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Low order strong controller design

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Abstract: Strong controller design (feedback stabilization by stable controllers) is studied for plants with restricted pole-zero structure. Low order strong controllers are designed for systems satisfying the parity interlacing property with constrained right half plane zeros, and unconstrained right half plane poles. Extension to infinite dimensional systems is also illustrated.

Keywords: Linear systems, strong stability, robust stability, time delay systems

1. INTRODUCTION

Strong stabilization refers to achieving closed-loop stability using a controller that is stable itself. There are many practical reasons for the importance of stable stabilizing controllers, see e.g., Özbay (2024) and references therein.

A linear, time-invariant (LTI) plant can be stabilized using a stable controller if and only if it satisfies the structural constraint of parity interlacing property (PIP), i.e., it has an even number of poles between any pair of its extended positive real axis zeros Youla et al. (1974). Although this PIP test for existence of strongly stabilizing controllers is very simple, designing strong controllers for plants that pass the PIP test proves to be very challenging. Various design methods were proposed in the literature that require some sequential interpolation (see e.g., Doyle et al. (1992); Vidyasagar (1985); Youla et al. (1974)). For the most recent results on this subject (including extension to infinite dimensional systems) see also Ünal (2022); Ünal & İftar (2020); Özer & İftar (2023); Stefanovski (2023); Vardulakis et al. (2023). Note that the order of the strong controller may end up being extremely large even for relatively simple low order plants Smith & Sondergeld (1986). Therefore, for low-order strong controller designs it is necessary to make some structural assumptions on the right half plane (RHP) poles and zeros.

Strong controller designs that explicitly formulate the controller parameters were explored in Gündeş & Özbay (2022) for several plant classes with restrictions on their zeros or poles in RHP. There are also several papers that design strong controllers exclusively for *very specific application*, e.g., the fourth-order linearized model of a two-link planar robot which has restrictions on the number of extended RHP zeros Xin & Liu (2013); Xin et al. (2023, 2025). Recently, Özbay et al. (2025) presented two alternative methods for a particular case of the same system. The present paper extends the above mentioned works to a larger class of systems.

The current paper proposes explicit strong controller designs for single-input single-output LTI plants that have an arbitrary number of RHP zeros at zero and at infinity, with a limited number of zeros on the positive real axis and the imaginary axis. No restrictions are imposed on the plant poles, or the plant zeros in the left half plane (LHP). The goal is to design very simple and low-order controllers with freedom in the design parameters that are explicitly stated, which may be chosen to satisfy other performance specifications or to place the closed-loop poles into desirable regions. To emphasize the low order property of these stable stabilizing controllers, we introduce the terminology of *strong controllers* to define stable controllers with order never exceeding the order of the plant they stabilize.

Notation: $\mathbb{C}$ denotes the set of complex numbers. The closed RHP is $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \Re(s) \geq 0\}$, and the open LHP is $\mathbb{C}_- = \{s \in \mathbb{C} \mid \Re(s) < 0\}$. Extended RHP is $\mathbb{C}_{+e} = \mathbb{C}_+ \cup \{\infty\}$. Real and positive real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively; $\mathcal{R}_p$ denotes the set of real proper rational functions. The space $\mathcal{H}_\infty$ is the set of all bounded analytic functions in $\mathbb{C}_+$. For $f \in \mathcal{H}_\infty$, the norm is defined as $\|f\|_\infty := \text{ess sup}_{\Re(s)>0}|f(s)|$; to simplify the notation we drop the subscript and write $\|f\|$ for $\|f\|_\infty$ when the meaning of the norm is clear from the context. The set of rational stable transfer functions is denoted by $\mathcal{S} = \mathcal{R}_p \cap \mathcal{H}_\infty$. The degree of the polynomial θ is denoted by $\deg(\theta)$. Again, to simplify the notation, (s) is dropped in transfer functions such as $P(s)$.

2. PROBLEM DESCRIPTION

Consider the standard LTI, unity-feedback system, with a plant $P \in \mathcal{R}_p$ and controller $C \in \mathcal{R}_p$ represented by their transfer functions. A stable controller $C \in \mathcal{S}$ that stabilizes the feedback system with plant P and has low order not exceeding the order of P is called a *strong controller*. The feedback system is stable if and only if $S = (1 + PC)^{-1}$, CS and SP are in $\mathcal{H}_\infty$. The plant transfer function is

$$P = \frac{n}{d} = \frac{n}{d_u d_s} = ND^{-1},\ N = \frac{n}{\theta}\ ,\ D = \frac{d}{\theta} \tag{1}$$

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where d is a monic polynomial, d_s is a monic Hurwitz polynomial, $\deg d \geq \deg n$. In $d = d_u d_s$, the roots of d_u and d_s correspond to the poles of P in the unstable region $\mathbb{C}_+$ and in the stable region $\mathbb{C}_-$, respectively; θ is any monic Hurwitz polynomial of the same order as $\deg d$. The relative degree of P is $q = \deg(d) - \deg(n)$. Since d and θ are monic, $D(\infty) = 1$. It is assumed that the plant satisfies the PIP. If $q \geq 1$ then the plant has a zero at $s = \infty$, and if the plant has another zero at $s = 0$, then PIP means that $D(0) > 0$. In this case, $\theta(s)$ can be chosen in such a way that $D(0) = 1$ as well.

