set-point control. Furthermore, the simplicity and transparency of the analysis makes it accessible to a broad audience.

2) Since the GAS property holds for all (positive) choices of the filter constants a_i and b_i, we do not impose any restriction on the bandwidth of the filters. As pointed out in the previous section, however, the tracking performance will depend on its frequency response characteristic.
3) Even though we have considered here a simpler model for the flexibility effects our result applies verbatim to the model used in [1].
4) For the sake of brevity, we have presented only the case where joint stiffness is exactly known. Stability robustness *vis a vis* joint stiffness and gravity forces uncertainty can be easily established and follows directly from the analysis of [1].

An alternative solution for replacing velocity measurement in regulation tasks of flexible joint robots has recently been proposed in [5]. As point of comparison between both results, it is interesting to note that the controller of [5] contains n second order relative degree zero filters with inputs the motor shaft position and the gravity compensating constant term and output the generated torques. In contrast with this, the controller proposed here consists only of n first order relative degree zero filters. On the other hand, the design procedure followed in [5] is based on energy shaping ideas which exploit the natural structure of the system in a more transparent way. In particular, the Lyapunov function used for the analysis in [5] is the systems total energy. Although for the present design we have a nice interpretation in terms of a feedback interconnection of passive subsystems (see Remark 1), a clearer physical understanding of the Lyapunov function is as yet unavailable. Some simulation studies comparing these schemes are reported in [9].

References

[1] P. Tomei, "A simple PD controller for robots with elastic joints," *IEEE Trans. Automat. Contr.*, vol. 36, no. 10, pp. 1208–1213, 1991.

[2] M. Spong and M. Vidyasagar, *Robot Dynamics and Control.* New York: Wiley & Sons, 1989.

[3] P. Lancaster and M. Tismenetsky, *The Theory of Matrices.* New York: Academic, 1985.

[4] R. Kelly, "A simple set-point robot controller by using only position measurements," in *Proc. IFAC World Congres,* Sydney, Australia, Jul. 1993.

[5] A. Ailon and R. Ortega, "An observer-based set-point controller for robot manipulators with flexible joints," *Syst. Contr. Lett.*, vol. 21, pp. 329–335, 1993.

[6] H. Berghuis and H. Nijmeijer, "Global regulation of robots using only position measurements," *Syst. Contr. Lett.*, vol. 21, pp. 289–293, 1993.

[7] P. R. Belanger, "Estimation of angular velocity and acceleration from shaft encoder measurements," in *Proc. IEEE Conf. Rob. and Automation*, Nice, France, 1992, pp. 585–592.

[8] H. Berghuis, R. Ortega, and H. Nijmeijer, "A robust adaptive robot controller," *IEEE Trans. Rob. and Automation*, vol. 9, no. 6, pp. 825–830, Dec. 1993.

[9] A. Loria, "Set point control of robots with flexible joints: A comparative study" Masters' thesis, Université de Technologie de Compiègne, France, June 1993.

[10] C. Desoer and M. Vidyasagar, *Feedback systems: Input–Output Properties.* New York: Academic, 1975.

Stabilizing Controller Design for Linear Systems with Sensor or Actuator Failures

A. Nazli Gündeş

***Abstract*— In this note, the stability of the standard linear, time-invariant, multi-input multi-output unity-feedback system is investigated in the presence of either sensor or actuator failures. Any diagonal stable perturbation is included in the failure descriptions. Stabilizing controllers are synthesized for two failure classes: the first class allows at most one failure at a time; the second class requires at least one connection without failure. A parameterization of all stabilizing controllers is achieved with prior knowledge of the failure. A controller design method requiring no knowledge of the failure is also presented; this method is restricted to plants for which certain closed-loop transfer functions can be made diagonal.**

I. Introduction

A feedback system is said to have complete integrity if it remains stable in the presence of sensor or actuator failures. If all sensors (or actuators) are disconnected simultaneously, the standard unity-feedback system becomes an open-loop cascade connection and, therefore, the plant and the controller both have to be stable. With a stable plant and controller, necessary and sufficient conditions for complete integrity were derived in [2]. For stable plants, a controller design method ensuring complete integrity was developed in [1]. A number of reliable stabilization results are also available; stabilization using two controllers was studied in [7], [9], and a methodology for the design of reliable control systems was developed in [8].

The standard linear, time-invariant (LTI), multi-input multi-output (MIMO) unity-feedback system with sensor or actuator failures was studied in [4]. Necessary and sufficient conditions were given for integrity with a prespecified maximum number of failures. It was shown that the integrity requirement imposes constraints on denominator matrices of coprime factorizations of the plant and the controller.

A controller design method ensuring integrity is developed in this note. In the standard integrity problem, failure means that a sensor or an actuator is completely disconnected. A more general failure description is used here, allowing the corresponding connection to be multiplied by any arbitrary stable transfer function (including zero) in case of failure. Requiring complete integrity against simultaneous failure of all sensors or of all actuators restricts the plants and the controllers to be stable. Instead, two classes of sensor or actuator failures are considered: the first allows only one failure at a time; in the second, any number of failures may occur but at least one connection must remain normal. A parameterization of all controllers ensuring integrity against either sensor or actuator failures is given in Theorem 3.5, which explicitly shows how the controller can be updated without solving for the class of all controllers for each new plant resulting from the failure. A controller design methodology, which does not require the failure to be known, is developed for plants that allow diagonalization of certain transfer functions of the nominal system (Proposition 3.6). The results apply to continuous-time and discrete-time systems.

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II. Preliminaries

Notation

Let $\mathcal{U}$ be a subset of the complex numbers $\mathbb{C}$, $\mathcal{U}$ is closed and symmetric about the real axis, and $\pm\infty \in \mathcal{U}$, $\mathbb{C}\backslash\mathcal{U}$ is nonempty. Let $\mathcal{R}_{\mathcal{U}}$, $\mathbb{R}_p(s)$, and $\mathbb{R}_{sp}(s)$ be the ring of proper rational functions with no poles in $\mathcal{U}$, the ring of proper rational functions, and the set of strictly proper rational functions of s (with real coefficients). The group of units of $\mathcal{R}_{\mathcal{U}}$ is $\mathcal{J}$ and the set of nonstrictly proper elements of $\mathcal{R}_{\mathcal{U}}$ is $\mathcal{I} = \mathcal{R}_{\mathcal{U}}\backslash\mathbb{R}_{sp}(s)$. The set of matrices with entries in $\mathcal{R}_{\mathcal{U}}$ is $\mathcal{M}(\mathcal{R}_{\mathcal{U}})$, M is called $\mathcal{R}_{\mathcal{U}}$-stable iff $M \in \mathcal{M}(\mathcal{R}_{\mathcal{U}})$, and $M \in \mathcal{M}(\mathcal{R}_{\mathcal{U}})$ is $\mathcal{R}_{\mathcal{U}}$-unimodular iff $\det M \in \mathcal{J}$. If $p, q \in \mathcal{R}_{\mathcal{U}}$, then $p \sim q$ iff $p = aq$ for some $a \in \mathcal{J}$.

$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ denotes a state-space representation of P. P has no uncontrollable $\mathcal{U}$-modes or equivalently, P is $\mathcal{R}_{\mathcal{U}}$-stabilizable, if $\operatorname{rank}[sI_n - \mathbf{AB}] = n$ for all $s \in \mathcal{U}$. Similarly, P has no unobservable $\mathcal{U}$ modes or equivalently, P is $\mathcal{R}_{\mathcal{U}}$-detectable, if $\operatorname{rank}[(sI_n - \mathbf{A})^T\mathbf{C}^T]^T = n$ for all $s \in \mathcal{U}$. If P has no hidden modes associated with eigenvalues in $\mathcal{U}$, i.e. if P is $\mathcal{R}_{\mathcal{U}}$-stabilizable and $\mathcal{R}_{\mathcal{U}}$-detectable, then P has no hidden $\mathcal{U}$ modes.

(N_P, D_P) denotes a right-coprime factorization (RCF) and $(\tilde{D}_P, \tilde{N}_P)$ denotes a left-coprime factorization (LCF) of $P \in \mathbb{R}_P(s)^{no\times ni}$, where $N_P, \tilde{N}_P \in \mathcal{R}_{\mathcal{U}}^{no\times ni}$, $D_P \in \mathcal{R}_{\mathcal{U}}^{ni\times ni}$, $\tilde{D}_P \in \mathcal{R}_{\mathcal{U}}^{no\times no}$, $P = N_P D_P^{-1} = \tilde{D}_P^{-1}\tilde{N}_P$, $\det D_P \sim \det \tilde{D}_P \in \mathcal{I}$. Similarly, (N_C, D_C) denotes a RCF and $(\tilde{D}_C, \tilde{N}_C)$ denotes a LCF of $C \in \mathbb{R}_p(s)^{ni\times no}$.

