

Maximizing Allowable Feedback Delay via Sensor Placement for the Acrobot System

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Abstract: The transfer function of a linearized model of the so-called acrobot system, around the upright equilibrium point, is unstable and it has two pairs of poles and a pair of zeros, pairwise symmetric around the origin. A method for strongly stabilizing controller design is considered for this system. The largest allowable feedback delay, by the proposed method, is investigated with respect to the location of the right half plane zero, which in turn depends on the sensor location. The optimal location of the zero is identified for the largest possible delay. The delay margins corresponding to the controllers obtained earlier in the literature are compared with the delay margin resulting from a third order stable controller derived in the paper. Furthermore, the trade-off between delay maximization and sensitivity minimization are discussed with a numerical example.

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1. INTRODUCTION

One of the fundamental robust control problems is the delay margin optimization, see e.g., Ma et al. (2019), Ma and Chen (2019), Qi et al. (2017), Ozbay and Gundes (2022). The delay margin (DM) indicates how much additional time delay is tolerated in a stable feedback system (positive DM). Such an uncertain delay may be due to unmodeled sensor/actuator dynamics or ignored communication, sampling, or computation times. For a given plant, maximizing the delay margin among all stabilizing controllers is a difficult problem in general, Qi et al. (2017), Xu et al. (2024).

A two-link under-actuated robotic system known as acrobot, Spong (1995), has received attention in the recent literature, see e.g. Wiebe et al. (2024) and Xin et al. (2025). Typical experimental set-up is shown in Fig. 1, where the torque input u is applied at the joint shown in red, θ_1 is the position of the first link, and θ_2 is the position of the second link, relative to the first one. It is assumed that a sensor is placed at the green diamond on the second link. The output of the sensor is the position $-y$ indicated in Fig. 1. Around the upright equilibrium, the linearized plant transfer function is $P(s) = Y(s)/U(s)$. The value of l_r determines the location of the zeros of the transfer function as shown in Section 4; see Xin and Liu (2013) for further details.

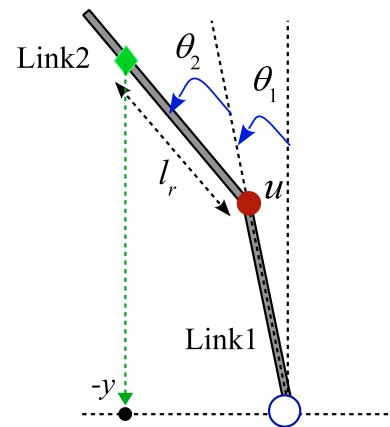


Fig. 1. Acrobot system.

The plant $P(s)$ is unstable with two real poles, p_1 and p_2 , in \mathbb{R}_+ , and two symmetric poles, $-p_1$ and $-p_2$. Moreover, it has two zeros, z and $-z$, symmetric around the origin. In certain specific configurations we have $z > 0$, and its location is dependent on the sensor placement on the upper link (Link 2), Xin et al. (2023). In Section 4 we will analyze the effect of z (hence the choice of l_r), on the largest allowable feedback delay.

Another important robust control problem is strong stabilization: finding a stable controller stabilizing the feedback system with a given unstable plant. There are many

techniques for the design of strongly stabilizing controllers, see e.g. Ozbay (2024) and references therein. For the unstable acrobot system, strongly stabilizing controllers are investigated in Xin and Liu (2013), Xin et al. (2023) and Xin et al. (2025). In this paper, a small gain based design method is applied to the acrobot system. Here, strongly stabilizing controllers are designed under a sufficient condition. Furthermore, possible time delays are also allowed in this method by incorporating solutions to one-block \mathcal{H}_∞ -control problem for systems with delay. Thus, the effect of the location of z on the largest allowable feedback delay (subject to stable controllers derived by the small gain approach) can be analyzed.

We would like to emphasize that the “largest allowable feedback delay” is not the same as the “largest achievable margin”. But strongly stabilizing controllers for the former might give good results for the latter as demonstrated in the paper.

In Section 2 a strong stabilization method is summarized. In Section 3 a computation procedure for the largest allowable feedback delay is described. In Section 4 the said procedure is applied to the acrobot model, and for some specific numerical values of p_1 and p_2 the largest allowable delay is computed for each z . Also in Section 4 a reduced order strongly stabilizing controller is obtained; the corresponding delay margin and sensitivity are analyzed and compared to previously published controllers from the literature. Concluding remarks are made in the last section.

2. A STRONGLY STABILIZING CONTROLLER DESIGN METHOD

Consider the standard unity feedback system shown in Fig. 2, where $P(s)$ is the plant and $C(s)$ is the controller.

