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A test for stabilizability by constant output-feedback

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Abstract

An elementary test is developed for stabilizability of a linear, time-invariant (LTI), single-input single-output (SISO) plant using output-feedback.

Introduction

Any LTI plant represented by a proper rational transfer function can be stabilized in the standard feedback configuration using dynamic output-feedback. Constant output feedback controllers have been considered in several earlier papers, see for example, [3]. In this paper, a test is proposed to check existence of static output-feedback controllers that stabilize a given LTI plant; the test is limited to scalar plant transfer functions.

A well-known method of checking that a given polynomial has all of its roots in the open-left-half-plane (OLHP) of the field of complex numbers $\mathbb{C}$ is the Routh-Hurwitz stability criterion [2]; closed-loop stability using constant feedback can be verified using this method by generating the Routh array symbolically in terms of the constant feedback variable K and then looking for simultaneous solutions of a number of strict inequalities in K. The method developed here relies on solving the roots of known polynomials independent of K to determine if closed-loop stability can be achieved using constant output-feedback.

Main Results

Let P denote the transfer-function of the plant; we assume that P is a *proper* rational function of s with real coefficients and write it as follows:

$$P(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}$$

where N, D is a coprime pair of polynomials, D is monic. Since P is proper, $\delta(N) = m \le \delta(D) = n$, where $\delta(N)$ denotes the degree of the polynomial N. An alternate way to write P is to decompose N and D so that

$$P(s) = \frac{h_N(s^2) + s\,g_N(s^2)}{h_D(s^2) + s\,g_D(s^2)}, \qquad (1)$$

where $h_N(s^2)$, $h_D(s^2)$, $g_N(s^2)$, $g_D(s^2)$ are polynomials of even powers of s [1].

If constant output-feedback K is used, the closed-loop characteristic equation is $1 + K\,P(s) = 0$ and the characteristic polynomial $\beta(s)$ is given by

$$\beta(s) = D(s) + K\,N(s) = (h_D(s^2) + K\,h_N(s^2))$$
$$+\, s\,(g_D(s^2) + K\,g_N(s^2)) =: h(s^2) + s\,g(s^2)\,.$$

A polynomial is said to be Hurwitz (sometimes called strictly Hurwitz) if and only if all of its roots are in the OLHP of $\mathbb{C}$. We now state an important condition so that the closed-loop characteristic polynomial $\beta(s)$ is Hurwitz.

Lemma 1: [1] Let $\beta(s) = h(s^2) + s\,g(s^2)$, where $h(s^2)$ and $g(s^2)$ are polynomials of even powers of s. Then $\beta(s)$ is Hurwitz if and only if the following four conditions hold:

1) The coefficients of $h(s^2)$ and $g(s^2)$ are strictly positive, with no missing coefficients.

2) Either $\delta(h) = \delta(g) = n-1$ or $\delta(h) = \delta(g) + 2 = n$.

3) All of the roots of h and g are purely imaginary (on the $j\omega$-axis) and are distinct.

4) Let h_j denote a root of h and let g_j denote a root of g. **a)** If $\delta(h) = \delta(g) = n-1$, then the roots of h and g are interlaced as follows on the $j\omega$-axis:

$$-jg_{n-1} < -jh_{n-1} < \cdots < -jg_1 < -jh_1 < 0$$
$$< jh_1 < jg_1 < \cdots < jh_{n-1} < jg_{n-1}\,. \qquad (2)$$

b) If $\delta(h) = \delta(g) + 2 = n$, then the roots of h and g are interlaced as follows on the $j\omega$-axis:

$$-jh_n < -jg_n < \cdots < -jh_1 < -jg_1 < 0$$
$$< jg_1 < jh_1 < \cdots < jg_n < jh_n\,. \qquad (3)$$

□

Based on the description of $P(s)$ in (1), we see that there are eight different types of $P(s)$. Conditions for stabilizability of these different types will now be considered as four separate cases.

<u>**Case 1:**</u> $g_N(s^2) = 0$ and $g_D(s^2) = 0$; hence,

$$P(s) = \frac{h_N(s^2)}{h_D(s^2)}\,. \qquad (4)$$

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Proposition 1: Plants whose transfer-functions are described as (4) cannot be stabilized using constant output-feedback. □

<u>Case 2:</u> Either $g_N(s^2) = 0$ and $h_D(s^2) = 0$ or $h_N(s^2) = 0$ and $g_D(s^2) = 0$; hence,

$$P(s) = \frac{h_N(s^2)}{s\, g_D(s^2)} \qquad (5)$$

$$\text{or } P(s) = \frac{s\, g_N(s^2)}{h_D(s^2)} \, . \qquad (6)$$

Proposition 2: Plants whose transfer-functions are described as (5) can be stabilized using constant output-feedback if and only if the polynomials $h_N(s^2)$ and $g_D(s^2)$ (or $-h_N(s^2)$ and $g_D(s^2)$) satisfy the four conditions of Lemma 1. Plants whose transfer-functions are described as (6) can be stabilized using constant output-feedback if and only if the polynomials $g_N(s^2)$ and $h_D(s^2)$ (or $-g_N(s^2)$ and $h_D(s^2)$) satisfy the four conditions of Lemma 1. □