In Section 3.2, n and d are allowed to be retarded quasi-polynomials. Such systems have finitely many poles and zeros in $\mathbb{C}_+$. Therefore, in light of Lemma 1 below, a strongly stabilizing controller can be designed based on finite dimensional techniques.

3. GENERAL CASE

We begin with some preliminary results to simplify the controller design.

Lemma 1. Let the plant be in the form $P = P_o P_N$ where $P_o \in \mathcal{R}_p$ and $P_N \in \mathcal{H}_\infty$ such that $P_N^{-1} \in \mathcal{H}_\infty$. If a strong controller C_o can be found for P_o then $C = C_o P_N^{-1}$ is a strong controller for P. Moreover, if $P_n \in \mathcal{S}$ is a reduced order approximation of P_N, with $P_n^{-1} \in \mathcal{S}$, then $C_a = C_o P_n^{-1}$ is a strong controller for P provided that the approximation error satisfies

$$\|T_o \; (P_N P_n^{-1} - 1)\| < 1 \quad \text{where} \quad T_o := \frac{P_o C_o}{1 + P_o C_o}. \tag{2}$$

Proof. See Section 3 of Özbay (2024). □

Lemma 1 is useful in obtaining reduced order strong controllers by approximating the invertible part of the plant. In fact, as demonstrated in Özbay (2024), P_N can be infinite dimensional (e.g., ratio of two quasi-polynomials), in which case a rational P_n is sought for a finite dimensional strong controller. An example is given in Section 3.2.

We consider plants as in (1), with no poles at $s = 0$. Let m be the multiplicity of the plant zero at $s = 0$ and q be the relative degree, i.e, the multiplicity of the plant's zero at $s = \infty$. If $m \geq 2$, part **a)** of Lemma 2, stated below, reduces the strong controller design to the case $m = 1$. Similarly, if $q \geq 2$, part **b)** of Lemma 2 can be used for reducing the strong controller design to the case $q = 1$.

To simplify the notation, define

$$A_\mu := \frac{s^{(m-1)}}{(s+\mu)^{(m-1)}} \; , \quad A_\rho := \frac{\rho^{(q-1)}}{(s+\rho)^{(q-1)}} \; . \tag{3}$$

Lemma 2. **a)** Let $F \in \mathcal{S}$ be a stable transfer-function such that $F(0) = 0$. Let $\mu \in \mathbb{R}_+$ and V_f satisfy (4):

$$\mu < (m-1)^{-1} \|s^{-1} F\|_\infty^{-1} \; , \quad V_f := 1 - (1 - A_\mu)\, F \; . \tag{4}$$

Then V_f is a unit in $\mathcal{S}$ for any μ that satisfies (4).

b) Let $G \in \mathcal{S}$ be a strictly proper, stable transfer-function. Let $\rho \in \mathbb{R}_+$ and W_g satisfy (5):

$$\rho > (q-1)\|sG\| \; , \quad W_g := 1 - (1 - A_\rho)\, G \; . \tag{5}$$

Then W_g is a unit in $\mathcal{S}$ for any ρ that satisfies (5).

Proof. **a)** By Lemma 3.3 in Gündeş & Özbay (2022), $\|\, s\,(1 - A_\mu)\,\| = (m-1)\mu$. Therefore, $\|s(1 - A_\mu)\, s^{-1} F\| \leq \|s(1 - A_\mu)\| \|s^{-1} F\| < 1$ for any μ satisfying (4) and it follows that V_f is a unit in $\mathcal{S}$.

b) By Lemma 3.1 in Gündeş & Özbay (2022) we have $\|\, s^{-1}\,(1 - A_\mu)\,\| = \dfrac{(q-1)}{\rho}$. Therefore, $\|s^{-1}(1 - A_\rho)\, sG\| \leq \|s^{-1}(1 - A_\rho)\| \|\, sG\| < 1$ for any ρ satisfying (5) and it follows that W_g is a unit in $\mathcal{S}$. □

Lemma 3. Let $m \geq 2$. Then, with a sufficiently small $\mu > 0$, $P(s)$ can be written as

$$P(s) = A_\mu(s) \; P_1(s)$$

where $P_1(s)$ has only one zero at $s = 0$, and all other $\mathbb{C}_+$ poles and zeros of P match those of P_1. Instead of $(m-1)$ zeros at $s = 0$, new $(m-1)$ zeros at $s = -\mu$ are introduced in P_1. Assume that a strong controller C_1 is found for P_1. Then $T_1 = \frac{P_1 C_1}{1 + P_1 C_1} \in \mathcal{S}$ and it satisfies $T_1(0) = 0$. If

$$\mu < (m-1)^{-1} \|s^{-1} T_1(s)\|^{-1},$$

then C_1 is a strong controller for P.