System Descriptions

Consider the LTI, MIMO feedback systems $\mathcal{S}(P, C)$ (the nominal system), $\mathcal{S}(F_S, P, C)$, $\mathcal{S}(P, F_A, C)$ (Figs. 1–3); P: $e_P \mapsto y_P$, C: $e_C \mapsto y_C$, F_S: $y_P \mapsto y_S$, and F_A: $y_C \mapsto y_A$ represent the plant's and the controller's transfer functions, the sensor, and the actuator connections. The $(n_o \times n_o)$ sensor-failure matrix F_S and the $(n_i \times n_i)$ actuator-failure matrix F_A are diagonal, $\mathcal{R}_{\mathcal{U}}$-stable, with entries nominally equal to one; the failure of the jth sensor or actuator is represented by a stable rational function (including zero, but different than one) in the jth diagonal entry of F_S or F_A. Nominally, $F_S = I_{no}$ and $F_A = I_{ni}$. Let $\mathcal{F}_{Sk} := \{\operatorname{diag}[f_1 \cdots f_{no}] \mid$ for $j = 1, \cdots, n_o$, $f_j \in \mathcal{R}_{\mathcal{U}}$ and at least $(n_o - k)$ of the $f_j = 1\}$ denote the class of sensor failures; the subscript k is the maximum possible number of sensor failures, $k \in \{1, \cdots, n_o\}$. The class $\mathcal{F}_{S1}$ is the set of all diagonal $\mathcal{R}_{\mathcal{U}}$-stable matrices where at most one of the diagonal entries may be different than one, representing the failure of at most one of the n_o sensors. The class $\mathcal{F}_{S(no-1)}$ represents failure of at most $(n_o - 1)$ of the n_o sensors and $\mathcal{F}_{Sno}$ includes the possibility of all sensors failing simultaneously. A special case of failure is when a sensor gets completely disconnected, represented by a zero in the corresponding diagonal entry of F_S. This sub-class is $\mathcal{F}_{Sk}^0 := \{\operatorname{diag}[f_1 \cdots f_{no}] \mid$ for $j = 1, \cdots, n_o$, $f_j = 1$ or $f_j = 0$, $\sum_{j=1}^{n_o} f_j \geq (n_o - k)\}$. Similarly, $\mathcal{F}_{Am} := \{\operatorname{diag}[f_1 \cdots f_{ni}] \mid$ for $j = 1, \cdots, n_i$, $f_j \in \mathcal{R}_{\mathcal{U}}$ and at least $(n_i - m)$ of the $f_j = 1\}$ denotes the class of actuator failures; the subscript m is the maximum possible number of actuator failure, $m \in \{1, \cdots, n_i\}$. The actuator-failure classes $\mathcal{F}_{A1}$, $\mathcal{F}_{A(ni-1)}$, $\mathcal{F}_{Ani}$ are also defined similarly, and $\mathcal{F}_{Am}^0 \subset \mathcal{F}_{Am}$ is defined as $\mathcal{F}_{Am}^0 := \{\operatorname{diag}[f_1 \cdots f_{ni}] \mid$ for $j = 1, \cdots, n_i$, $f_j = 1$ or $f_j = 0$, $\sum_{j=1}^{n_i} f_j \geq (n_i - m)\}$.

In $\mathcal{S}(P, C)$, $\mathcal{S}(F_S, P, C)$, and $\mathcal{S}(P, F_A, C)$, let $u := [u_P^T u_C^T]^T$, $y := [y_P^T y_C^T]^T$, and let H: $u \mapsto y$, H_S: $u \mapsto y$, and H_A: $u \mapsto y$ be the closed-loop transfer functions. It is assumed that $P \in \mathbb{R}_p(s)^{no\times ni}$, $C \in \mathbb{R}_p(s)^{ni\times no}$, $\mathcal{S}(P, C)$, $\mathcal{S}(F_S, P, C)$, and $\mathcal{S}(P, F_A, C)$ are well posed; equivalently, $H \in \mathcal{M}(\mathbb{R}_p(s))$, $H_S \in \mathcal{M}(\mathbb{R}_p(s))$, $H_A \in \mathcal{M}(\mathbb{R}_p(s))$, and P and C have no hidden $\mathcal{U}$ modes.

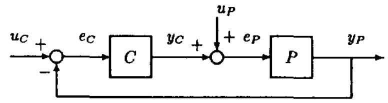

Fig. 1. The system $\mathcal{S}(P, C)$.

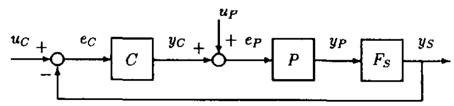

Fig. 2. The system $\mathcal{S}(F_S, P, C)$.

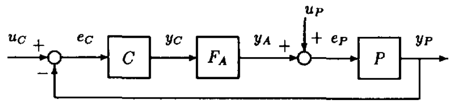

Fig. 3. The system $\mathcal{S}(P, F_A, C)$.

Definitions 2.1 ($\mathcal{R}_{\mathcal{U}}$-stability, integrity, failure hidden-$\mathcal{U}$ modes):

a)

i) The system $\mathcal{S}(F_S, P, C)$ is said to be $\mathcal{R}_{\mathcal{U}}$-stable iff $H_S \in \mathcal{M}(\mathcal{R}_{\mathcal{U}})$.

ii) For $k = 1, \cdots, n_o$, $\mathcal{S}(F_S, P, C)$ is said to have k-sensor integrity iff it is $\mathcal{R}_{\mathcal{U}}$-stable for all $F_S \in \mathcal{F}_{Sk}$.

iii) The plant P is said to have no k-sensor failure hidden $\mathcal{U}$ modes iff for all $F_S \in \mathcal{F}_{Sk}$, $F_S P$ is $\mathcal{R}_{\mathcal{U}}$-detectable, i.e., for all $F_S \in \mathcal{F}_{Sk}$, $\operatorname{rank}[(sI_n - \mathbf{A})^T(F_S\mathbf{C})^T]^T = n$ for all $s \in \mathcal{U}$.

iv) The controller C is said to have no k-sensor failure hidden $\mathcal{U}$ modes iff for all $F_S \in \mathcal{F}_{Sk}$, CF_S is $\mathcal{R}_{\mathcal{U}}$-stabilizable.

b)

i) $\mathcal{S}(P, F_A, C)$ is said to be $\mathcal{R}_{\mathcal{U}}$-stable iff $H_A \in \mathcal{M}(\mathcal{R}_{\mathcal{U}})$.

ii) For $m = 1, \cdots, n_i$, $\mathcal{S}(P, F_A, C)$ is said to have m-actuator integrity iff it is $\mathcal{R}_{\mathcal{U}}$-stable for all $F_A \in \mathcal{F}_{Am}$.

iii) P is said to have no m-actuator failure hidden $\mathcal{U}$ modes iff for all $F_A \in \mathcal{F}_{Am}$, PF_A is $\mathcal{R}_{\mathcal{U}}$-stabilizable, i.e., for all $F_A \in \mathcal{F}_{Am}$, $\operatorname{rank}[sI_n - \mathbf{A}\,\mathbf{B}F_A] = n$ for all $s \in \mathcal{U}$.

iv) C is said to have no m-actuator failure hidden $\mathcal{U}$ modes iff for all $F_A \in \mathcal{F}_{Am}$, $F_A C$ is $\mathcal{R}_{\mathcal{U}}$-detectable.

Lemma 2.2 (Conditions for Integrity) [4]:

a) $\mathcal{S}(F_S, P, C)$ has k-sensor integrity ($\mathcal{S}(P, F_A, C)$ has m-actuator integrity) if and only if $D_S = \begin{bmatrix} \tilde{D}_P & -\tilde{N}_P N_C \\ F_S & D_C \end{bmatrix}$ is $\mathcal{R}_{\mathcal{U}}$-unimodular for all $F_S \in \mathcal{F}_{Sk}$ ($D_A = \begin{bmatrix} D_P & -F_A \\ \tilde{N}_C N_P & \tilde{D}_C \end{bmatrix}$ is $\mathcal{R}_{\mathcal{U}}$-unimodular for all $F_A \in \mathcal{F}_{Am}$).

b) If $\mathcal{S}(F_S, P, C)$ has k-sensor integrity ($\mathcal{S}(P, F_A, C)$ has m-actuator integrity), then P and C have no k-sensor-failure hidden $\mathcal{U}$ modes (m-actuator failure hidden $\mathcal{U}$ modes, respectively).

III. Controller Synthesis

In $\mathcal{S}(P, C)$, C is an $\mathcal{R}_{\mathcal{U}}$-stabilizing controller iff $C \in \mathbb{R}_p(s)^{ni\times no}$ and $\mathcal{S}(P, C)$ is $\mathcal{R}_{\mathcal{U}}$-stable. The set $\mathbf{S}(P) := \{C \mid C \in \mathbb{R}_p(s)^{ni\times no}$ and $\mathcal{S}(P, C)$ is $\mathcal{R}_{\mathcal{U}} -$ stable$\}$ is the set of all $\mathcal{R}_{\mathcal{U}}$-stabilizing controllers.