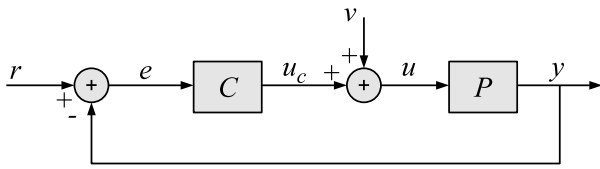


Fig. 2. Feedback system.

The feedback system is stable if the following closed loop transfer functions are in \mathcal{H}_∞ :

$$S = (1+PC)^{-1}, \quad CS = C(1+PC)^{-1}, \quad PS = P(1+PC)^{-1}.$$

As illustrated in Xin et al. (2023), the plant transfer function for the acrobot model is

$$P(s) = \frac{k_o(s-z)(s+z)}{(s-p_1)(s-p_2)(s+p_1)(s+p_2)}, \quad (1)$$

where k_o is a constant and $0 < p_1 < p_2$. The plant satisfies the parity interlacing property (i.e. there exists a strongly stabilizing controller) if and only if $0 < z < p_1$ or $z > p_2$. In what follows we assume that $z > 0$ satisfies this condition.

For strong stabilization purposes one can take the controller to be in the form

$$C(s) = \frac{(s+p_1)(s+p_2)}{k_o(s+z)(\varepsilon s+1)} C_o(s) \quad (2)$$

where ε is a small positive number (typically, smaller than $\frac{0.01}{p_2}$), and $C_o \in \mathcal{H}_\infty$ is a stabilizing controller for the “core unstable plant”

$$P_o(s) = \frac{(s-z)}{(s-p_1)(s-p_2)(\varepsilon s+1)}, \quad (3)$$

see e.g. Ozbay (2024). Note that $P(s)C(s) = P_o(s)C_o(s)$.

Define a coprime factorization $P_o = N/D$, where

$$N(s) = \frac{(s-z)}{(s+p_1)(s+p_2)(\varepsilon s+1)},$$

$$D(s) = \frac{(s-p_1)(s-p_2)}{(s+p_1)(s+p_2)}.$$

Choose an arbitrary $W_o \in \mathcal{H}_\infty$ such that $W_o^{-1} \in \mathcal{H}_\infty$. Then, a given $C_o \in \mathcal{H}_\infty$ is stabilizing P_o if $W_o U$ is invertible in \mathcal{H}_∞ , where

$$U := D + NC_o.$$

Now, by a slight modification of Lemma III.4 of Zeren and Ozbay (1999), we conclude that there exists a strongly stabilizing C_o for the core unstable plant P_o if the following sufficient condition is satisfied:

$$\gamma_0 := \inf_{Q \in \mathcal{H}_\infty} \|R - NQ\|_\infty < 1, \quad (4)$$

where $R(s) := 1 - W_o(s)D(s)$.

Given R and N , if $\gamma_0 < 1$, then we can find a $Q \in \mathcal{H}_\infty$ solving the one block \mathcal{H}_∞ -control problem (4). In this case, $C_o = Q/W_o$ is a strongly stabilizing controller for P_o ; see Ozbay (2024) for details.

Clearly, the choice of W_o plays an important role in solvability of the problem (4). In Section 4 we restrict W_o to the class of first order stable transfer functions, which are invertible in \mathcal{H}_∞ .

3. DELAY MAXIMIZATION AMONG STRONGLY STABILIZING CONTROLLERS

Let $P_o = N/D$ and W_o be such that $\gamma_0 < 1$, i.e. (4) holds. Define \mathcal{C}_s as the subset of \mathcal{H}_∞ whose elements are in the form $C_o = Q/W_o$ where $Q \in \mathcal{H}_\infty$ satisfies

$$\gamma(Q, 0) < 1, \quad \text{with } \gamma(Q, 0) := \|R - NQ\|_\infty. \quad (5)$$

Suppose a $Q \in \mathcal{H}_\infty$ satisfies (5). Then, by the continuity argument (note that N is strictly proper), there exists $\tau > 0$ such that

$$\gamma(Q, h) := \|R - e^{-hs}NQ\|_\infty < 1, \quad \forall h \in [0, \tau]. \quad (6)$$

The largest possible τ satisfying (6) is defined as the delay margin corresponding to the controller $C_o = Q/W_o$ in the set \mathcal{C}_s , and it is denoted by $\tau(C_o)$. We also define the largest achievable delay margin by controllers in \mathcal{C}_s as

$$DM_{max} = \sup_{C_o \in \mathcal{C}_s} \tau(C_o). \quad (7)$$

Finding the optimal C_{opt} in \mathcal{C}_s achieving DM_{max} is currently an unsolved problem.

