then $\mathbf{D}_A$ and $\mathbf{D}_F$ are biproper, and hence, these systems are well posed.

III. Simultaneously Stabilizing Controllers

A well-posed LTI system interconnection is said to be $\mathcal{R}$-stable iff the transfer-function from all exogenous inputs to all closed-loop signals is $\mathcal{R}$-stable, i.e., all transfer-functions are in $\mathcal{M}(\mathcal{R})$.

The nominal system $\mathcal{S}(P,C)$ is said to be $\mathcal{R}$-stable iff the transfer-function H from $u := [u_P^T, u_C^T]^T$ to $y := [y_P^T, y_C^T]^T$ is $\mathcal{R}$-stable [8]. Similarly, when $\mathbf{G}_A$ and $\mathbf{G}_F$ are $\mathcal{R}$-stable, the systems $\mathcal{S}(P+\mathbf{G}_A, C)$ and $\mathcal{S}(P(I+\mathbf{G}_F P)^{-1}, C)$ are said to be $\mathcal{R}$-stable iff the transfer-function $H : u \mapsto y$ is $\mathcal{R}$-stable [4].

Let (N,D) be an RCF and $(\tilde{D},\tilde{N})$ be an LCF of the plant $P \in R_p^{n_o\times n_i}$. Let (N_C, D_C) and $(\tilde{D}_C, \tilde{N}_C)$ be an RCF and LCF of the controller $C \in R_p^{n_i\times n_o}$. Let $\mathbf{G}_A \in \mathcal{M}(\mathcal{R})$. By (1), since

$H = (N_R \mathbf{D}_A^{-1} N_L + E)$ is a bicoprime factorization, the system $\mathcal{S}(P + \mathbf{G}_A, C)$ is $\mathcal{R}$-stable if and only

$$\mathbf{D}_A := \tilde{D}_C D + \tilde{N}_C (N + \mathbf{G}_A D) \quad \text{is} \quad \mathcal{R}\text{-unimodular.} \tag{2}$$

Let $\mathbf{G}_F \in \mathcal{M}(\mathcal{R})$ be such that the feedback-perturbed plant $P(I + \mathbf{G}_F P)^{-1} \in \mathcal{M}(R_p)$. With $\mathbf{D}_F$ replacing $\mathbf{D}_A$, the system $\mathcal{S}(P(I + \mathbf{G}_F P)^{-1}, C)$ is $\mathcal{R}$-stable if and only if

$$\mathbf{D}_F := \tilde{D}_C (D + \mathbf{G}_F N) + \tilde{N}_C N \quad \text{is} \quad \mathcal{R}\text{-unimodular.} \tag{3}$$

The controller C is said to be an $\mathcal{R}$-stabilizing controller for $P \in R_p^{n_o \times n_i}$ iff $C \in R_p^{n_i \times n_o}$ and the nominal system $\mathcal{S}(P, C)$ is $\mathcal{R}$-stable. The controller C is an $\mathcal{R}$-stabilizing controller for P if and only if C is given by ([8], [5])

$$C = (V - Q\tilde{N})^{-1}(U + Q\tilde{D}) = (\tilde{U} + DQ)(\tilde{V} - NQ)^{-1} \tag{4}$$

for some $\mathcal{R}$-stable Q such that $(V - Q\tilde{N})$ is biproper (which holds for all $Q \in \mathcal{M}(\mathcal{R})$ when P is strictly proper), where $U, V, \tilde{U}, \tilde{V}$ are $\mathcal{R}$-stable matrices such that

$$\begin{bmatrix} V & U \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -\tilde{U} \\ N & \tilde{V} \end{bmatrix} = I. \tag{5}$$

Lemma 3.1 (Simultaneously Stabilizing Controllers): Let (N, D) be an RCF and $(\tilde{D}, \tilde{N})$ be an LCF of the plant $P \in R_p^{n_o \times n_i}$. Let $\mathbf{G}_A \in \mathcal{M}(\mathcal{R})$.

a) Consider the system $\mathcal{S}(P + \mathbf{G}_A, C)$. The $\mathcal{R}$-stabilizing controller C for the nominal plant P is also an $\mathcal{R}$-stabilizing controller for the additively-perturbed plant $(P + \mathbf{G}_A)$ if and only if C is given by (4), where $Q \in \mathcal{M}(\mathcal{R})$ is such that $(V - Q\tilde{N})$ is biproper and

$$\mathbf{D}_A = I + (U + Q\tilde{D})\mathbf{G}_A D \quad \text{is} \quad \mathcal{R}\text{-unimodular.} \tag{6}$$

b) Consider the system $\mathcal{S}(P(I + \mathbf{G}_F P)^{-1}, C)$. The $\mathcal{R}$-stabilizing controller C for the nominal plant P is also an $\mathcal{R}$-stabilizing controller for the feedback-perturbed plant $P(I + \mathbf{G}_F P)^{-1}$ if and only if C is given by (4), where $Q \in \mathcal{M}(\mathcal{R})$ is such that $(V - Q\tilde{N})$ is biproper and

$$\mathbf{D}_F = I + (V - Q\tilde{N})\mathbf{G}_F N \quad \text{is} \quad \mathcal{R}\text{-unimodular.} \tag{7}$$