Definitions 3.1 (Controllers with Integrity):

a) In $\mathcal{S}(F_S, P, C)$, C is called a controller with k-sensor integrity iff $C \in \mathbf{R}_p(s)^{ni\times no}$ and $\mathcal{S}(F_S, P, C)$ has k-sensor integrity. $\mathbf{S}_{Sk}(P) := \{C \mid C \in \mathbf{R}_p(s)^{ni\times no}$ and $\mathcal{S}(F_S, P, C)$ has k-sensor integrity$\}$ is called the set of all controllers with k-sensor integrity.

b) In $\mathcal{S}(P, F_A, C)$, C is called a controller with m-actuator integrity iff $C \in \mathbf{R}_p(s)^{ni\times no}$ and $\mathcal{S}(P, F_A, C)$ has m-actuator-integrity. $\mathbf{S}_{Am}(P) := \{C \mid C \in \mathbf{R}_p(s)^{ni\times no}$ and $\mathcal{S}(P, F_A, C)$ has $m-$actuator integrity$\}$ is called the set of all controllers with m-actuator integrity. □

Let (N_P, D_P) and $(\tilde{D}_P, \tilde{N}_P)$ be any RCF and LCF of P; then there exist $U_P, \tilde{U}_P, V_P, \tilde{V}_P \in \mathcal{M}(\mathcal{R_U})$, such that $V_P D_P + U_P N_P = I_{ni}$, $\tilde{D}_P\tilde{V}_P + \tilde{N}_P\tilde{U}_P = I_{no}$, $V_P\tilde{U}_P = U_P\tilde{V}_P$. In $\mathcal{S}(P, C)$, $C \in \mathbf{R}_p(s)^{ni\times no}$ is an $\mathcal{R_U}$-stabilizing controller if and only if some RCF (N_C, D_C) and some LCF $(\tilde{D}_C, \tilde{N}_C)$ of C satisfy ([10], [3])

$$\begin{bmatrix}\tilde{D}_C & \tilde{N}_C\\ -\tilde{N}_P & \tilde{D}_P\end{bmatrix}\begin{bmatrix}D_P & -N_C\\ N_P & D_C\end{bmatrix} = \begin{bmatrix}I_{ni} & 0\\ 0 & I_{no}\end{bmatrix}. \tag{3.1}$$

The set $\mathbf{S}(P)$ of all $\mathcal{R_U}$-stabilizing controllers in $\mathcal{S}(P, C)$ is: $\mathbf{S}(P) = \{C = (V_P - Q\tilde{N}_P)^{-1}(U_P + Q\tilde{D}_P) = (\tilde{U}_P + D_PQ)(\tilde{V}_P - N_PQ)^{-1} \mid Q \in \mathcal{R}_{\mathcal{U}}^{ni\times no}, \det(V_P - Q\tilde{N}_P) \sim \det(\tilde{V}_P - N_PQ) \in \mathcal{I}\}$. For any $Q \in \mathcal{M}(\mathcal{R_U})$, $\det(V_P - Q\tilde{N}_P) \sim \det(\tilde{V}_P - N_PQ)$. If P is strictly proper, then $\det(V_P - Q\tilde{N}_P) \in \mathcal{I}$ (equivalently, $\det(\tilde{V}_P - N_PQ) \in \mathcal{I}$) for all $Q \in \mathcal{M}(\mathcal{R_U})$. When $P \in \mathcal{R}_{\mathcal{U}}^{no\times ni}$, the set $\mathbf{S}(P)$ becomes $\mathbf{S}(P) = \{C = (I_{ni} - QP)^{-1}Q = Q(I_{no} - PQ)^{-1} \mid Q \in \mathcal{R}_{\mathcal{U}}^{ni\times no}, \det(I_{ni} - QP) = \det(I_{no} - PQ) \in \mathcal{I}\}$.

Proposition 3.2 (Parameterization of Controllers with Integrity): a) In $\mathcal{S}(F_S, P, C)$, the set $\mathbf{S}_{Sk}(P)$ of all controllers with k-sensor integrity is

$$\begin{aligned}\mathbf{S}_{Sk}(P) = \{C &= (V_P - Q\tilde{N}_P)^{-1}(U_P + Q\tilde{D}_P)\\ &= (\tilde{U}_P + D_PQ)(\tilde{V}_P - N_PQ)^{-1} \mid Q \in \mathcal{R}_{\mathcal{U}}^{ni\times no},\\ &\quad \det(V_P - Q\tilde{N}_P) \sim \det(\tilde{V}_P - N_PQ) \in \mathcal{I},\\ &\quad \text{and for all } F_S \in \mathcal{F}_{Sk},\\ &\quad F_S + (\tilde{V}_P - N_PQ)\tilde{D}_P(I_{no} - F_S)\\ &= I_{no} - N_P(U_P + Q\tilde{D}_P)(I_{no} - F_S) \text{ is } \mathcal{R_U} - \text{unimodular}\}.\end{aligned} \tag{3.2}$$

b) In $\mathcal{S}(P, F_A, C)$, the set $\mathbf{S}_{Am}(P)$ of all controllers with m-actuator integrity is: $\mathbf{S}_{Am}(P) = \{C = (V_P - Q\tilde{N}_P)^{-1}(U_P + Q\tilde{D}_P) = (\tilde{U}_P + D_PQ)(\tilde{V}_P - N_PQ)^{-1} \mid Q \in \mathcal{R}_{\mathcal{U}}^{ni\times no}, \det(V_P - Q\tilde{N}_P) \sim \det(\tilde{V}_P - N_PQ) \in \mathcal{I}$ and for all $F_A \in \mathcal{F}_{Am}$, $F_A + D_P(V_P - Q\tilde{N}_P)(I_{ni} - F_A) = I_{ni} - (\tilde{U}_P + D_PQ)\tilde{N}_P(I_{ni} - F_A)$ is $\mathcal{R_U}$ − unimodular$\}$.

c) Let $P \in \mathcal{M}(\mathcal{R_U})$; then $\mathbf{S}_{Sk}(P) = \{C = Q(I_{no} - PQ)^{-1} \mid Q \in \mathcal{R}_{\mathcal{U}}^{ni\times no}, \det(I_{no} - PQ) \in \mathcal{I}$, and for all $F_S \in \mathcal{F}_{Sk}$, $F_S + (I_{no} - PQ)(I_{no} - F_S) = I_{no} - PQ(I_{no} - F_S)$ is $\mathcal{R_U}$ − unimodular$\}$ and $\mathbf{S}_{Am}(P) = \{C = (I_{ni} - QP)^{-1}Q \mid Q \in \mathcal{R}_{\mathcal{U}}^{ni\times no}, \det(I_{ni} - QP) \in \mathcal{I}$, and for all $F_A \in \mathcal{F}_{Am}$, $F_A + (I_{ni} - QP)(I_{ni} - F_A) = I_{ni} - QP(I_{ni} - F_A)$ is $\mathcal{R_U}$ − unimodular$\}$.

Proof: We prove a); b) and c) are similar. For each $k = 1, \cdots, n_o$, $\mathbf{S}_{Sk}(P) \subset \mathbf{S}(P)$ because $F_S = I_{no} \in \mathcal{F}_{Sk}$. Using (3.1), performing elementary row operations (in $\mathcal{R_U}$) on the matrix D_S in Lemma 2.2, $C \in \mathbf{S}_{Sk}(P)$ if and only if $F_S + D_C\tilde{D}_P(I_{no} - F_S) = F_S + (I_{no} - N_P\tilde{N}_C)(I_{no} - F_S) = I_{no} - N_P\tilde{N}_C(I_{no} - F_S)$ is $\mathcal{R_U}$-unimodular, where $\tilde{N}_C = U_P + Q\tilde{D}_P$ and $D_C = \tilde{V}_P - N_PQ$. □

The conditions imposed on denominator matrices of coprime factorizations of P due to the requirement of having no failure hidden $\mathcal{U}$ modes are stated in Lemma 3.3 for the classes $\mathcal{F}_{S1}$, $\mathcal{F}_{S(no-1)}$, $\mathcal{F}_{A1}$, and $\mathcal{F}_{A(ni-1)}$; similar conditions are necessary for C [4].