Proof. With the controller C_1 and plant P the sensitivity function is $S = (1 + PC_1)^{-1} = (1 + P_1 C_1)^{-1} V_f^{-1}$ where

$$V_f = 1 + T_1(s)\,(A_\mu(s) - 1)\,.$$

From Lemma 2-(a) we have $V_f^{-1} \in \mathcal{S}$. □

Lemma 4. Let $q \geq 2$. Then, with a sufficiently large $\rho > 0$, $P(s)$ can be written as

$$P(s) = A_\rho(s) \; P_1(s)$$

where $P_1(s)$ has relative degree one (only one zero at $s = \infty$), and all other $\mathbb{C}_+$ poles and zeros of P match those of P_1. Instead of $(q-1)$ zeros of P at infinity, $(q-1)$ zeros at $s = -\rho$ are introduced in P_1. Assume that a strong controller C_1 is found for P_1. Then, $T_1 = \frac{P_1 C_1}{1 + P_1 C_1} \in \mathcal{S}$ is strictly proper. If

$$\rho > (q-1)\|s \; T_1(s)\|, \tag{6}$$

then C_1 is a strong controller for P.

Proof. With the controller C_1 and plant P the sensitivity function is $S = (1 + PC_1)^{-1} = (1 + P_1 C_1)^{-1} W_g^{-1}$ where

$$W_g = 1 + T_1(s)\,(A_\rho(s) - 1)\,.$$

From Lemma 2-(b) we have $W_g^{-1} \in \mathcal{S}$. □

Remark. If $m = 0$ or $m = 1$, then there is no need for A_μ; and similarly if $q = 0$ or $q = 1$ then there is no need for A_ρ. It is much easier to design reduced order strong controller for such systems, see e.g., Gündeş & Özbay (2022).

Corollary 1. When $m \geq 2$ and $q \geq 2$, Lemma 3 and Lemma 4 can be applied sequentially to design a strongly stabilizing controller for P. More precisely, re-write P as

$$P(s) = A_\mu(s) A_\rho(s) \; P_1(s) = \frac{A_\mu(s) A_\rho(s) N_1(s)}{D(s)} \quad \text{where}$$

$$N_1 = A_\mu^{-1} A_\rho^{-1} \; N(s), \quad P_1 = N_1 / D \; . \tag{7}$$

Note that P_1 has a single zero at $s = 0$ and its relative degree is 1. First, design a strong controller C_1 for P_1, choosing ρ large enough, applying Lemma 4, we have $W_1^{-1} \in \mathcal{H}_\infty$ where

$$W_1 := 1 + T_1(A_\rho - 1), \quad \text{with} \quad T_1 = \frac{P_1 C_1}{1 + P_1 C_1}.$$

Moreover, with C_1 applied to P, the sensitivity function $S = (1 + PC_1)^{-1}$ is

$$S = S_1\ W_1^{-1}\ [1 + T_1 A_\rho W_1^{-1}(A_\mu - 1)]^{-1},\ S_1 := 1 - T_1.$$

Using Lemma 4, we choose $\mu > 0$ to satisfy

$$\mu < (m-1)^{-1}\|s^{-1}T_1 A_\rho W_1^{-1}\|^{-1}. \tag{8}$$

Then, $S \in \mathcal{H}_\infty$, and $PS \in \mathcal{H}_\infty$ because $P_1 S_1 \in \mathcal{H}_\infty$. □

In general, strong controller design amounts to finding an interpolation unit $U \in \mathcal{S}$. The number of interpolation points is the same as the number of plant's RHP zeros. There are various algorithms for the construction of the unit, see e.g., Stefanovski (2023); Yücesoy & Özbay (2019). Here we investigate low order interpolating unit, which necessitates constraints on the RHP zeros.

3.1 Strictly proper plants with $q \geq 1$, $m \geq 1$ and one positive zero

Consider a plant with no poles at $s = 0$, and multiple poles in $\mathbb{C}_+$. Assume that $m \geq 1$ and $q \geq 1$, and $P(z) = 0$ at a single point $z > 0$ such that the PIP is satisfied (all other poles and zeros of P are in $\mathbb{C}_-$). Then, P_1 defined in (7) can be factored as $P_1 = N_1/D$ where $D \in \mathcal{S}$, with $D(\infty) = 1$, $D(0) = 1$ and $D(z) > 0$, and the zeros of $N_1 \in \mathcal{H}_\infty$ are 0, z and ∞, all with multiplicity one. A strongly stabilizing controller C_1 can be constructed by finding a $U_1 \in \mathcal{S}$ such that $U_1^{-1} \in \mathcal{S}$ and $U_1(0) = D(0) = 1$, $U_1(\infty) = D(\infty) = 1$, and $U_1(z) = D(z)$. Then,

$$C_1 = (U_1 - D)/N_1 = A_\mu A_\rho (U_1 - D)/N \tag{9}$$

is a strong controller for P_1. Note that in (9) internally $(m-1)$ zeros of A_μ cancels $(m-1)$ zeros of N at $s = 0$. Moreover, with this controller we have $T_1 = (1 - DU_1^{-1})$ and $S_1 = DU_1^{-1}$. Furthermore, the feedback system with plant P and controller C_1 is strongly stable if ρ satisfies (6) and μ satisfies (8). Since there are only three interpolation conditions, we seek a second order U_1,