□

Proof: The system $\mathcal{S}(P + \mathbf{G}_A, C)$ is $\mathcal{R}$-stable if and only if (2) holds. The controller C is an $\mathcal{R}$-stabilizing controller for the nominal plant P if and only if C is given by (4); using $\tilde{D}_C = (V - Q\tilde{N})$, $\tilde{N}_C = (U + Q\tilde{D})$ in (2), by (5), $\mathbf{D}_A$ becomes $\mathbf{D}_A = (I + (U + Q\tilde{D})\mathbf{G}_A D)$. To ensure that $\mathbf{D}_A$ is $\mathcal{R}$-unimodular and that the controller C is proper, the controller-parameter $Q \in \mathcal{M}(\mathcal{R})$ must be such that (6) holds and the denominator $\tilde{D}_C$ is biproper. Similarly, in the system $\mathcal{S}(P(I + \mathbf{G}_F P)^{-1}, C)$, condition (7) follows from ensuring that the matrix $\mathbf{D}_F$ is $\mathcal{R}$-unimodular where C is given by (4). □

We now show that there exists a simultaneously $\mathcal{R}$-stabilizing controller $C \in R_p^{n_i \times n_o}$ for the nominal plant and the additively-perturbed plant (similarly, for the nominal and the feedback-perturbed plant).

Theorem 3.2 (Guaranteed Existence and a Class of Simultaneously $\mathcal{R}$-Stabilizing Controllers): Let (N, D) be an RCF and $(\tilde{D}, \tilde{N})$ be an LCF of the plant $P \in R_{sp}^{n_o \times n_i}$. Let $U, V, \tilde{U}, \tilde{V} \in \mathcal{M}(\mathcal{R})$ be as in (5).

a) In the system $\mathcal{S}(P + \mathbf{G}_A, C)$, where $\mathbf{G}_A \in \mathcal{M}(\mathcal{R})$ is a known $\mathcal{R}$-stable additive-perturbation, the nominal plant P and the additively perturbed plant $(P + \mathbf{G}_A)$ can be simultaneously $\mathcal{R}$-stabilized. Furthermore, a set of simultaneously $\mathcal{R}$-stabilizing controllers for P and $(P + \mathbf{G}_A)$ is given by

$$\{C = (V - (\bar{Q}_a + Q_a)\tilde{N})^{-1}(U + (\bar{Q}_a + Q_a)\tilde{D}) \mid Q_a \in \mathcal{R}^{n_i \times n_o}, \ \| Q_a \| < \| \tilde{D}\mathbf{G}_A D \bar{\mathbf{D}}_A^{-1} \|^{-1}\}, \tag{8}$$

$$\bar{Q}_a := U\mathbf{G}_A \sum_{\ell=2}^{k} \frac{r_\ell}{k^\ell} (DU\mathbf{G}_A)^{\ell-2}\tilde{U}, \tag{9}$$

$$\bar{\mathbf{D}}_A := I + (U + \bar{Q}_a\tilde{D})\mathbf{G}_A D = \left(I + \frac{U\mathbf{G}_A D}{k}\right)^k = I + (U\mathbf{G}_A D) + \sum_{\ell=2}^{k} \frac{r_\ell}{k^\ell}(U\mathbf{G}_A D)^\ell \tag{10}$$

where $r_\ell = \frac{k!}{(k-\ell)!\ell!}$ are the binomial coefficients and k is any integer such that $k > \| U\mathbf{G}_A D \|$.

b) In the system $\mathcal{S}(P(I + \mathbf{G}_F P)^{-1}, C)$, where $\mathbf{G}_F \in \mathcal{M}(\mathcal{R})$ is a known $\mathcal{R}$-stable feedback-perturbation, the nominal plant P and the feedback-perturbed plant $P(I + \mathbf{G}_F P)^{-1}$ can be simultaneously $\mathcal{R}$-stabilized. Furthermore, a set of simultaneously $\mathcal{R}$-stabilizing controllers for P and $P(I + \mathbf{G}_F P)^{-1}$ is given by

$$\{C = (V - (\bar{Q}_f + Q_f)\tilde{N})^{-1}(U + (\bar{Q}_f + Q_f)\tilde{D}) \mid Q_f \in \mathcal{R}^{n_i \times n_o}, \ \| Q_f \| < \| \tilde{N}\bar{\mathbf{G}}_F N \bar{\mathbf{D}}_F^{-1} \|^{-1}\}, \tag{11}$$

$$\bar{Q}_f := -V\mathbf{G}_F \sum_{\ell=2}^{k} \frac{r_\ell}{k^\ell} (NV\mathbf{G}_F)^{\ell-2}\tilde{V}, \tag{12}$$

$$\bar{\mathbf{D}}_F := I + (V - \bar{Q}_f\tilde{N})\mathbf{G}_F N = \left(I + \frac{V\mathbf{G}_F N}{k}\right)^k = I + (V\mathbf{G}_F N) + \sum_{\ell=2}^{k} \frac{r_\ell}{k^\ell}(V\mathbf{G}_F N)^\ell \tag{13}$$

where $r_\ell = \frac{k!}{(k-\ell)!\ell!}$ are the binomial coefficients and k is any integer such that $k > \| V\mathbf{G}_F N \|$.