Lemma 3.3 (Tests for Failure Hidden $\mathcal{U}$ Modes) [4]:

a) Consider $\mathcal{S}(F_S, P, C)$. Let $(\tilde{D}_P, \tilde{N}_P)$ be any LCF of P.

i) P has no k-sensor failure hidden $\mathcal{U}$ modes if and only if $(F_S, \tilde{D}_P)$ is a right coprime pair for all $F_S \in \mathcal{F}_{Sk}$.

ii) Let $F_S \in \mathcal{F}_{S1}$; P has no one-sensor failure hidden $\mathcal{U}$ modes if and only if there exists an $\mathcal{R_U}$-unimodular $L_1 \in \mathcal{R}_{\mathcal{U}}^{no\times no}$ such that

$$L_1\tilde{D}_P = \begin{bmatrix}1 & \tilde{d}_{1,2} & \cdots & \tilde{d}_{1,no}\\ 0 & \tilde{d}_{2,2} & \cdots & \tilde{d}_{2,no}\\ & & \ddots & \vdots\\ & \mathbf{O} & & \tilde{d}_{no,no}\end{bmatrix};$$

$$\left(\begin{bmatrix}\tilde{d}_{1,1+j}\\ \vdots\\ \tilde{d}_{j,1+j}\end{bmatrix}, \tilde{d}_{1+j,1+j}\right) \text{ is right-coprime,}$$

$$j = 1, \cdots, n_o - 1. \tag{3.3}$$

iii) Let $F_S \in \mathcal{F}_{S(no-1)}$; P has no $(n_o - 1)$-sensor failure hidden $\mathcal{U}$ modes if and only if there exists an $\mathcal{R_U}$-unimodular $L_{(n_o-1)} \in \mathcal{R}_{\mathcal{U}}^{no\times no}$ such that

$$L_{(n_o-1)}\tilde{D}_P = \begin{bmatrix}1 & & \mathbf{O} & \tilde{d}_{1,no}\\ & \ddots & & \vdots\\ & & 1 & \tilde{d}_{no-1,no}\\ & \mathbf{O} & & \tilde{d}_{no,no}\end{bmatrix};$$

$$(\tilde{d}_{j,no}, \tilde{d}_{no,no}) \text{ is coprime,} \qquad j = 1, \cdots, n_o - 1. \tag{3.4}$$

b) Consider $\mathcal{S}(P, F_A, C)$. Let (N_P, D_P) be any RCF of P.

i) P has no m-actuator failure hidden $\mathcal{U}$ modes if and only if (D_P, F_A) is a left coprime pair for all $F_A \in \mathcal{F}_{Am}$.

ii) Let $F_A \in \mathcal{F}_{A1}$; P has no one-actuator failure hidden $\mathcal{U}$ modes if and only if there exists an $\mathcal{R_U}$-unimodular $R_1 \in \mathcal{R}_{\mathcal{U}}^{ni\times ni}$ such that

$$D_PR_1 = \begin{bmatrix}1 & 0 & & \mathbf{O}\\ d_{2,1} & d_{2,2} & & \\ \vdots & \vdots & \ddots & \\ d_{ni,1} & d_{ni,2} & \cdots & d_{ni,ni}\end{bmatrix};$$

$$(d_{j+1,j+1}, [d_{1+j,1} \cdots d_{1+j,j}]) \text{ is left-coprime,}$$

$$j = 1, \cdots, n_i - 1. \tag{3.5}$$

iii) Let $F_A \in \mathcal{F}_{A(ni-1)}$; P has no $(n_i - 1)$-actuator failure hidden $\mathcal{U}$ modes if and only if there exists an $\mathcal{R_U}$-unimodular $R_{(n_i-1)} \in \mathcal{R}_{\mathcal{U}}^{ni\times ni}$ such that

$$D_PR_{(n_i-1)} = \begin{bmatrix}1 & & & \mathbf{O}\\ & \ddots & & \\ \mathbf{O} & & 1 & \\ d_{ni,1} & \cdots & d_{ni,ni-1} & d_{ni,ni}\end{bmatrix};$$

$$(d_{ni,ni}, d_{ni,j}) \text{ is coprime,} \qquad j = 1, \cdots, n_i - 1. \tag{3.6}$$

Remarks:

i)

a) Let $F_S \in \mathcal{F}_{S1}$; let P have no one-sensor failure hidden $\mathcal{U}$ modes; let $L_1 \in \mathcal{R}_{\mathcal{U}}^{no \times no}$ be $\mathcal{R}_{\mathcal{U}}$-unimodular as in (3.3). Then the coprimeness condition in (3.3) holds if and only if, for $j = 2, \cdots, n_o$, $l = 1, \cdots, j$, there exist $\tilde{y}_{j,l} \in \mathcal{R}_{\mathcal{U}}$ such that $\sum_{l=1}^{j} \tilde{y}_{j,l}\tilde{d}_{l,j} = 1$; equivalently, for any $\tilde{q}_{j,l} \in \mathcal{R}_{\mathcal{U}}$, $j = 2, \cdots, n_o$, $l = 1, \cdots, j-1$,

$$\left([\tilde{y}_{j,1} \ \cdots \ \tilde{y}_{j,j}] + [\tilde{q}_{j,1} \ \cdots \ \tilde{q}_{j,j-1}] \cdot \begin{bmatrix} \tilde{d}_{j,j} & & \mathbf{O} & -\tilde{d}_{1,j} \\ & \ddots & & \vdots \\ \mathbf{O} & & \tilde{d}_{j,j} & -\tilde{d}_{j-1,j} \end{bmatrix} \right) \begin{bmatrix} \tilde{d}_{1,j} \\ \vdots \\ \tilde{d}_{j,j} \end{bmatrix} = 1.$$

Let

$$Y_{S1} := \begin{bmatrix} 1 & 0 & & \mathbf{O} \\ \tilde{y}_{2,1} & \tilde{y}_{2,2} & & \\ \vdots & \vdots & \ddots & \\ \tilde{y}_{no,1} & \tilde{y}_{no,2} & \cdots & \tilde{y}_{no,no} \end{bmatrix} L_1 \tag{3.7}$$

(see (3.8) at the bottom of the page) where, for $j = 2, \cdots, n_o$, $l = 1, \cdots, j-1$, $\tilde{q}_{j,l} \in \mathcal{R}_{\mathcal{U}}$. For all $Q_{S1} \in \mathcal{M}(\mathcal{R}_{\mathcal{U}})$ as in (3.8) and for all $F_S \in \mathcal{F}_{S1}$, $I_{no} - (I_{no} - (Y_{S1} + Q_{S1})\tilde{D}_P)(I_{no} - F_S)$ is $\mathcal{R}_{\mathcal{U}}$-unimodular. Also, for all $F_S \in \mathcal{F}_{S1}$,

$$M_{S1} := I_{no} - (I_{no} - Y_{S1}\tilde{D}_P)(I_{no} - F_S) \quad \text{is } \mathcal{R}_{\mathcal{U}} - \text{unimodular.} \tag{3.9}$$

b) Let $F_S \in \mathcal{F}_{S(no-1)}$; let P have no $(n_o - 1)$-sensor failure hidden $\mathcal{U}$ modes; let $L_{(n_o-1)} \in \mathcal{R}_{\mathcal{U}}^{no \times no}$ be $\mathcal{R}_{\mathcal{U}}$-unimodular as in (3.4). Then the coprimeness condition in (3.4) holds if and only if, for $j = 1, \cdots, n_o - 1$, there exist $\tilde{x}_{j,j}$, $\tilde{x}_j \in \mathcal{R}_{\mathcal{U}}$ such that $\tilde{x}_{j,j}\tilde{d}_{no,no} + \tilde{x}_j\tilde{d}_{j,no} = 1$; equivalently, for any $\tilde{q}_{j+1} \in \mathcal{R}_{\mathcal{U}}$, $j = 1, \cdots, n_o$, $(\tilde{x}_{j,1} - \tilde{q}_{j+1}\tilde{d}_{j,no})\tilde{d}_{no,no} + (\tilde{x}_j + \tilde{q}_{j+1}\tilde{d}_{no,no})\tilde{d}_{j,n_o} = 1$. Let (see (3.10) at the bottom of the next page)

$$Q_{S(no-1)} := \operatorname{diag}[\tilde{q}_1 \ \ \tilde{q}_2\tilde{d}_{2,no} \ \cdots \ \tilde{q}_{no-1}\tilde{d}_{no-1,no} \ \ \tilde{q}_{no}] \cdot \begin{bmatrix} 0 & 0 & \cdots & 1 \\ d_{no,no} & 0 & \cdots & -d_{1,no} \\ & \ddots & \ddots & \vdots \\ \mathbf{O} & & d_{no,no} & -d_{no-1,no} \end{bmatrix} L_{no-1} \tag{3.11}$$

where, for $j = 1, \cdots, n_o$, $\tilde{q}_j \in \mathcal{R}_{\mathcal{U}}$. For all $Q_{S(no-1)} \in \mathcal{M}(\mathcal{R}_{\mathcal{U}})$ as in (3.11) and for all $F_S \in \mathcal{F}_{S(no-1)}$, $I_{no} - (I_{no} - (Y_{S(no-1)} + Q_{S(no-1)})\tilde{D}_P)(I_{ni} - F_S)$ is $\mathcal{R}_{\mathcal{U}}$-unimodular. Also, for all $F_S \in \mathcal{F}_{S(no-1)}$

$$M_{S(no-1)} := I_{no} - (I_{no} - Y_{S(no-1)}\tilde{D}_P)(I_{no} - F_S) \quad \text{is } \mathcal{R}_{\mathcal{U}} - \text{unimodular.} \tag{3.12}$$