$$U_1(s) = \frac{s^2 + as + b}{s^2 + cs + b}\quad, a, b, c > 0.$$

Clearly, $U_1(0) = U_1(\infty) = 1$ as required. Now we need $U_1(z) = D(z)$ which is equivalent to

$$(1 - D(z))(z^2 + b) + a\ z = zD(z)\ c. \tag{10}$$

If $D(z) \leq 1$ pick any $a, b > 0$ and solve for c. If $D(z) > 1$ pick any $c, b > 0$ and solve for a. In conclusion, when the plant zeros are at $s = 0$, $s = z > 0$ and $s = \infty$, and PIP is satisfied, it is always possible to find a second order interpolating unit U_1 independent of the pole locations.

Remark. We try to find the lowest order U_1 so that the order of the controller $C_1 = (U_1 - D)/N_1$ is reduced. Note that with three interpolation conditions $U_1(z) = D(z)$ and $U_1(0) = U_1(\infty) = 1$ it is not possible to find a first order unit U_1 in $\mathcal{S}$; so the lowest possible order for U_1 is two. Moreover, in this case, two parameters (a, b or c, b) are free, the third parameter is determined from (10). The free parameters can be optimized for other design objectives.

Example 1. Consider a numerical example for illustration:

$$P(s) = \frac{s^4\ (s-1)}{(s-4)^2(s^2-4s+8)^2(s+1)^6}\ , m = 4,\ q = 7.$$

$$P_1(s) = \frac{s(s+\mu)^3(s/\rho+1)^6\ (s-1)}{(s-4)^2(s^2-4s+8)^2(s+1)^6}\ ,\quad z = 1.$$

The PIP is satisfied: there are no poles in $(0, z)$ and two poles in (z, ∞). Define $N_1 = P_1 D$ with

$$D(s) = \frac{(s-4)^2(s^2-4s+8)^2}{(s+4)^2(s^2+4s+8)^2},$$

which is inner and satisfies $D(0) = D(\infty) = 1$, $D(1) = (3/13)^2 > 0$. We can choose $b = 1$, $a = 20D_1(1)$ leading to $c = 20 + 2(D(1)^{-1} - 1)$. Then, the lower bound of ρ is computed from (6) as 382.5512 taking $\rho = 10^3$ in the second step we compute the upper bound of μ, (8), as 0.00594, choosing $\mu = 10^{-3}$ we satisfy all conditions. Finally, the controller $C_1 = (U_1 - D)/N_1$ is

$$C_1(s) = \frac{-22.49\ (s+1)^6\ (s-34.5)\ nc_1(s)\ nc_2(s)}{(s/\rho+1)^6\ (s+\mu)^3\ (s+0.018)\ (s+55.54)}$$

$nc_1 = s^2 - 1.824s + 4.142$, $nc_2 = s^2 - 2.955s + 16.41$. Note that the plant is 12th order, the controller is 11th order. The feedback system is stable with the stable controller C_1 (all closed loop poles are to the left of -0.0007). The freedom in the selection of the parameters a and b can be used for performance objectives, such as sensitivity minimization, and robust stability. For this example, HIFOO (see e.g., Gümüşsoy et al. (2009) and their references), or `systune` of Matlab, with their default values, could not find an 11th order strongly stabilizing controller. On the other hand, an initialization of HIFOO or `systune` with our controller can lead to other strong controllers with better spectral abscissa and/or $\mathcal{H}_2$ or $\mathcal{H}_\infty$-norm). □

3.2 Time delay systems with Im-axis zeros

In this section we illustrate how to obtain reduced order strong controllers for a class of infinite dimensional systems using the method of the previous section. This class include time delay systems with finitely many poles and zeros in $\mathbb{C}_+$, so that the plant can be written as $P = P_o P_\infty$, with finite dimensional strictly proper P_o containing all poles and zeros of P in $\mathbb{C}_+$ and infinite dimensional $P_\infty \in \mathcal{H}_\infty$ with $P_\infty^{-1} \in \mathcal{H}_\infty$. As an example, consider

$$P(s) = \frac{(s + ke^{-h_1 s})}{(s - 2 + e^{-h_2 s})\ (s - 1 + 2e^{-h_3 s})} \tag{11}$$

where $h_2 = 0.75$, $h_3 = \pi/3$, and $h_1 = \pi/2k$ with $k > 0$. The relative degree of P is $q = 1$. The plant is unstable with three poles in $\mathbb{C}_+$, at $p_{1,2} \approx 0.49 \pm j1.08$ and $p_3 \approx 1.726$, all other poles (infinitely many) are in $\mathbb{C}_-$. The plant has two zeros on the Im-axis, at $z_{1,2} = \pm jk$, and all other zeros are in $\mathbb{C}_-$. The dominant poles and zeros are illustrated in Fig. 1, for $k = 5$.