<u>Case 3:</u> $g_N(s^2) = 0$ or $h_N(s^2) = 0$ or $g_D(s^2) = 0$ or $h_D(s^2) = 0$; hence,

$$P(s) = \frac{h_N(s^2)}{h_D(s^2) + s\, g_D(s^2)} \qquad (7)$$

$$\text{or } P(s) = \frac{s\, g_N(s^2)}{h_D(s^2) + s\, g_D(s^2)} \qquad (8)$$

$$\text{or } P(s) = \frac{h_N(s^2) + s\, g_N(s^2)}{h_D(s^2)} \qquad (9)$$

$$\text{or } P(s) = \frac{h_N(s^2) + s\, g_N(s^2)}{s\, g_D(s^2)} \, . \qquad (10)$$

To determine stabilizability using constant output-feedback for the four plant types of case 3, we calculate the value of K such that $1 + KP$ is strictly proper and the values of K such that the roots of the closed-loop characteristic polynomial $\beta(s)$ are purely imaginary; we call these *the critical K values* and calculate them as follows: If $P(s)$ is strictly proper, then $1 + KP$ is never strictly proper; if $n = m$, then $1 + KP$ is strictly proper for the critical K value $K_s = \frac{-1}{a_m} = -\frac{D(\infty)}{N(\infty)}$. Now $\beta(s)$ has roots at the origin for $K_o = -\frac{b_0}{a_0} = -\frac{D(0)}{N(0)}$. The calculation of the purely imaginary roots and the corresponding K values for the four types of $P(s)$ in (7)-(10) are all similar; for example, in (7),

$$K_i = -\frac{h_D(s^2)}{h_N(s^2)} \;, \quad s = j\omega_i \;, \qquad (11)$$

where $j\omega_i$ is a purely imaginary root of g_D ; the critical K values corresponding to distinct ω_i satisfying $g_D((j\omega_i)^2) = 0$ need not be distinct. Note that there are at most $n+1$ critical K values. At the critical K values we determined, we find the number of right-half-plane (RHP) roots of $\beta(s)$; this can be achieved by calculating the roots of $\beta(s)$ at these critical K's.

In Proposition 3 below we give the conditions for stabilizability using constant output-feedback for the plant type in (7) of case 3; types (8)-(10) are omitted here due to their similarity.

Proposition 3: Let $K_1 < K_2 < \cdots < K_\ell$, $\ell \leq n+1$, be the well-ordered critical K values of $\beta(s)$. Then $P(s)$ given by (7) can be stabilized using constant output-feedback if and only if the following two conditions hold:

1) All of the roots of the polynomial g_D are purely imaginary and distinct.

2) There exist two consecutive critical K values K_j and K_{j+1} at which the number of RHP roots of $\beta(s)$ is zero and for any arbitrary constant K_a , where $K_j < K_a < K_{j+1}$, the number of RHP roots of $\beta(s)$ is zero.

Furthermore, if conditions **1** and **2** hold, then the closed-loop system is stable for all K such that $K_j < K < K_{j+1}$. □

<u>Case 4:</u> None of the terms, $g_N(s^2)$, $h_N(s^2)$, $g_D(s^2)$, $h_D(s^2)$ is zero; hence,

$$P(s) = \frac{h_N(s^2) + s\, g_N(s^2)}{h_D(s^2) + s\, g_D(s^2)} \, . \qquad (12)$$

The procedure to determine stabilizability using constant output-feedback for this case is very similar to case 3. Again, we calculate the critical K values for $\beta(s)$. The critical values K_s and K_o are found exactly the same way as in case 3 above. The purely imaginary roots and the corresponding K values are

$$K_i = -\frac{h_D(s^2)}{h_N(s^2)} = -\frac{g_D(s^2)}{g_N(s^2)} \;, \quad s = j\omega_i \;,$$

where $j\omega_i$ is a purely imaginary root of $g_N h_D - h_N g_D$; the critical K values corresponding to distinct ω_i satisfying

$$g_N((j\omega_i)^2)\, h_D((j\omega_i)^2) - h_N((j\omega_i)^2)\, g_D((j\omega_i)^2) = 0$$

need not be distinct. Note that there are at most $n+1$ critical K values. At these critical K values, we find the number of right-half-plane (RHP) roots of $\beta(s)$.

Proposition 4: Let $K_1 < K_2 < \cdots < K_\ell$, $\ell \leq n+1$, be the well-ordered critical K values of $\beta(s)$. Then $P(s)$ given by (12) can be stabilized using constant output-feedback if and only if there exist two consecutive critical K values K_j and K_{j+1} at which the number of RHP roots of $\beta(s)$ is zero and for any arbitrary constant K_a , where $K_j < K_a < K_{j+1}$, the number of RHP roots of $\beta(s)$ is zero. Furthermore, the closed-loop system is stable for all K such that $K_j < K < K_{j+1}$. □

Although case 4 is the most general case and all of the others can be treated as special cases using the procedure for case 4, we considered cases 1, 2, 3 separately to take advantage of the simplifications provided by Lemma 1.

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