Below, we investigate a possible solution by finding a strongly stabilizing controller for the plant $e^{-h_o s} P_o(s)$ for a fixed h_o (the largest possible h_o satisfying $\gamma(Q, h_o) < 1$). Of course, such a controller will stabilize all plants for the

delay values in the neighborhood of h_o but that interval may not include all of $[0, h_o)$. So, h_o may not qualify as a lower bound for the DM_{max} .

On the other hand, for each fixed $h > 0$ we can compute

$$\gamma_h := \inf_{Q \in \mathcal{H}_\infty} \gamma(Q, h) \quad (8)$$

see e.g. Özbay et al. (2018). Moreover, γ_h is an increasing function with h ; and $\gamma_0 < 1$. Therefore, a bi-section search algorithm can be constructed to find the largest h for which γ_h remains less than 1:

$$h_{max} := \max\{h \geq 0 \text{ such that } \gamma_h < 1\}. \quad (9)$$

We call h_{max} the “maximum tolerable delay”, i.e., the largest allowable feedback delay h for which we can find a strongly stabilizing controller for $e^{-hs}P_o(s)$, using the small gain argument $\gamma_h < 1$.

The “delay maximizing” controller $C_{opt} = Q_{opt}/W_o$ can be found from $Q_{opt} \in \mathcal{H}_\infty$ which satisfies

$$\|R - e^{-hs}NQ_{opt}\|_\infty < 1$$

with $h = (h_{max} - \delta_h)$ for arbitrarily small δ_h (note that for $\delta_h = 0$ the above inequality becomes equality). Clearly, the optimal controller C_{opt} is infinite dimensional. But it may be possible to approximate it by a relatively low order controller without significantly compromising from the maximum delay achieved. An example is provided in the next section.

In the next section, the computational procedure described above is applied to the acrobot model. We allow z to be variable, hence for each value of z in the intervals $z \in (0, p_1)$ and $z \in (p_2, \infty)$ we can compute $h_{max}(z)$.

Finally, we may want to find the best z which maximizes $h_{max}(z)$. Thus, we can determine the optimal sensor location so that the plant’s \mathbb{C}_+ zero is at the point where the function $h_{max}(z)$ is maximized.

4. NUMERICAL RESULTS FOR THE ACROBOT SYSTEM

We consider a benchmark example from the literature. In Xin et al. (2023) the acrobot system model parameters lead to the following poles

$$p_1 = 2.24, \quad p_2 = 6.101;$$

the location of the \mathbb{C}_+ zero, z , varies with the value of l_r as shown in Fig. 3. The values of p_1 and p_2 given above are independent of l_r . For the values of $l_r \in (0.6667, 0.9987)$ the zeros are on the Im-axis. Moreover, for $l_r \in (1.0664, 2)$ the parity interlacing property is violated (length of the second link is 2). Therefore, in what follows we consider two regions $l_r \in (0, 0.6667)$ and $l_r \in (0.9987, 1.0664)$. That corresponds to $z \in (0.1, 2.05)$ and $z \in (6.103, 97.8)$, respectively.

While Xin and Liu (2013) and Gundes and Ozbay (2022) consider $z = 1.281$, the more recent work Xin et al. (2023) take $z = 1.7321$. Note that the values of z in these works satisfy the parity interlacing property since $z < p_1$. Therefore, we will pay a special attention to the region $z \in (0, p_1)$ in computing the optimal z location for delay maximization problem.

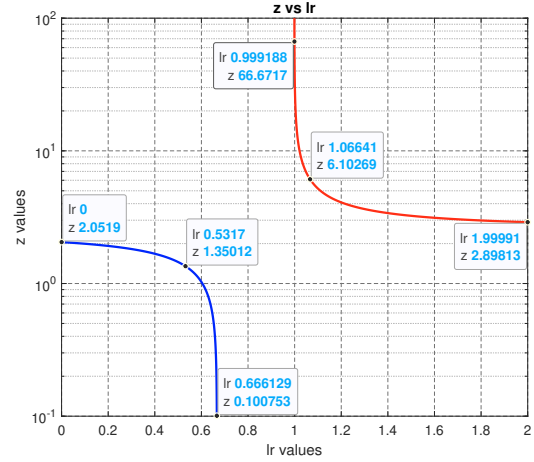


Fig. 3. Location of the \mathbb{C}_+ zero, z , versus l_r .

In this section we take a first order W_o , in the form

$$W_o(s) = \frac{s + a}{s + b}, \quad (10)$$

$$b = 1 + 20e^{-1.7z}, \quad (11)$$

$$a = \frac{(z + b)(z + p_1)(z + p_2)}{(z - p_1)(z - p_2)} - z. \quad (12)$$

In other words, for varying z the “best” choice for a and b changes.