The plant can be factored as $P(s) = P_o(s)P_\infty(s)$, where

$$P_o(s) \approx \frac{s^2 + k^2}{(s - 1.726)(s^2 - 0.985s + 1.411)}$$

and $P_\infty = P/P_o$ is an infinite dimensional transfer function such that $P_\infty, P_\infty^{-1} \in \mathcal{H}_\infty$. Following Lemma 1, we first find a reduced order strong controller C_o for P_o. Next, construct an approximation $P_n \in \mathcal{H}_\infty$ for P_∞ such that $P_n^{-1} \in \mathcal{H}_\infty$. Then, the controller $C = C_o P_n^{-1}$ is strongly stabilizing P if $\|T_o\ (P_\infty P_n^{-1} - 1)\| < 1$. Note that as $k \searrow 0$ the plant is "close to violating PIP", because it has an odd number poles on the positive real axis. Therefore, it

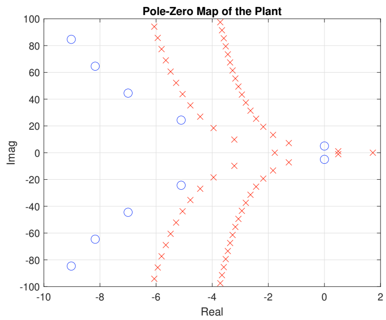

Fig. 1. Dominant poles and zeros of the plant (11).

is expected that for small values of k it will be difficult to find a reduced order C_o for P_o. Define

$$D_o(s) = \frac{(s-1.726)(s^2-0.985s+1.411)}{(s+1.726)(s^2+0.985s+1.411)}$$

and $N_o = P_o D_o$. As before, C_o is constructed from a unit $U_o \in \mathcal{H}_\infty$ such that $U_o(\pm jk) = D_o(\pm jk)$ and $U_o(\infty) = D_o(\infty) = 1$. Since there are three interpolation conditions, we seek a second order U_o in the form

$$U_o(s) = \frac{s^2 + aks + bk^2}{s^2 + cks + dk^2}, \; a, b, c, d > 0.$$

$U_o(\infty) = 1$ is automatically satisfied. Let $D(jk) = x_k + jy_k$ for real numbers x_k and y_k. Since D is inner we have $|D(jk)| = 1$. Interpolation conditions are equivalent to

$$a = \frac{(d_1 - b_1 x_k)}{y_k} \;, \quad c = \frac{(d_1 x_k - b_1)}{y_k} \;, \qquad (12)$$

where $d_1 := d-1$ and $b_1 := b-1$. It can be verified that for $k \geq 3.4136$ we have $x_k > 0$ and $y_k > 0$, hence with $b = 1$ we obtain $a > 0$ and $c > 0$, for all $d > 1$. For $k < 0.57$, a 2nd order unit U_o cannot be found with our method. As a numerical example, let $k = 5$, $b = 1$ and $d = 1.25$. Then, $U_o(s) = \frac{s^2+1.42s+25}{s^2+0.6737s+31.25}$,

$$C_o(s) = \frac{U_o(s) - D_o(s)}{N_o(s)} = \frac{6.1685(s^2 - 0.09303s + 0.8882)}{s^2 + 0.6737s + 31.25}.$$

Again, P_o is third order and C_o is second order.

We now approximate P_∞ with a second order P_n:

$$P_n(s) = \frac{s^2 + 13.63s + 48.7}{s^2 + 8.5s + 100}.$$

Bode plots of P_∞ and P_n are shown in Fig. 2. Moreover, with the controller C_o given above we have

$$\|T_o(P_\infty P_n^{-1} - 1)\| = 0.7214 < 1.$$

Thus, $C_o P_n^{-1}$ is a 4th order strongly stabilizing controller for the infinite dimensional P, given by (11). Here, $b > 0$ and $d > 0$ can be further optimized so that (i) equations (12) give $a > 0$, $c > 0$, (ii) stability condition (2) holds, and (iii) other design objectives (such as sensitivity minimization, stability margin maximization, disturbance rejection, reference tracking) are met.

4. SPECIAL CASES

We now consider some special plants. In view of Lemma 3, and Corollary 1, in this section, we may assume w.l.o.g. that $m = 1$ or $m = 0$. Since these are special cases, proofs of controller construction are omitted.

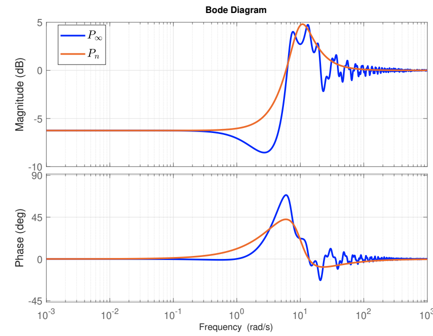

Fig. 2. Bode plots of P_∞ and P_n.

4.1 Plants with zeros at $s = 0$ with no finite zero in $\mathbb{C}_+$

Consider a special case of the plant in the form

$$P(s) = \frac{s^m n_s}{d_u d_s} \;. \qquad (13)$$

i) Biproper plants: In (13), suppose $q = 0$, $m = 1$. Since P has no zeros at infinity, PIP holds without any other constraints. Strong controllers for the case $m = 0$ were proposed in Gündeş & Özbay (2022) (Proposition 1). In Proposition 2 we only consider $m = 1$ for biproper plants.