Proof: We prove the additive-perturbation case in detail: By Lemma 3.1-ii), C is an $\mathcal{R}$-stabilizing controller for both P and $(P + \mathbf{G}_A)$ if and only if C is given by (4), where $Q \in \mathcal{M}(\mathcal{R})$ is such that (6) holds. Substituting $Q = \bar{Q}_a$ of (9) into (4), since $\tilde{U}\tilde{D} = DU$ by (5), we obtain

$$\bar{C}_a := (V - \bar{Q}_a\tilde{N})^{-1}(U + \bar{Q}_a\tilde{D}) = (V - \bar{Q}_a\tilde{N})^{-1}U\left(I + \sum_{\ell=2}^{k} \frac{r_\ell}{k^\ell}(\mathbf{G}_A DU)^{\ell-1}\right). \tag{14}$$

Since P is strictly proper, $(V - Q\tilde{N})$ is biproper for all $Q \in \mathcal{M}(\mathcal{R})$ and hence, $\bar{C}_a$ is proper. With $Q = \bar{Q}_a$, using the binomial expansion in (6), since the integer $k > \| U\mathbf{G}_A D \|$, $\mathbf{D}_A = I + (U + \bar{Q}_a\tilde{D})\mathbf{G}_A D = I + (U + U\mathbf{G}_A \sum_{\ell=2}^{k} r_\ell k^{-\ell} (DU\mathbf{G}_A)^{\ell-2}\tilde{U}\tilde{D})\mathbf{G}_A D = (I + \frac{U\mathbf{G}_A D}{k})^k = \bar{\mathbf{D}}_A$ is $\mathcal{R}$-unimodular. By Lemma 3.1, the controller $\bar{C}_a$ in (14) simultaneously $\mathcal{R}$-stabilizes P and $(P + \mathbf{G}_A)$. For any $Q_a \in \mathcal{R}^{n_i \times n_o}$ satisfying (8), $(I + Q_a\tilde{D}\mathbf{G}_A D\bar{\mathbf{D}}_A^{-1})$ is $\mathcal{R}$-unimodular. Therefore condition (6) is still satisfied for $Q = (\bar{Q}_a + Q_a)$ since $\mathbf{D}_A = \bar{\mathbf{D}}_A + Q_a\tilde{D}\ \mathbf{G}_A D = (I + Q_a\tilde{D}\mathbf{G}_A D\ \bar{\mathbf{D}}_A^{-1})\mathbf{D}_A$ is $\mathcal{R}$-unimodular. Since P is strictly proper, the controllers in set (8) are proper for any choice of Q. By Lemma 3.1, the controllers in (8) simultaneously $\mathcal{R}$-stabilize P and $(P + \mathbf{G}_A)$.

Set (11) for the feedback-perturbation case follows similarly by choosing $Q = \bar{Q}_f$ in condition (7) of Lemma 3.1-ii): $\mathbf{D}_F$ becomes

$\mathbf{\tilde{D}}_F$ given in (13), which is $\mathcal{R}$-unimodular, and

$$\bar{C}_f := (V - \bar{Q}_f\tilde{N})^{-1}(U + \bar{Q}_f\tilde{D}) = V\left(I + \sum_{\ell=2}^{k} \frac{r_\ell}{k^\ell}(\mathbf{G}_F NV)^{\ell-1}\right)^{-1}(U + \bar{Q}_f\tilde{D}) \quad (15)$$

is a simultaneously $\mathcal{R}$-stabilizing controller for P and $P(I + \mathbf{G}_F P)^{-1}$. □

Remark 3.3:

a) Simultaneously $\mathcal{R}$–Stabilizing Controllers When the Plant is Not Strictly Proper: In Theorem 3.2, it was assumed for simplicity that the plant is strictly proper; $(V - Q\tilde{N})$ is biproper for all $Q \in \mathcal{M}(\mathcal{R})$ when $P \in R_{\text{sp}}^{n_o \times n_i}$, and hence, the controllers are proper for any choice of Q. For proper plants which are not strictly proper, sets (8) and (11) can be modified to ensure that the simultaneously $\mathcal{R}$-stabilizing controllers are proper. For any $P \in R_{\text{p}}^{n_o \times n_i}, U$ in (5) can be chosen strictly proper; to see that there exists a solution with $U \in \mathcal{M}(R_{\text{sp}})$, let $X, Y, \tilde{X}, \tilde{Y} \in \mathcal{M}(\mathcal{R})$ satisfy (5). Let $\hat{Q} \in \mathcal{R}^{n_i \times n_o}$ be any $\mathcal{R}$-stable matrix such that $\hat{Q}(\infty) = -X(\infty)\tilde{D}^{-1}(\infty)$. Define $U := (X + \hat{Q}\tilde{D})$, $V := (Y - \hat{Q}\tilde{N})$, $\tilde{U} := (\tilde{X} + D\hat{Q})$, $\tilde{V} := (\tilde{Y} - N\hat{Q})$; then by construction, $U(\infty) = 0$.