ii)

a) Let $F_A \in \mathcal{F}_{A1}$, i.e,. let $m = 1$; let P have no one-actuator failure hidden $\mathcal{U}$ modes; let $R_1 \in \mathcal{R}_{\mathcal{U}}^{ni \times ni}$ be $\mathcal{R}_{\mathcal{U}}$-unimodular as in (3.5). Then the coprimeness condition in (3.5) holds if and only if, for $j = 2, \cdots, n_i$, $l = 1, \cdots, j$, there exist $y_{l,j} \in \mathcal{R}_{\mathcal{U}}$ such that $\sum_{l=1}^{j} d_{j,l}y_{l,j} = 1$; equivalently, for any $q_{l,j} \in \mathcal{R}_{\mathcal{U}}$

$$j = 2, \cdots, n_i, \qquad l = 1, \cdots, j-1,$$

$$[d_{j,1} \ \cdots \ d_{j,j}] \left(\begin{bmatrix} y_{1,j} \\ \vdots \\ y_{j,j} \end{bmatrix} + \begin{bmatrix} d_{j,j} & & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & & d_{j,j} \\ -d_{j,1} & \cdots & -d_{j,j-1} \end{bmatrix} \cdot \begin{bmatrix} q_{1,j} \\ \vdots \\ q_{j-1,j} \end{bmatrix} \right) = 1.$$

Let

$$Y_{A1} := R_1 \begin{bmatrix} 1 & y_{1,2} & \cdots & y_{1,ni} \\ 0 & y_{2,2} & \cdots & y_{2,ni} \\ & & \ddots & \vdots \\ & \mathbf{O} & & y_{ni,ni} \end{bmatrix}, \tag{3.13}$$

(see also (3.14) at the bottom of the next page) where, for $j = 2, \cdots, n_i$, $l = 1, \cdots, j-1$, $q_{l,j} \in \mathcal{R}_{\mathcal{U}}$. For all $Q_{A1} \in \mathcal{M}(\mathcal{R}_{\mathcal{U}})$ as in (3.14) and for all $F_A \in \mathcal{F}_{A1}$, $I_{ni} - (I_{ni} - D_P(Y_{A1} + Q_{A1}))(I_{ni} - F_A)$ is $\mathcal{R}_{\mathcal{U}}$-unimodular. Also, for all $F_A \in \mathcal{F}_{A1}$

$$M_{A1} := I_{ni} - (I_{ni} - F_A)(I_{ni} - D_P Y_{A1}) \text{ is } \mathcal{R}_{\mathcal{U}} - \text{unimodular.} \tag{3.15}$$

b) Let $F_A \in \mathcal{F}_{A(ni-1)}$, i.e., let $m = (n_i - 1)$. Let P have no $(n_i - 1)$-actuator failure hidden $\mathcal{U}$ modes. Let $R_{(n_i-1)} \in \mathcal{R}_{\mathcal{U}}^{ni \times ni}$ be $\mathcal{R}_{\mathcal{U}}$-unimodular as in (3.6). Then the coprimeness condition in (3.6) holds if and only if, for $j = 1, \cdots, n_i - 1$, there exist $x_{j,j}$, $x_j \in \mathcal{R}_{\mathcal{U}}$ such that $d_{ni,ni}x_{j,j} + d_{ni,j}x_j = 1$; equivalently, for any $q_{j+1} \in \mathcal{R}_{\mathcal{U}}$, $j = 1, \cdots, n_i$, $d_{ni,ni}(x_{j,j} - q_{j+1}d_{ni,j}) + d_{ni,j}(x_j + q_{j+1}d_{ni,ni}) = 1$. Let (see (3.16) at the bottom of the next page)

$$Q_{A(ni-1)} := R_{(ni-1)} \begin{bmatrix} 0 & d_{ni,ni} & & \mathbf{O} \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & d_{ni,ni} \\ 1 & -d_{ni,1} & \cdots & -d_{ni,ni-1} \end{bmatrix} \cdot \operatorname{diag}[q_1 \ \ d_{ni,2}q_2 \ \cdots \ d_{ni,ni-1}q_{ni-1} \ \ q_{ni}] \tag{3.17}$$

where, for $j = 1, \cdots, n_i$, $q_j \in \mathcal{R}_{\mathcal{U}}$. The diagonal entries of $D_P Y_{A(ni-1)}$ are all equal to one, the entries

$$Q_{S1} := \begin{bmatrix} 0 & \tilde{q}_{1,2} & \tilde{q}_{1,3} & \tilde{q}_{1,4} & \cdots & \tilde{q}_{1,no} \\ \tilde{q}_{2,1}[\tilde{d}_{2,2} & -\tilde{d}_{1,2}] & \tilde{q}_{2,3} & \tilde{q}_{2,4} & \cdots & \tilde{q}_{2,no} \\ [\tilde{q}_{3,1} & \tilde{q}_{3,2}][\tilde{d}_{3,3}I_2 & \begin{matrix} -\tilde{d}_{1,3} \\ -\tilde{d}_{2,3} \end{matrix}] & \tilde{q}_{3,4} & \cdots & \tilde{q}_{3,no} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ [\tilde{q}_{no,1} & \tilde{q}_{no,2} & \tilde{q}_{no,3} & \cdots \ \tilde{q}_{no,no-2} & \tilde{q}_{no,no-1}] & \left[\tilde{d}_{no,no}I_{no-1} \ \begin{matrix} -\tilde{d}_{1,no} \\ \vdots \\ -\tilde{d}_{no-1,no} \end{matrix}\right] \end{bmatrix} L_1 \tag{3.8}$$

immediately above the diagonal ones and the $(n_i, 1)$ entry may not be zero and all other entries are zero. For all $Q_{A(ni-1)} \in \mathcal{M}(\mathcal{R_U})$ as in (3.17) and for all $F_A \in \mathcal{F}_{A(ni-1)}$, $I_{ni} - (I_{ni} - D_P(Y_{A(ni-1)} + Q_{A(ni-1)}))(I_{ni} - F_A)$ is $\mathcal{R_U}$-unimodular. Also, for all $F_A \in \mathcal{F}_{A(ni-1)}$

$$M_{A(ni-1)} := I_{ni} - (I_{ni} - F_A)(I_{ni} - D_P Y_{A(ni-1)}) \text{ is } \mathcal{R_U}\text{-unimodular.} \tag{3.18}$$

Theorem 3.5 (All Controllers with Integrity):

a) In $\mathcal{S}(F_S, P, C)$: If $F_S \in \mathcal{F}_{S1}$, let P have no one-sensor failure hidden $\mathcal{U}$ modes; let Y_{Sk}, M_{Sk} be as in (3.7), (3.9).If $F_S \in \mathcal{F}_{S(no-1)}$, let P have no $(n_o - 1)$-sensor failure hidden $\mathcal{U}$ modes; let Y_{Sk}, M_{Sk} be as in (3.10), (3.12). Then for $k = 1$ or $(n_o - 1)$, the set of all controllers with k-sensor integrity is

$$\begin{aligned}\mathbf{S}_{Sk}(P) = \{\tilde{D}_C^{-1}\tilde{N}_C = N_C D_C^{-1} \mid \tilde{D}_C &= V_P + U_P M_{Sk}^{-1} Y_{Sk} \tilde{N}_P \\ &\quad - Q_S(I_{no} - \tilde{D}_P(I_{no} - F_S) M_{Sk}^{-1} Y_{Sk})\tilde{N}_P,\\[4pt] \tilde{N}_C = U_P M_{Sk}^{-1}(I_{no} - Y_{Sk}\tilde{D}_P)& \\ + Q_S \tilde{D}_P (I_{no} - (I_{no} - F_S)&(I_{no} - Y_{Sk}\tilde{D}_P))^{-1},\\[4pt] N_C = \tilde{U}_P(I_{no} - \tilde{D}_P Y_{Sk}) &+ D_P Q_S,\\[4pt] D_C = Y_{Sk} + F_S \tilde{V}_P(I_{no} - \tilde{D}_P Y_{Sk}) &+ F_S N_P Q_S,\\[4pt] Q_S \in \mathcal{R}_{\mathcal{U}}^{ni \times no}, \det \tilde{D}_C \sim \det D_C \in \mathcal{I}\}.&\end{aligned} \tag{3.19}$$

If the plant is strictly proper, then in (3.19), for any $Q_S \in \mathcal{R}_{\mathcal{U}}^{ni \times no}$, $\det \tilde{D}_C = \det(V_P + U_P M_{Sk}^{-1} Y_{Sk} \tilde{N}_P - Q_S(I_{no} - \tilde{D}_P\ (I_{no} - F_S) M_{Sk}^{-1} Y_{Sk})\tilde{N}_P) \sim \det D_C = \det(Y_{Sk} + F_S \tilde{V}_P(I_{no} - \tilde{D}_P Y_{Sk}) + F_S N_P Q_S) \in \mathcal{I}$.