Proposition 2. Let $P \in \mathcal{R}_p$ as (13), with $q = 0$, have no zeros in $\mathbb{C}_{+e}$ other than $m = 1$ zero at $s = 0$. Choose $U(s) = D(0)$; then a stable controller is $C_s(s) = (D(0) - D(s))/N(s)$. Controller's order does not exceed $(\deg d - 1)$.

ii) Strictly proper plants: In (13), suppose that the plant is strictly proper, and has $q > 0$ zeros at infinity in addition to $m = 1$ zero at $s = 0$, but has no other zeros in $\mathbb{C}_+$. Since controllers for the case $m = 0$ were already proposed in Gündeş & Özbay (2022) (Proposition 2, Proposition 3), we only consider $m = 1$. In this case, PIP holds if and only if $D(0) = D(\infty) = 1$.

Proposition 3. Let $P \in \mathcal{R}_p$ satisfy PIP and assume $q \geq 1$ and $m = 1$. The plant has no other finite zeros in $\mathbb{C}_+$. In this case, interpolation conditions are $U_1(0) = 1$ and $U_1(\infty) = 1$. So, we can choose $U_1(s) = 1$ and apply Lemma 4 with $C_1(s) = (1 - D_1(s))/N_1(s)$. If $m \geq 1$ then apply Corollary 1. The order of the controller C_1 does not exceed $\deg(d) - 1$.

Example 2. The design in Proposition 3 is applied to a 5th order plant as in (13), where $1 \leq m \leq 4$, $1 \leq q \leq 4$, $m + q \leq 5$, $\deg(d_s) = 1$, and $\deg(n_s) = (5 - m - q)$:

$$P = \frac{s^m n_s}{(s^2+16)(s-5)(s-8)d_s} \;. \qquad (14)$$

Let $\theta = (s+4)^3(s+10)d_s$. Then $D(0) = 1$. Define $\chi := (35s^2 + 112s + 752)$. **a)** Suppose $q = 1$. (i) If $m = 1$, then $\deg(n_s) = 3$. A strong controller is $C_{s1} = \chi\, d_s/n_s$. (ii) If $2 \leq m \leq 4$, (for $D(0) = 1$) choose $\mu < \frac{1}{1.25(m-1)}$. If $m = 2$, then $\deg(n_s) = 2$. Choosing $\mu = 0.6$, a strong controller (with $D(0) = 1$) is $C_s = (s + 0.6)^{-1} C_{s1}$.
b) Suppose $q > 1$. Then (5) is satisfied for $\rho > 35(q-1)$. If $m = 1$, $q = 2$, then $\deg(n_s) = 2$. Choosing $\rho = 70$, a strong controller $C_s = 70(s+70)^{-1} C_{s1}$. For $m \geq 2$ μ is chosen so that (8) holds. For example with $q = 2$,

and $m = 3$, choosing $\rho = 70$, (8) holds for $\mu < 0.2837$. Choosing $\mu = 0.25$, a strong controller is $C_s = 70(s + 70)^{-1}(s + 0.25)^{-2}C_{s1}$. The freedom of choice in θ, ρ, μ can be explored to satisfy performance specifications. All stable controllers in this example are 3rd order, i.e., two less than the plant's order. □

4.2 Special case: a plant with two RHP poles

Case 1. Zeros at zero and infinity with two RHP poles. Let the plant transfer function be

$$P(s) = \frac{n}{d} = \frac{s^m n_s}{(s-p_1)(s-p_2)d_s} = \frac{s^m n_s}{d_u d_s} , \qquad (15)$$

$\deg(n_s) \le \deg(d_s) - m + 2 = \deg(d) - m$; $\deg(d_u) = 2$. The two RHP poles are the roots of d_u. These two poles may be distinct or the same; they may be real or a complex-conjugate pair. To simplify the notation, define

$$v := p_1 + p_2 \ , \ w := p_1 p_2 \ ; \qquad (16)$$

then $d_u = (s^2 - vs + w)$. If p_1, p_2 is a complex-conjugate pair, then $v = 2\Re(p_1)$, and $w = |p_1|^2$.

For plants as (15), we show that some coprime factorizations simplify the design and result in even lower order strong controllers. Consider the coprime factorization with $D = d_u/\theta_s$, where $\theta_s = (s + \eta_1)(s + \eta_2)$, with $\eta_1 \in \mathbb{R}_+$, $\eta_2 = \frac{w}{\eta_1}$, so that $D(0) = D(\infty) = 1$, and $U_1(s) = 1$.

a) If $\deg(n_s) = \deg(d_s) - m + 2$, i.e., P in (15) is not strictly proper, then applying Proposition 2, the strong controller becomes

$$C_s = A_\mu \frac{(1-D)}{N} = \frac{(\eta_1 + \eta_2 + v)d_s}{(s+\mu)^{(m-1)} n_s} \ ; \qquad (17)$$

$\mu \in \mathbb{R}_+$ satisfies $\mu < w(m-1)^{-1}(\eta_1 + \eta_2 + v)^{-1}$. The order of C_s in (17) is one less than the plant's order.