In the additive-perturbation case, with U strictly proper, $\bar{Q}_a$ in (9) is also strictly proper; hence, $(V - \bar{Q}_a\tilde{N})$ is biproper since $(V, -\bar{Q}_a \tilde{N})(\infty)D(\infty) = I - (U + \bar{Q}_a\tilde{D})(\infty)N(\infty) = I$; this shows that the controller $\bar{C}_a$ in (8) is in fact strictly proper. To ensure that all controllers in set (8) are proper, the choice of Q_a should be restricted to those for which the denominator $(V - (\bar{Q}_a + Q_a)\tilde{N})$ is biproper, in addition to the constraint that $\| Q_a \| < \| \tilde{D}\, \mathbf{G}_A D, \mathbf{\tilde{D}}_A^{-1} \|^{-1}$. Any $Q_a \in \mathcal{M}(\mathcal{R})$ which is strictly proper would satisfy this biproperness condition on the controller's denominator. In the feedback-perturbation case, to ensure that all controllers in set (11) are proper, Q_f should be such that $(V - (\bar{Q}_f + Q_f)\tilde{N})$ is biproper, in addition to $\| Q_f \| < \| \tilde{N}\mathbf{G}_F N\mathbf{\tilde{D}}_F^{-1} \|^{-1}$.

b) Existence Proof Using the Parity-Interlacing-Property: Two given plants can be simultaneously $\mathcal{R}$-stabilized if and only if a "pseudoplant," which is formed by these two plants and the associated Bezout identity (5), is strongly $\mathcal{R}$-stabilizable (see, for example, [7] and [8]). Applying this well-known result to the additive-perturbation case, by (6), P and $(P + \mathbf{G}_A)$ can be simultaneously $\mathcal{R}$-stabilized if and only if the pseudoplant $P_{Sa} := \tilde{D}\mathbf{G}_A D\,(I + U\mathbf{G}_A D)^{-1}$ can be strongly $\mathcal{R}$-stabilized. In the feedback-perturbation case, by (7), P and $P(I + \mathbf{G}_F P)^{-1}$ can be simultaneously $\mathcal{R}$-stabilized if and only if the pseudoplant $P_{Sf} := \tilde{N}\mathbf{G}_F\, N(I + V\mathbf{G}_F N)^{-1}$ can be strongly $\mathcal{R}$-stabilized. Since this is a result of independent interest, we show that P_{Sa} (and similarly P_{Sf}) is always strongly $\mathcal{R}$-stabilizable using the parity-interlacing property:

Consider P_{Sa}; by (5), $DU = \tilde{U}\tilde{D}$ implies that the pair $((I + U\,\mathbf{G}_A D), \tilde{D}\mathbf{G}_A D)$ is right-coprime since $(I - U\mathbf{G}_A D)(I + U\,\mathbf{G}_A D) + (U\mathbf{G}_A\tilde{U})(\tilde{D}\mathbf{G}_A D) = I$. Without loss of generality, it can be assumed that $(I + U\mathbf{G}_A D)$ is nonsingular; this can be guaranteed by choosing U strictly proper in (5) as explained in the previous remark. Now P_{Sa} is strongly $\mathcal{R}$-stabilizable if and only if the pair $((I + U\mathbf{G}_A D), \tilde{D}\mathbf{G}_A D)$ satisfies the parity-interlacing property. By (5), for any blocking $\mathcal{U}$-zero $s_0 \in \mathcal{U} \cap \mathbb{R}$ of $(\tilde{D}\mathbf{G}_A D)$, $(\tilde{U}\tilde{D}\mathbf{G}_A D)(s_0) = (DU\mathbf{G}_A D)(s_0) = 0$. Since the pair $((I + U\mathbf{G}_A D), \tilde{D}\mathbf{G}_A D)$ is right-coprime, $\det(I + U\mathbf{G}_A D)(s_0) \neq 0$; therefore, $\det(I + U\mathbf{G}_A D)(s_0) = \det(I + \frac{1}{2}U\mathbf{G}_A D)^2(s_0)$ is positive at all real-axis blocking $\mathcal{U}-$ zeros of $(\tilde{D}\,\mathbf{G}_A D)$ and hence, the parity-interlacing-property is satisfied. Similarly, consider P_{Sf}; by (5), $NV = \tilde{V}\tilde{N}$ implies that the pair $((I + V\mathbf{G}_F N), \tilde{N}\mathbf{G}_F N)$ is right-coprime since $(I - V\mathbf{G}_F N)(I + V\mathbf{G}_F N) + (V\mathbf{G}_F\tilde{V})(\tilde{N}\mathbf{G}_F N) = I$. As in the case of P_{Sa}, it can be shown that the pair $((I + V\mathbf{G}_F N), \tilde{N}\mathbf{G}_F N)$ satisfies the parity-interlacing property since $\det(I + V\mathbf{G}_F N)(s_0) = \det(I + \frac{1}{2}V\mathbf{G}_F N)^2(s_0)$ is positive at all real-axis blocking $\mathcal{U}$-zeros of $(\tilde{N}\mathbf{G}_F N)$.