b) In $\mathcal{S}(P, F_A, C)$: If $F_A \in \mathcal{F}_{A1}$, let P have no one-actuator failure hidden $\mathcal{U}$ modes; let Y_{Am}, M_{Am} be as in (3.13), (3.15). If $F_A \in \mathcal{F}_{A(ni-1)}$, let P have no $(n_i - 1)$-actuator failure hidden $\mathcal{U}$ modes; let Y_{Am}, M_{Am} be as in (3.16), (3.18). Then for $m = 1$ or $(n_i - 1)$, the set of all controllers with m-actuator-integrity is

$$\begin{aligned}\mathbf{S}_{Am}(P) = \{N_C D_C^{-1} = \tilde{D}_C^{-1}\tilde{N}_C \mid N_C &= (I_{ni} - D_P Y_{Am})\\ &\cdot M_{Am}^{-1}\tilde{U}_P + (I_{ni} - (I_{ni} - D_P Y_{Am})(I_{ni} - F_A))^{-1} D_P Q_A,\\[4pt] D_C = \tilde{V}_P + N_P Y_{Am} M_{Am}^{-1}\tilde{U}_P& \\ - N_P(I_{ni} - Y_{Am} M_{Am}^{-1}&(I_{ni} - F_A)D_P)Q_A,\\[4pt] \tilde{D}_C = Y_{Am} + (I_{ni} - Y_{Am} D_P)&V_P F_A - Q_A \tilde{N}_P F_A,\\[4pt] \tilde{N}_C = (I_{ni} - Y_{Am} D_P)&U_P + Q_A \tilde{D}_P,\\[4pt] Q_A \in \mathcal{R}_{\mathcal{U}}^{ni \times no}, \det D_C \sim& \det \tilde{D}_C \in \mathcal{I}\}.\end{aligned} \tag{3.20}$$

If the plant is strictly proper, then in (3.20), for any $Q_A \in \mathcal{R}_{\mathcal{U}}^{ni \times no}$, $\det D_C = \det(\tilde{V}_P + N_P Y_{Am} M_{Am}^{-1}\ \tilde{U}_P - N_P(I_{ni} - Y_{Am} M_{Am}^{-1}(I_{ni} - F_A)D_P)) \sim \det \tilde{D}_C = \det(Y_{Am} + (I_{ni} - Y_{Am} D_P)V_P F_A - Q_A \tilde{N}_P F_A) \in \mathcal{I}$. □

Proof: We prove a); b) is similar [6]. C is a controller with k-sensor integrity if and only if, for some LCF $(\tilde{D}_C, \tilde{N}_C)$, $(I_{no} - N_P \tilde{N}_C(I_{no} - F_S))$ (equivalently, $(I_{ni} - \tilde{N}_C(I_{no} - F_S)N_P) = \tilde{D}_C D_P + \tilde{N}_C F_S N_P$) is $\mathcal{R_U}$-unimodular for all $F_S \in \mathcal{F}_{Sk}$. For $k = 1$ or $k = (n_o - 1)$, for the right-coprime pair $(F_S N_P, D_P)$ see (3.21) at the bottom of the next page. Using (3.21), all solutions of $(\tilde{D}_C, \tilde{N}_C)$ satisfying $\tilde{D}_C D_P + \tilde{N}_C F_S N_P = I_{ni}$, for all $F_S \in \mathcal{F}_{Sk}$, is given by (3.19). Since $C \in \mathcal{M}(\mathbf{R}_p(s))$ is and only if $\det \tilde{D}_C \in \mathcal{I}$, Q_S is any $\mathcal{R_U}$-stable matrix such that $\det \tilde{D}_C \in \mathcal{I}$. An RCF (N_C, D_C) is also obtained from (3.21). □

$$Y_{S(no-1)} := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ -\tilde{d}_{2,no}\tilde{x}_1 & 1 & & 0 & 0 & -\tilde{d}_{2,no}\tilde{x}_{1,1} \\ 0 & -\tilde{d}_{3,no}\tilde{x}_2 & & 0 & 0 & -\tilde{d}_{3,no}\tilde{x}_{2,2} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\tilde{d}_{no-1,no}\tilde{x}_{no-2} & 1 & -\tilde{d}_{no-1,no}\tilde{x}_{no-2,no-2} \\ 0 & 0 & \cdots & 0 & \tilde{x}_{no-1} & \tilde{x}_{no-1,no-1} \end{bmatrix} L_{(no-1)}, \tag{3.10}$$

$$Q_{A1} := R_1 \begin{bmatrix} \begin{matrix} 0 \\ q_{2,1} \end{matrix} & \begin{bmatrix} d_{2,2} \\ -d_{2,1} \end{bmatrix} q_{1,2} & \begin{bmatrix} d_{3,3} I_2 \\ \\ -d_{3,1} - d_{3,2} \end{bmatrix} \begin{bmatrix} q_{1,3} \\ q_{2,3} \end{bmatrix} & \cdots & \begin{bmatrix} \\ d_{ni,ni} I_{ni-1} \\ \\ -d_{ni,1} \cdots - d_{ni,ni-1} \end{bmatrix} \begin{bmatrix} q_{1,ni} \\ q_{2,ni} \\ q_{3,ni} \\ \vdots \\ q_{ni-2,ni} \\ q_{ni-1,ni} \end{bmatrix} \\ q_{3,1} & q_{3,2} & & & \\ q_{4,1} & q_{4,2} & q_{4,3} & \cdots & \\ \vdots & \vdots & \vdots & & \\ q_{ni,1} & q_{ni,2} & q_{ni,3} & \cdots & \end{bmatrix} \tag{3.14}$$

$$Y_{A(ni-1)} := R_{(ni-1)} \begin{bmatrix} 1 & -x_1 d_{ni,2} & 0 & \cdots & 0 & 0 \\ 0 & 1 & -x_2 d_{ni,3} & & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & x_{ni-1} \\ 0 & -x_{1,1} d_{ni,2} & -x_{2,2} d_{ni,3} & \cdots & -x_{ni-2,ni-2} d_{ni,ni-1} & x_{ni-1,ni-1} \end{bmatrix} \tag{3.16}$$

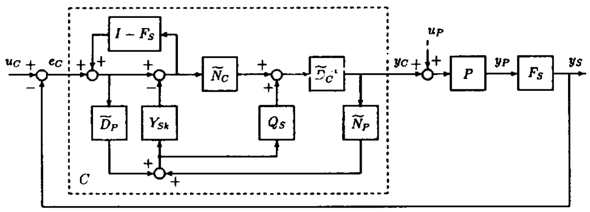

Fig. 4. $\mathcal{S}(F_S, P, C)$, where C has k-sensor integrity for $k = 1$ or $k = n_o - 1$.

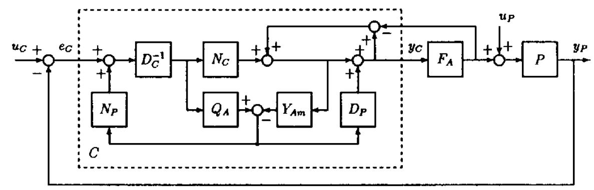

Fig. 5. $\mathcal{S}(P, F_A, C)$, where C has m-actuator integrity for $m = 1$ or $m = n_i - 1$.

If $P \in \mathcal{M}(\mathcal{R_U})$, then $\tilde{D}_P$, D_P, Y_{Sk}, M_{Sk}, Y_{Am}, M_{Am} can be taken as identity matrices. Then (3.19) and (3.20) become $\mathbf{S}_{Sk}(P) = \{C = (I_{ni} - Q_S F_S P)^{-1} Q_S \mid Q_S \in \mathcal{R}_{\mathcal{U}}^{ni \times no}, \det(I_{ni} - Q_S F_S P) \in \mathcal{I}\}$, $\mathbf{S}_{Am}(P) = \{C = (I_{ni} - Q_A P F_A)^{-1} Q_A \mid Q_A \in \mathcal{R}_{\mathcal{U}}^{ni \times no}, \det(I_{ni} - Q_A P F_A) \in \mathcal{I}\}$. These can be obtained by replacing P with $F_S P$ and with PF_A in the parameterization of all $\mathcal{R_U}$-stabilizing controllers.

A block diagram of (3.19) is shown in Fig. 4 for the $\mathcal{R_U}$-stable system $\mathcal{S}(F_S, P, C)$, where $C = \tilde{D}_C^{-1}\tilde{N}_C$ is any controller with k-sensor-integrity. Only one of the internal blocks of the controller depends on F_S and all others are the same as those for the nominal system. Q_S represents the free parameter. Similarly, Fig. 5 shows the block-diagram of the $\mathcal{R_U}$-stable system $\mathcal{S}(P, F_A, C)$, where $C = N_C D_C^{-1}$ is any controller with m-actuator integrity. The actual inputs and outputs of the failure-matrix F_A are used in the controller.