b) If $\deg(n_s) = \deg(d_s) - m - q + 2$, i.e., P in (15) has relative degree $q > 0$, then applying Proposition 3, the strong controller becomes

$$C_s = \frac{A_\rho(\eta_1 + \eta_2 + v)d_s}{(s+\mu)^{(m-1)} n_s} \ ; \qquad (18)$$

$\rho \in \mathbb{R}_+$ satisfies $\rho > (q-1)(\eta_1 + \eta_2 + v)$, $\mu \in \mathbb{R}_+$ satisfies $\mu < (m-1)^{-1} \| [A_\rho \theta_s + (1 - A_\rho)d_u]^{-1} A_\rho(\eta_1 + \eta_2 + v) \|^{-1}$. The order of C_s in (18) is two less than the plant's order.

Example 3. Consider the linearized plant model representing "acrobot" and "pendubot" systems Xin et al. (2023):

$$P(s) = \frac{\lambda\,(s-z)(s+z)}{(s-p_1^2)(s-p_2^2)} , \qquad (19)$$

$d_u = (s - p_1)(s - p_2)$, $d_s = (s + p_1)(s + p_2)$. Let $p_1 = 2.24$, $p_2 = 6.101$. If $z = 0$, then $n_s = \lambda$; P has $m = 2$ zeros at $s = 0$ and $q = 2$ zeros at infinity. Using $\eta_1 = p_1$, $\theta_s = d_s$ in the coprime factorization, Proposition 3 can be used to obtain the strong controller, where $\rho > 16.682$. Choosing $\rho = 17$, (8) holds for $0 < \mu < 0.8192$. Choosing $\mu = 0.8$, $C_s = \frac{2\rho v d_s}{n_s(s+\rho)(s+\mu)} = \frac{283.594(s+p_1)(s+p_2)}{\lambda(s+17)(s+0.8)}$.

Case 2: One positive zero with two RHP poles Let the plant transfer function be

$$P(s) = \frac{(s-z)s^m n_s}{(s-p_1)(s-p_2)d_s} = \frac{(s-z)s^m n_s}{d_u d_s} , \qquad (20)$$

$\deg(n_s) + q = \deg(d_s) - m + 1 = \deg(d) - m - 1$. We assume P in (20) satisfies PIP. If both $m = 0$ and $q = 0$, then PIP is always satisfied. If $p_1 = (\alpha + jp)$, $p_2 = (\alpha - jp)$, then $d_u(z) = (z - p_1)(z - p_2) = (z - \alpha)^2 + p^2 > 0$ implies $D(z) > 0$ for any coprime factorization $P = ND^{-1}$. If $m > 0$ or $q > 0$ and the RHP poles are real, then PIP is satisfied if and only if $D(z) > 0$; for P in (20), this means $z < \min\{p_1 , p_2\}$ or $z > \max\{p_1 , p_2\}$. For P as (20), consider a coprime factorization, where $\theta_s = (s+\eta_1)(s+\eta_2)$ is as given in Case 1, so that $D(0) = D(\infty) = 1$:

$$D = \frac{(s-p_1)(s-p_2)}{\theta_s} \ , \ N = \frac{(s-z)s^m n_s}{\theta_s d_s} \ . \qquad (21)$$

Then $(1 - \Phi D) = (1 - D(z)^{-1}D)(1 - D)$ for Φ as (22):

$$\Phi := 1 + D(z)^{-1}(1 - D) \ , \quad \Phi^{-1}(z) = D(z) \ . \qquad (22)$$

Proposition 4. Let $P \in \mathcal{R}_p$ as (20) satisfy PIP. Factorize P as (21). Define Φ as (22).

a) If P has no zeros at $s = 0$ or at infinity, i.e., both $m = 0$ and $q = 0$, then a strong controller

$$C_s = \frac{[D(z) - D]}{N} = \frac{-(\eta_1 + \eta_2 + v)(zs - w)d_s}{\theta_s(z)\, n_s} \ . \qquad (23)$$

The order of C_s in (23) is one less than the plant's order.

b) Suppose $m \le 1$ and m, q are not both zero. Assume $D(z) > 0$. Define $G := [1 - \Phi D]$ in Lemma 2. If $q > 1$, choose any $\rho \in \mathbb{R}_+$ satisfying (5), i.e., $\rho > (q-1)\| s(1 - \Phi D) \|$. A strong controller is

$$C_s = A_\rho \frac{[1 - \Phi D]}{\Phi N} \ . \qquad (24)$$

c) Suppose $m > 1$. Assume $D(z) > 0$.
(i) If $q \le 1$, for any $\mu \in \mathbb{R}_+$ satisfying $\mu < ((m-1)\|s^{-1}(1 - \Phi D)\|)^{-1}$, a strong controller is

$$C_s = A_\mu \frac{[1 - \Phi D]}{\Phi N} \ . \qquad (25)$$

(ii) If $q > 1$, define W as W_g in (5), with $G = [1 - \Phi D]$. Choose any $\mu \in \mathbb{R}_+$ satisfying $\mu < ((m-1)\|s^{-1}(1 - \Phi D W^{-1})\|)^{-1}$. A strong controller is

$$C_s = A_\mu A_\rho \frac{[1 - \Phi D]}{\Phi N} \ . \qquad (26)$$

The strong controllers are the same order as the plant when $q = 0$, and one less than the plant's order when $q \ge 1$. □