This approach also clearly verifies the existence of simultaneously $\mathcal{R}$-stabilizing controllers for P and $(P + \mathbf{G}_A)$ (and similarly for P and $P(I + \mathbf{G}_F P)^{-1}$). The proof given for Theorem 3.2, however, has the advantage of explicitly constructing (a class) of simultaneously $\mathcal{R}$-stabilizing controllers.

c) Comparison with the Interpolation Method for Scalar Plants: As shown in the previous remark, the pseudoplant P_{Sa} (and similarly P_{Sf}) always satisfies the parity-interlacing-property; however, the simultaneously $\mathcal{R}$-stabilizing controller still remains to be constructed. One approach for SISO plants is to obtain a simultaneously $\mathcal{R}$-stabilizing controller by interpolation: Let $P = ND^{-1} \in R_p$. Define $m := 1 + (U + QD)\mathbf{G}_A D$; then by (6), $m(s_0) = 1$ for all $s_0 \in \mathcal{U} \cap \mathbb{R}$ such that $(\mathbf{G}_A D^2)(s_0) = 0$. For appropriate choice of W (not necessarily finite-dimensional), $m = e^{(W\mathbf{G}_A D^2)}$ is a unit of $\mathcal{R}$ (see interpolation in the disc algebra [8]). A parameterization of all $\mathcal{R}$-stabilizing controllers is obtained as[1]

$$C = (V - QN)^{-1}(U + QD) = \frac{m - 1}{\mathbf{G}_A - (m - 1)P}, \quad m = e^{(W\mathbf{G}_A D^2)}. \quad (16)$$

The controller parameters in (16), which are obtained by interpolation, are not finite dimensional. On the other hand, the controller parameters Q in (8), and similarly (15), in Theorem 3.2 are finite dimensional; they are obtained explicitly without interpolation. Furthermore, the finite-dimensional controller parameters in Theorem 3.2 also apply to MIMO plants.

d) Existence of Simultaneously $\mathcal{R}$-Stabilizing Controllers by the Small-Gain Condition: By Lemma 3.1-ii), there exists a controller C which $\mathcal{R}$-stabilizes the nominal plant and the additively-perturbed plant simultaneously if and only if there is an $\mathcal{R}$-stable Q such that $\mathbf{D}_A$ in (6) is $\mathcal{R}$-unimodular. A sufficient condition that is often used in this type of problem is the small-gain condition (see, for example, [8], [1]): If we could always find an $\mathcal{R}$-stable Q such that

$$\| (U + Q\tilde{D})\mathbf{G}_A D \| < 1 \quad (17)$$

then the small-gain condition would imply that $\mathbf{D}_A = I + (U + Q\tilde{D})\mathbf{G}_A D$ is $\mathcal{R}$-unimodular and the existence of simultaneously $\mathcal{R}$-stabilizing controllers is concluded. As shown in Example 3.5, since there is no restriction on the perturbation $\mathbf{G}_A$, such Q may not exist. Since Theorem 3.2 ensures that there always exist controllers which $\mathcal{R}$-stabilize P and $(P + \mathbf{G}_A)$ simultaneously, the small-gain condition is too conservative for this problem.

Comment 3.4 (Robustness of the Simultaneously $\mathcal{R}$-Stabilizing Controllers): In the usual robust stabilization problem, $\mathbf{G}_A$ and $\mathbf{G}_F$ would be unknown perturbations, which have "sufficiently small-gain" and consequently, it is possible to find one $\mathcal{R}$-stabilizing controller for the entire class of plants resulting from small perturbations of the nominal plant. We explain the additive-perturbation case briefly; the feedback-perturbation case is similar: In the system $\mathcal{S}(P + \mathbf{G}_A, C)$, by Lemma 3.1-ii), the simultaneously $\mathcal{R}$-stabilizing controller C is given by (4), where $Q \in \mathcal{M}(\mathcal{R})$ is such that (6) holds. Note that the matrix $\mathbf{D}_A$ in (6) is $\mathcal{R}$-unimodular if and only if $(I + \mathbf{G}_A D(U + Q\tilde{D}))$ is $\mathcal{R}$-unimodular. Suppose that the $\mathcal{R}$-stable additive-perturbation $\mathbf{G}_A$ satisfies

$$\| \mathbf{G}_A \| < \| DU \|^{-1}. \quad (18)$$

[1] The parameterization in (16) using the unit $m = e^{(W\mathbf{G}_A D^2)}$ was suggested by Dr. V. Blondel.