Controller Design Method Based on Decoupling

Since these conditions are necessary for k-sensor integrity and m-actuator integrity, assume that the plant P has no k-sensor failure hidden $\mathcal{U}$ modes (for $k = 1$ or $k = (n_o - 1)$) for $\mathcal{S}(F_S, P, C)$ and that it has no m-actuator failure hidden $\mathcal{U}$ modes (for $m = 1$ or $m = (n_i - 1)$) for $\mathcal{S}(P, F_A, C)$. The controllers in Proposition 3.6.a) for $\mathcal{S}(F_S, P, C)$ are based on $\mathcal{R_U}$-stabilizing controllers which diagonalize the map H_{pc}: $u_C \mapsto y_P$ of the nominal system $\mathcal{S}(P, C)$, where $H_{pc} = PC(I_{no} + PC)^{-1}$. It is not required here that H_{pc} is nonsingular with this controller. A sufficient condition for the existence of $\mathcal{R_U}$-stabilizing controllers such that H_{pc} is diagonal and nonsingualr is that P is full row rank and does not have any $\mathcal{U}$-poles coinciding with zeros [5]. Similarly, the controllers in Proposition 3.6.b) for $\mathcal{S}(P, F_A, C)$ are similarly based on $\mathcal{R_U}$-stabilizing controllers diagonalizing the map H_{cp}: $u_P \to y_C$ of the nominal system $\mathcal{S}(P, C)$.

Proposition 3.6 (A Set of Controllers with Integrity):

a) In $\mathcal{S}(F_S, P, C)$, let $C_{S\mathcal{D}} \in \mathbf{S}(P)$ be any $\mathcal{R_U}$-stabilizing controller for P such that the transfer function H_{pc} of the nominal system $\mathcal{S}(P, C)$ is diagonal. Let $(\tilde{D}_{S\mathcal{D}}, \tilde{N}_{S\mathcal{D}})$ be an LCF and $(N_{S\mathcal{D}}, D_{S\mathcal{D}})$ be an RCF of $C_{S\mathcal{D}}$ satisfying (3.1). If $F_S \in \mathcal{F}_{S1}$, let P have no one-sensor-failure hidden-$\mathcal{U}$-modes; let Y_{Sk} be defined by (3.7). If $F_S \in \mathcal{F}_{S(no-1)}$, let P have no $(n_o - 1)$-sensor failure hidden $\mathcal{U}$ modes; let Y_{Sk} be defined by (3.10). Then, for $k = 1$ or $k = (n_o - 1)$, a controller with k-sensor integrity is

$$\begin{aligned} C &= \tilde{D}_C^{-1}\tilde{N}_C = (\tilde{D}_{S\mathcal{D}} + \tilde{N}_{S\mathcal{D}}(Y_{Sk} + Q_{Sk})\tilde{N}_P)^{-1} \\ &\quad \cdot (\tilde{N}_{S\mathcal{D}} - \tilde{N}_{S\mathcal{D}}(Y_{Sk} + Q_{Sk})\tilde{D}_P) \\ &= N_C D_C^{-1} = (N_{S\mathcal{D}} - D_P \tilde{N}_{S\mathcal{D}}(Y_{Sk} + Q_{Sk})) \\ &\quad \cdot (D_{S\mathcal{D}} + N_P \tilde{N}_{S\mathcal{D}}(Y_{Sk} + Q_{Sk}))^{-1} \\ &= C_{S\mathcal{D}}(I_{no} + (Y_{Sk} + Q_{Sk})\tilde{N}_P C_{S\mathcal{D}})^{-1}(I_{no} - (Y_{Sk} + Q_{Sk})\tilde{D}_P) \end{aligned} \tag{3.22}$$

for any $Q_{Sk} \in \mathcal{R}_{\mathcal{U}}^{no \times no}$ satisfying the following: If $k = 1$, $Q_{Sk} = Q_{S1}$ is as in (3.8); if $k = (n_o - 1)$, $Q_{Sk} = Q_{S(no-1)}$ is as in (3.11); in addition, C should be proper, equivalently, $\det(\tilde{D}_{S\mathcal{D}} + \tilde{N}_{S\mathcal{D}}(Y_{Sk} + Q_{Sk})\tilde{N}_P) \sim \det(D_{S\mathcal{D}} + N_P \tilde{N}_{S\mathcal{D}}(Y_{Sk} + Q_{Sk})) \in \mathcal{I}$, which holds for all $Q_{Sk} \in \mathcal{M}(\mathcal{R_U})$ if P is strictly proper.

b) In $\mathcal{S}(P, F_A, C)$, let $C_{A\mathcal{D}} \in \mathbf{S}(P)$ be any $\mathcal{R_U}$-stabilizing controller for P such that the transfer function H_{cp} of the nominal system $\mathcal{S}(P, C)$ is diagonal. Let $(\tilde{D}_{A\mathcal{D}}, \tilde{N}_{A\mathcal{D}})$ be an LCF and $(N_{A\mathcal{D}}, D_{A\mathcal{D}})$ be an RCF of $C_{A\mathcal{D}}$ satisfying (3.1). If $F_A \in \mathcal{F}_{A1}$, let P have no one-actuator failure hidden $\mathcal{U}$ modes; let Y_{Am} be defined by (3.13). If $F_A \in \mathcal{F}_{A(ni-1)}$, let P have no $(n_i - 1)$-actuator failure hidden $\mathcal{U}$ modes; let Y_{Am} be defined by (3.16). Then, for $m = 1$ or $m = (n_i - 1)$, a controller with m-actuator integrity is

$$\begin{aligned} C &= \tilde{D}_C^{-1}\tilde{N}_C = (\tilde{D}_{A\mathcal{D}} + (Y_{Am} + Q_{Am})N_{A\mathcal{D}}\tilde{N}_P)^{-1} \\ &\quad \cdot (\tilde{N}_{A\mathcal{D}} - (Y_{Am} + Q_{Am})N_{A\mathcal{D}}\tilde{D}_P) \\ &= N_C D_C^{-1} = (N_{A\mathcal{D}} - D_P(Y_{Am} + Q_{Am})N_{A\mathcal{D}}) \\ &\quad \cdot (D_{A\mathcal{D}} + N_P(Y_{Am} + Q_{Am})N_{A\mathcal{D}})^{-1} \\ &= (I_{ni} - D_P(Y_{Am} + Q_{Am})) \\ &\quad \cdot (I_{ni} + C_{A\mathcal{D}} N_P(Y_{Am} + Q_{Am}))^{-1} C_{A\mathcal{D}} \end{aligned} \tag{3.23}$$

for any $Q_{Am} \in \mathcal{R}_{\mathcal{U}}^{ni \times ni}$ satisfying the following: If $m = 1$, $Q_{Am} = Q_{A1}$ is as in (3.14); if $m = (n_i - 1)$, $Q_{Am} = Q_{A(ni-1)}$ is as in (3.17); in addition, C should be proper, equivalently, $\det(\tilde{D}_{A\mathcal{D}} + (Y_{Am} + Q_{Am})N_{A\mathcal{D}}\tilde{N}_P) \sim \det(D_{A\mathcal{D}} + N_P(Y_{Am} + Q_{Am})N_{A\mathcal{D}}) \in \mathcal{I}$, which holds for all $Q_{Am} \in \mathcal{M}(\mathcal{R_U})$ if P is strictly proper.

Proof: We prove a); b) is similar [6]. In (3.2), choose (V_P, U_P) as $(\tilde{D}_{S\mathcal{D}}, \tilde{N}_{S\mathcal{D}})$ and $(\tilde{U}_P, \tilde{V}_P)$ as $(N_{S\mathcal{D}}, D_{S\mathcal{D}})$. Choose Q as $Q = -\tilde{N}_{S\mathcal{D}}(Y_{Sk} + Q_{Sk})$, where Q_{Sk} is Q_{S1} (for one-sensor integrity) or it is $Q_{S(no-1)}$ (for $(n_o - 1)$-sensor integrity), defined by (3.8) and (3.11). Since $H_{pc} = N_P \tilde{N}_{S\mathcal{D}}$ is diagonal, for $k = 1$ or $k = (n_o - 1)$, $I_{no} - N_P(U_P + Q\tilde{D}_P)(I_{no} - F_S) = I_{no} - N_P\tilde{N}_{S\mathcal{D}}(I_{no} - (Y_{Sk} + Q_{Sk})\tilde{D}_P)(I_{no} - F_S)$ is $\mathcal{R_U}$-unimodular for all $F_S \in \mathcal{F}_{Sk}$. Therefore the controller given by (3.22) is in $\mathbf{S}_{Sk}(P)$.

$$\begin{bmatrix} V_P + U_P M_{Sk}^{-1} Y_{Sk}\tilde{N}_P & U_P M_{Sk}^{-1}(I_{no} - Y_{Sk}\tilde{D}_P) \\ -(I_{no} - \tilde{D}_P(I_{no} - F_S)M_{Sk}^{-1}Y_{Sk})\tilde{N}_P & \tilde{D}_P(I_{no} - (I_{no} - F_S)(I_{no} - Y_{Sk}\tilde{D}_P))^{-1} \end{bmatrix} \cdot \begin{bmatrix} D_P & -\tilde{U}_P(I_{no} - \tilde{D}_P Y_{Sk}) \\ F_S N_P & Y_{Sk} + F_S \tilde{V}_P(I_{no} - \tilde{D}_P Y_{Sk}) \end{bmatrix} = \begin{bmatrix} I_{ni} & 0 \\ 0 & I_{no} \end{bmatrix}. \tag{3.21}$$

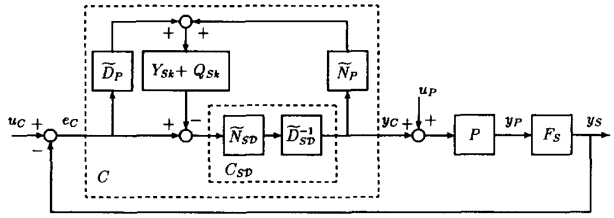

Fig. 6. $\mathcal{S}(F_S, P, C)$, where $N_P \tilde{N}_{SD}$ is diagonal and C has k-sensor integrity for $k = 1$ or $k = n_o - 1$.