The factorization in (21) is used in Proposition 4, with the only assumption that PIP holds. If we impose the additional restriction in (27) below on p_1, p_2, z, then a different controller design is obtained, with order reduced to two less than the strictly proper plant's order:

$$z \ < \ \frac{p_1\, p_2}{p_1 + p_2} = \frac{w}{v} \ . \qquad (27)$$

Condition (27) implies $z < \min\{p_1 , p_2\}$. For P in (20), consider another coprime factorization as (21), but with $\theta_s := (s^2 + \beta_1 s + \beta_2)$, where β_1, $\beta_2 \in \mathbb{R}_+$ are such that

$$f := p_1 p_2 - z(p_1 + p_2) = \beta_1 z + \beta_2 \ . \qquad (28)$$

Condition (27) implies $f > 0$; for example, $\beta_1 = 0.5 f z^{-1}$, $\beta_2 = 0.5 f$ is a possible choice. For this factorization, $D(z) = 1$. Then $(1 - D(z)^{-1}D)$ is strictly proper, $(1 - \Psi D) = (1 - D(0)^{-1}D)(1 - D)$ for Ψ as (29):

$$\Psi := 1 + D(0)^{-1}(1 - D) \ , \quad \Psi^{-1}(0) = D(0) = \frac{w}{\beta_2} \ . \qquad (29)$$

Proposition 5. Let $P \in \mathcal{R}_p$ be as (20); assume that (27) holds. Factorize P so that $D(z) = 1$. Define Ψ as (29).

a) Suppose $m = 0$.
(i) If $q \leq 1$, then a strong controller is

$$C_s = \frac{[1-D]}{N} = \frac{(v+\beta_1)d_s}{n_s} \ . \quad (30)$$

(ii) If $q > 1$, choose any $\rho \in \mathbb{R}_+$ such that $\rho > (q-1)\|s(1-D)\|$). A strong controller is

$$C_s = A_\rho \frac{[1-D]}{N} = A_\rho \frac{(v+\beta_1)d_s}{n_s} \ . \quad (31)$$

The order of C_s in (30) is one (resp. two) less than the plant's order when $q = 0$ (resp. $q = 1$). The order of C_s in (31) is two less than the plant's order.

b) Define $G := [1 - \Psi D]$ in Lemma 2. Suppose $m = 1$. If $q > 1$, choose any $\rho \in \mathbb{R}_+$ such that $\rho > (q-1)\|s(1-\Psi D)\|$. A strong controller

$$C_s = A_\rho \frac{[1-\Psi D]}{\Psi N} \ . \quad (32)$$

c) Suppose $m > 1$.
(i) If $q \leq 1$, choose any $\mu \in \mathbb{R}_+$ satisfying $\mu < ((m-1)\|s^{-1}(1-\Psi D)\|)^{-1}$. A strong controller is

$$C_s = A_\mu \frac{[1-\Psi D]}{\Psi N} \ . \quad (33)$$

The order of C_s in (33) is the same as the plant's order when $q = 0$, and it is one less than the plant's order when $q = 1$.
(ii) If $q > 1$, choose any $\rho \in \mathbb{R}_+$ as in part **(b)**. Define $\widetilde{W}$ as W_g in (5), with $G = [1 - \Psi D]$. Choose any $\mu \in \mathbb{R}_+$ satisfying $\mu < ((m-1)\|s^{-1}(1-\Psi D\widetilde{W}^{-1})\|)^{-1}$. A strong controller is

$$C_s = A_\mu A_\rho \frac{[1-\Psi D]}{\Psi N} \ . \quad (34)$$

The order of C_s in (34) is one less than the plant's order.

Example 4. Consider again the model (19) in Example 3, with $z = 1.281$. The poles are as in Example 3. Then $n_s = \lambda(s+z)$. Since condition (27) holds, the design in Proposition 5 (a-ii) can be applied: With $f = 2.9814$, let $\beta_1 = 1$; then $\beta_2 = 1.7004$ and $\theta_s = (s^2 + s + 1.7004)$. Choosing $\rho = 20$ that satisfies $\rho > 17.4372$, the controller in (31) becomes $C_s = \frac{186.82(s+p_1)(s+p_2)}{\lambda(s+z)(s+20)}$.

Case 3: Acrobot systems with Im-axis zeros and two RHP poles. See Özbay et al. (2025).

5. CONCLUSIONS

A simple reduced order strong controller design method is given for plants with restricted number of zeros in $\mathbb{C}_+$, and arbitrary number of $\mathbb{C}_+$ poles, subject to PIP. Extension to infinite dimensional systems with finitely many poles and zeros in $\mathbb{C}_+$ is also illustrated with a retarded time delay example. In this method there are free controller parameters. Numerical optimization of these parameters for sensitivity minimization will be shown in the expanded version of the paper.

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