Then by the small-gain theorem, $(I + \mathbf{G}_A DU)$ is $\mathcal{R}$-unimodular (equivalently, $M_a := (I + U\mathbf{G}_A D)$ is $\mathcal{R}$-unimodular); therefore (6) is satisfied for $Q = 0$ and hence, the controller $C = V^{-1}U$ simultaneously $\mathcal{R}$-stabilizes any number of additively-perturbed plants $(P + \mathbf{G}_A)$ for all $\mathbf{G}_A$ satisfying (18). Furthermore, (6) is satisfied by choosing any $\mathcal{R}$-stable Q of appropriate dimensions such that $\| Q \| < \| \tilde{D}\mathbf{G}_A D M_a^{-1} \|^{-1}$; for any Q having norm that satisfies this condition, the corresponding controllers in the set of all $\mathcal{R}$-stabilizing controllers (4) $\mathcal{R}$-stabilize any number of additively-perturbed plants, where the perturbations are unknown but have "sufficiently small gain" satisfying (18).

The problem considered here does not restrict the known perturbation $\mathbf{G}_A$ or $\mathbf{G}_F$ to have small gain, and yet, Theorem 3.2 shows that it is possible to simultaneously $\mathcal{R}$-stabilize the nominal plant together with a perturbed plant for a known perturbation $\mathbf{G}_A$ or $\mathbf{G}_F$. It is still possible to consider additional small unknown uncertainties around the nominal plant and around either the additively-perturbed version $(P + \mathbf{G}_A)$ or the feedback-perturbed version $P(I + \mathbf{G}_F P)^{-1}$ and obtain robustly $\mathcal{R}$-stabilizing controllers which simultaneously $\mathcal{R}$-stabilize the two systems. We explain this robustness property for the additive-perturbation case in detail: Consider the simultaneously $\mathcal{R}$-stabilizing controller $\bar{C}_a$ in (14) for P and $(P + \mathbf{G}_A)$, obtained by setting $Q_a = 0$ in set (8); let $\bar{\mathbf{D}}_A$ and $\bar{Q}_a$ be as in (10) and (9). The controller $\bar{C}_a$ $\mathcal{R}$-stabilizes the entire class of additively-perturbed plants $(P + \mathbf{G}_A + \Delta_a)$ for all unknown $\mathcal{R}$-stable additive uncertainties Δ_a satisfying

$$\| \Delta_a \| < \| D\bar{\mathbf{D}}_A^{-1}(U + \bar{Q}_a\tilde{D}) \|^{-1}. \tag{19}$$

If the unknown $\mathcal{R}$-stable additive-uncertainty Δ_a satisfies an alternate small-gain condition, namely

$$\| \Delta_a \| < \| D(U + \bar{Q}_a\tilde{D}) \|^{-1} \tag{20}$$

then the controller $\bar{C}_a$ simultaneously $\mathcal{R}$-stabilizes P and $(P + \mathbf{G}_A)$ and also all additively-perturbed plants $(P + \Delta_a)$. We modify the gain of Δ_a further to obtain robust $\mathcal{R}$-stability for all sufficiently small additive-uncertainties around both P and $(P + \mathbf{G}_A)$: For all $\Delta_a \in \mathcal{M}(\mathcal{R})$ satisfying

$$\| \Delta_a \| < \min \left\{ \| D\bar{\mathbf{D}}_A^{-1}(U + \bar{Q}_a\tilde{D}) \|^{-1}, \; \| D(U + \bar{Q}_a\tilde{D}) \|^{-1} \right\} \tag{21}$$

the controller $\bar{C}_a$ in (8) simultaneously $\mathcal{R}$-stabilizes P, $(P + \mathbf{G}_A)$, $(P + \mathbf{G}_A + \Delta_a)$, $(P + \Delta_a)$.

All three robustness claims above follow from Lemma 3.1-ii): Replacing $\mathbf{G}_A$ with $(\mathbf{G}_A + \Delta_a)$ in (2), the matrix $\mathbf{D}_A$ becomes $\mathbf{D}_A = I + (U + \bar{Q}_a\tilde{D})\,(\mathbf{G}_A + \Delta_a)D = (I + (U + \bar{Q}_a\tilde{D})\Delta_a D\bar{\mathbf{D}}_A^{-1})\bar{\mathbf{D}}_A$. Recall that $\bar{\mathbf{D}}_A = I + (U + \bar{Q}_a\tilde{D})\mathbf{G}_A D$ given in (10) is $\mathcal{R}$-unimodular. By the small-gain theorem, if Δ_a satisfies (19), then $[I + \Delta_a D\bar{\mathbf{D}}_A^{-1}(U + \bar{Q}_a\tilde{D})]$ is $\mathcal{R}$-unimodular, equivalently, $\mathbf{D}_A$ is $\mathcal{R}$-unimodular and hence, $\bar{C}_a$ is a $\mathcal{R}$-stabilizing controller for $(P + \mathbf{G}_A + \Delta_a)$. For all Δ_a satisfying (20), the matrix $(I + \Delta_a D(U + \bar{Q}_a\tilde{D}))$ is $\mathcal{R}$-unimodular and hence, with $\mathbf{G}_A = 0$, the matrix $\mathbf{D}_A$ is again $\mathcal{R}$-unimodular. If Δ_a satisfies (21), then $\mathbf{D}_A$ is $\mathcal{R}$-unimodular with or without $\mathbf{G}_A$ and hence, all additive-perturbations Δ_a around P and around $(P + \mathbf{G}_A)$ are $\mathcal{R}$-stabilized by using $\bar{C}_a$.