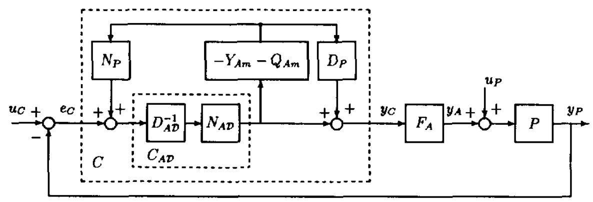

Fig. 7. $\mathcal{S}(P, F_A, C)$, where $N_{AD} \tilde{N}_P$ is diagonal and C has m-sensor integrity for $m = 1$ or $m = n_i - 1$.

Corollary 3.7 (A Set of Controllers with Integrity for $\mathcal{R}_\mathcal{U}$-Stable Plants)

a) Let $P \in \mathcal{M}(\mathcal{R}_\mathcal{U})$ in $\mathcal{S}(F_S, P, C)$. Let $C = (I_{ni} - N_{SD} P)^{-1} \tilde{N}_{SD} \in \mathbf{S}(P)$ be any $\mathcal{R}_\mathcal{U}$-stabilizing controller such that the transfer function H_{pc} of $\mathcal{S}(P, C)$ is diagonal. Then, for $k = 1$ or $k = (n_o - 1)$, $C = \tilde{N}_{SD} Q_{Sk} (I_{no} - P\tilde{N}_{SD} Q_{Sk})^{-1}$ is a controller with k-sensor integrity for any $Q_{Sk} \in \mathcal{R}_\mathcal{U}^{no \times no}$ satisfying the following: If $k = 1$, the diagonal entries of $Q_{Sk} = Q_{S1} \in \mathcal{M}(\mathcal{R}_\mathcal{U})$ are all zero; if $k = (n_o - 1)$, the sub-diagonal entries and $(1, n_o)$ entry of $Q_{Sk} = Q_{S(no-1)} \in \mathcal{M}(\mathcal{R}_\mathcal{U})$ are arbitrary and all others are zero; in addition, $\det(I_{no} - P\tilde{N}_{SD} Q_{Sk}) \in \mathcal{I}$, which holds for all $Q_{Sk} \in \mathcal{M}(\mathcal{R}_\mathcal{U})$ if P is strictly proper.

b) Let $P \in \mathcal{M}(\mathcal{R}_\mathcal{U})$ in $\mathcal{S}(P, F_A, C)$. Let $C = (I_{ni} - N_{AD} P)^{-1} N_{AD} \in \mathbf{S}(P)$ be an $\mathcal{R}_\mathcal{U}$-stabilizing controller such that the transfer-function H_{cp} of $\mathcal{S}(P, C)$ is diagonal. Then, for $m = 1$ or $m = (n_i - 1)$, $C = (I_{ni} - Q_{Am} N_{AD} P)^{-1} Q_{Am} N_{AD}$ is a controller with m-actuator integrity for any $Q_{Am} \in \mathcal{R}_\mathcal{U}^{ni \times ni}$ satisfying the following: If $m = 1$, the diagonal entries of $Q_{Am} = Q_{A1} \in \mathcal{M}(\mathcal{R}_\mathcal{U})$ are all zero; if $m = (n_i - 1)$, the super-diagonal entries and $(n_i, 1)$ entry of $Q_{Am} = Q_{A(ni-1)} \in \mathcal{M}(\mathcal{R}_\mathcal{U})$ are arbitrary all others are zero; in addition, $\det(I_{ni} - Q_{Am} N_{AD} P) \in \mathcal{I}$, which holds for all $Q_{Am} \in \mathcal{M}(\mathcal{R}_u)$ if P is strictly proper. □

The controllers in (3.22)–(3.23) are shown in Figs. 6–7; these controllers are independent of F_S and F_A. In Fig. 6, C_{SD} is an $\mathcal{R}_\mathcal{U}$-stabilizing controller for P such that $H_{pc} = N_P \tilde{N}_{SD}$ of the nominal system $\mathcal{S}(P, C)$ is diagonal but H_{pc} of $\mathcal{S}(F_S, P, C)$ is not diagonal: From (3.22), $H_{pc} = PC(I_{no} + F_S PC) = N_P \tilde{N}_{SD} (I_{no} - (Y_{Sk} + Q_{Sk}) \tilde{D}_P)(I_{no} - (I_{no} - F_S) N_P \tilde{N}_{SD} (I_{no} - (Y_{Sk} + Q_{Sk}) \tilde{D}_P))^{-1}$. Even when none of the sensors fail $(F_S = I_{no})$, $H_{pc} = N_P \tilde{N}_{SD} (I_{no} - (Y_{Sk} + Q_{Sk}) \tilde{D}_P)$, whose diagonal entries are all zero. Similar comments apply to Fig. 7.

IV. Conclusions

An algebraic design method was developed to ensure stability in the presence of either sensor or actuator failures. Two failure classes were considered. All controllers which guarantee stability were characterized and a design method was developed based on diagonalizing certain transfer functions. Future work will extend the failure classes to time-varying and nonlinear perturbations.

References

[1] C. A. Desoer and A. N. Gündeş, "Stability under sensor or actuator failures," in *Proc. Conf. Decision and Contr.*, 1988, pp. 2148–2149.

[2] M. Fujita and E. Shimemura, "Integrity against arbitrary feedback-loop failure in linear multivariable control systems," *Automatica*, vol. 24, no. 765, 1988.

[3] A. N. Gündeş and C. A. Desoer, *Algebraic Theory of Linear Feedback Systems with Full and Decentralized Compensators* (Lecture Notes in Control and Information Sciences, vol. 142). New York: Springer-Verlag, 1990.

[4] A. N. Gündeş, "Stability of feedback systems with sensor or actuator failures: Analysis," *Inter. J. Contr.*, vol. 56, no. 4, pp. 735–753, October 1992.

[5] ——, "Parameterization of decoupling controllers in the unity-feedback system," *IEEE Trans. Automat. Contr.*, vol. 37, no. 10, pp. 1572–1576, Oct. 1992.

[6] ——, "Stabilizing controller design for systems with sensor or actuator failures," Systems, Control, Robotics Techn. Rep. No. UCD-EECS-SCR 92/4, Univ. California, Davis, June 1992.

[7] D. D. Siljak, "Reliable control using multiple control systems," *Int. J. Contr.*, vol. 31, no. 2, pp. 303–329, 1980.

[8] R. J. Veillette, J. V. Medanić, and W. R. Perkins, "Design of reliable control systems," *IEEE Trans. Automat. Contr.*, vol. 37, no. 3, pp. 290–304, 1992.

[9] M. Vidyasagar and N. Viswanadham, "Reliable stabilization using a multi-controller configuration," *Automatica*, vol. 21, no. 5, pp. 599–602, 1985.

[10] M. Vidyasagar, *Control System Synthesis: A Factorization Approach.* Cambridge, MA: M.I.T. Press, 1985.

Robust Control of Robot Manipulators with Parametric Uncertainty

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***Abstract*— This note proposes a robust control law for n-link robot manipulators with parametric uncertainty whose upper bound is not assumed to be known. The proposed robust control based on the Corless–Leitmann approach includes a simple estimation law for the upper bound on the parametric uncertainty and an additional control input to be updated as a function of the estimated value. Using the Lyapunov stability theory, the uniform ultimate boundedness of the tracking error is proved.**

I. Introduction

Recently Spong [1] proposed a simple robust nonlinear control law for n-link robot manipulators with parametric uncertainty using the Lyapunov based theory of guaranteed stability. In this scheme, the Leitmann [2] or Corless–Leitmann [3] approach was used to design a robust controller. The novelty of the result in [1] is the fact that the uncertainty bounds needed to derive the control law and to prove

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