For the feedback-perturbation case, for all unknown feedback-uncertainties $\Delta_f \in \mathcal{M}(\mathcal{R})$ satisfying

$$\| \Delta_f \| < \min \left\{ \| N\bar{\mathbf{D}}_F^{-1}(V - \bar{Q}_f\tilde{N}) \|^{-1}, \; \| N(V - \bar{Q}_f\tilde{N}) \|^{-1} \right\} \tag{22}$$

the controller $\bar{C}_f$ in (15), which simultaneously $\mathcal{R}$-stabilizes P and $P(I + \mathbf{G}_F P)^{-1}$ also $\mathcal{R}$-stabilizes $P(I + (\mathbf{G}_F + \Delta_f)P)^{-1}$ and $P(I + \Delta_f P)^{-1}$.

Example 3.5: Let the region of instability $\mathcal{U}$ be the extended closed right-half plane. Consider the diagonal plant

$$P = \operatorname{diag}\left[\frac{-(s-3)}{s(s+1)}, \frac{18}{s-12}\right].$$

Then (5) is satisfied with

$$\begin{aligned} N = \tilde{N} &= \operatorname{diag}\left[\frac{-(s-3)}{(s+1)(s+3)}, \frac{18}{s+6}\right], \\ D = \tilde{D} &= \operatorname{diag}\left[\frac{s}{s+3}, \frac{s-12}{s+6}\right], \\ U = \tilde{U} &= I, \\ V = \tilde{V} &= \operatorname{diag}\left[\frac{s+5}{s+1}, 1\right]. \end{aligned}$$

Let the additive perturbation $\mathbf{G}_A$ be

$$\mathbf{G}_A = (g_{ij}) \in \mathcal{R}^{2\times 2}, g_{22}(0) = 0, | g_{12}(0) | \geq 0.5. \tag{23}$$

Since

$$\begin{aligned} g_{22}(0) &= 0 \\ (U + Q\tilde{D})\mathbf{G}_A D(0) &= \begin{bmatrix} 0 & -2g_{12}(0) \\ 0 & 0 \end{bmatrix} \end{aligned}$$

since $| g_{12}(0) | \geq 0.5$, the norm $\| (U + Q\tilde{D})\mathbf{G}_A \, D \| \geq 1$ for all $\mathcal{R}$-stable Q, and hence, (17) can never be satisfied. As commented in Remark 3.3-iii), choosing Q which makes the gain of $((U + Q\tilde{D})\mathbf{G}_A D)$ smaller than one is not always possible. Now to illustrate the method in the proof of Theorem 3.2 for finding simultaneously $\mathcal{R}$-stabilizing controllers, suppose that the perturbation $\mathbf{G}_A$ in (23) is

$$\mathbf{G}_A = \begin{bmatrix} 0 & \frac{3}{4(s+1)} \\ \frac{1}{s+1} & 0 \end{bmatrix}.$$

Then $\| U\mathbf{G}_A D \| = 1.5$. Taking $k > \| U\mathbf{G}_A D \|$ as $k = 2$, $\bar{Q}_a$ in (9) becomes $\bar{Q}_a = \frac{1}{4}\mathbf{G}_A$. By (14), an $\mathcal{R}$-stabilizing controller $\bar{C}_a$ for P and $(P + \mathbf{G}_A)$ is $\bar{C}_a = (V - \frac{1}{4}\mathbf{G}_A\tilde{N})^{-1}(I + \frac{1}{4}\mathbf{G}_A D)$.

IV. Conclusion

We showed that for any proper P and stable $\mathbf{G}_A$ (or $\mathbf{G}_F$), there exist proper controllers which simultaneously $\mathcal{R}$-stabilize P and $(P + \mathbf{G}_A)$ (or P and $P(I + \mathbf{G}_F P)^{-1}$). We gave a class of simultaneously $\mathcal{R}$-stabilizing controllers for the additive and feedback-perturbation cases. We showed that it is possible to design the simultaneously $\mathcal{R}$-stabilizing controller to ensure robust stability for additional small unknown uncertainties around the nominal plant and the known perturbation.

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Resource Optimization and $(\min, +)$ Spectral Theory

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***Abstract*—We show that certain resource optimization problems relative to timed event-graphs reduce to linear programs. The auxiliary variables which allow this reduction can be interpreted in terms of eigenvectors in the $(\min, +)$ algebra.**

I. Introduction

Timed event-graphs (TEG's) are a subclass of timed Petri nets which can be used to model deterministic discrete-event dynamic systems subject to saturation and synchronization phenomena, typically, flexible manufacturing systems, multiprocessor systems, and transportation networks [1]–[3], [5], [16], [17]. The most remarkable result about TEG's [1], [3], [4] is certainly the following: a TEG functioning at maximal speed reaches, after a finite time, a periodic regime. More precisely, let x denote the counter function of a given transition of the graph. That is, $x(t)$ represents the number of firings of the transition up to time t, usually the number of parts of a certain type produced up to time t, the number of messages sent up to time $t \cdots$. Then, there exists a constant λ (the periodic throughput) and $c \in \mathbb{N} \setminus \{0\}$, $T \in \mathbb{N}$ such that

$$t \geq T \Rightarrow x(t+c) = c \times \lambda + x(t). \qquad (1)$$

The denomination of periodic throughput is justified, because we get from (1)

$$\lambda = \lim_{t \to \infty} \frac{x(t)}{t} = \text{mean} \frac{\text{number of events}}{\text{time}}.$$

We shall consider the case where some markings are unknown: we assume that the initial markings (number of tokens) of some places are given by some indeterminates $q_1, \cdots, q_k \in \mathbb{N}$. Typically, the indeterminate q_i associated with place P_i represents an unknown quantity of resources (number of machines, pallets, processors, storage places, buffers) which corresponds to the (unknown) initial marking of this place P_i (see the example in Section IV). Then, the periodic throughput $\lambda = \lambda(q)$ becomes a (nondecreasing) function of the resource indeterminates $q_1, \cdots, q_k$. Given a linear cost

$$J(q) = p_1 q_1 + \cdots + p_k q_k \qquad (2)$$

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(p_i is the price of one unit of resource i) and a minimal required periodic throughput $\bar{\lambda}$, we consider the following resource optimization problem

$$(\mathcal{RO}) \quad \min\{J(q) \mid q \in \mathbb{N}^k, \quad \lambda(q) \geq \bar{\lambda}\}$$

which consists of minimizing the cost of the resources needed to obtain (at least) the periodic throughput $\bar{\lambda}$. A slightly different resource optimization problem was first considered by Cohen *et al.* in [4] where an iterative algorithm was given to find a minimal allocation of resources saturating the bottleneck process. The particular problem $(\mathcal{RO})$ has been previously considered by Hillion and Proth [16], Laftit *et al.* [17], Proth and Xie [19], and by the author in [12]. In [16], it was noticed that $(\mathcal{RO})$ is an integer linear programming problem with, unfortunately, as many constraints as elementary circuits in the graph. In [17] and [19], the authors obtained a nice reduction to an auxiliary linear program—with real and integer variables—involving essentially as many constraints as edges in the graph, so that the exact solution can be obtained for much larger systems. This result, however, was only given for a restricted class of cost functions and of TEG's. The purpose of this note is to extend the results of [17] and [19] to general TEG's and general cost functions: the linear program that we give is exactly the same as in [17] and [19], but without undesirable restrictions. As a by-product, using the duality between holding times and initial markings in TEG's, we obtain an analogous reduction for an extended resource optimization problem (which involves the possibility of selecting a higher performance equipment instead of buying more machines with a given performance). The simple proof proposed here relies on an elementary key result of the $(\min, +)$ spectral theory: we show that the throughput constraint $\lambda(q) \geq \bar{\lambda}$ is equivalent to the existence of a finite "subeigenvector" of a particular matrix (subeigenvectors are analogous to potentials in scheduling theory [2] and to excessive functions in potential theory). Then, this potential inequality translates to a set of linear constraints. These results are taken from the thesis of the author, up to some subsidiary extensions. We also mention that the related problem of the symbolic computation of the periodic throughput $\lambda(q)$ has been dealt with in [12] and [13].

II. A Subeigenvector Lemma

We first recall some $(\min, +)$ spectral theory. The traditional term "$(\min, +)$-algebra" refers to the set $\mathbf{R} \cup \{+\infty\}$ equipped with min (denoted by $\oplus$) and addition (denoted by $\otimes$). The zero element is written $\varepsilon \stackrel{\text{def}}{=} +\infty$, and we set $e \stackrel{\text{def}}{=} 0$ for the unit. We denote by $\mathbf{R}_{\min}$ this algebraic structure. There is a natural order relation on $\mathbf{R}_{\min}$ given by

$$a \preceq b \iff a \oplus b = \min(a,b) = b \iff a \geq b.$$

This is precisely the dual of the usual order (e.g., $2 \succeq 3$). The $(\min, +)$ notation extends to matrices in the obvious way. We shall write, for instance

$$(AB)_{ij} = (A \otimes B)_{ij} = \bigoplus_k A_{ik} \otimes B_{kj} = \min_k (A_{ik} + B_{kj})$$

and consequently $A^k = A \otimes \cdots \otimes A$ (k times). The spectral radius $\rho(C)$ [1], [6], [9], [14], of a $n \times n$ matrix C with entries in $\mathbf{R}_{\min}$