For the sake of completeness we first provide $h_{max}(z)$ for $z > p_2$, see Fig. 4. We observe that for $z > p_2$ the attainable maximum delay increases with z ; some interesting values are $h_{max}(10) = 0.035$, $h_{max}(20) = 0.103$, $h_{max}(30) = 0.135$, $h_{max}(50) = 0.163$, and $h_{max}(70) = 0.174$.

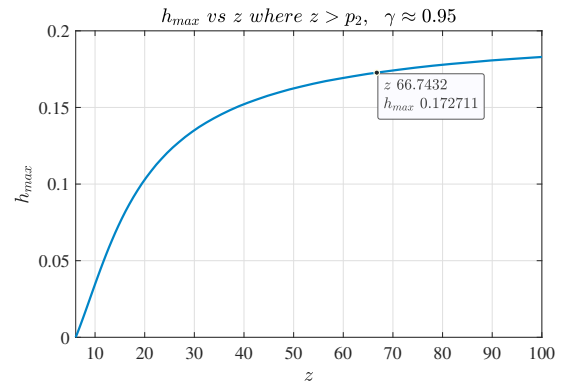


Fig. 4. Achievable maximal delay with respect to the zero location $z > p_2$.

We now restrict our attention to the region $z \in (0, p_1)$, which is the most studied situation in the literature. See Fig. 5 for the plot of $h_{max}(z)$ in this region. The maximum delay attainable in this method is 0.17275 sec., it holds for the best $z = 1.35$.

In Xin et al. (2023) Example 1, with $z = 1.7321$ two alternative second order controllers are obtained, the resulting delay margins are 0.027 sec., and 0.021 sec. For the same example, the third order controller of Gundes and Ozbay

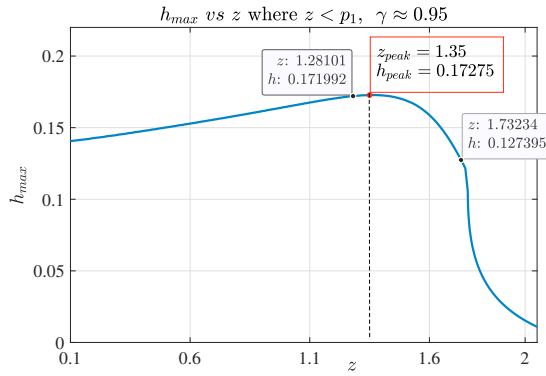


Fig. 5. Achievable maximal delay with respect to the zero location $z < p_1$.

(2022) (when its free parameters are chosen judiciously) gives a delay margin of 0.04 sec. On the other hand, Fig. 5 shows that it is possible to obtain a strongly stabilizing controller (which is infinite dimensional) leading to a maximal delay of 0.127 sec., but we do not know this is actually the “delay margin”.

In Xin and Liu (2013), and in Gundes and Ozbay (2022), the case $z = 1.281$, is considered. The second order stable controller of Gundes and Ozbay (2022) leads to a delay margin of 0.0167 sec.; in the same paper, when the third order controller parameters are optimized the delay margin improves to 0.0992 sec., which is exactly the same as the delay margin for the controller obtained earlier in Xin and Liu (2013) for the same z value. However, Fig. 5 shows that for $z = 1.281$, it is possible to obtain a maximal delay of 0.172 sec., using infinite dimensional controllers, resulting from the method outlined in Section 3. Again, we do not know if 0.172 sec. is the delay margin corresponding to this infinite dimensional controller.

Nevertheless, using approximations of the stable controller obtained in this method, it is possible to find a third order strongly stabilizing controller, which gives a delay margin of 0.16 sec.,

$$C_o(s) = 0.6258 \frac{(s + 14.6)}{(\varepsilon s + 1)}, \quad \varepsilon = 0.001 \quad (13)$$

(note that with (13) the controller $C(s)$ defined by (2) is third order), more precisely,

$$C(s) = \frac{0.6258 (s + 14.6)(s + p_1)(s + p_2)}{k_o (s + z)(\varepsilon s + 1)^2} =: C_3(s). \quad (14)$$

Here, the two poles at $(-1/\varepsilon)$ is included to make the controller proper. The Nyquist plot of $P(s)C_3(s)e^{-hs}$ with a delay value close to the margin, $h = 0.1575$ sec., is shown in Fig. 6; since (-1) is encircled twice in the counter clockwise direction, and PC has two poles in \mathbb{C}_+ , the feedback system is stable.

It is important to note that a 13th order approximation of the optimal infinite dimensional controller (for $h_{max} = 0.172$) leads to a stable feedback system for all plants $P(s)e^{-hs}$, for all $h \in (0.1, 0.17)$; yet the system becomes unstable for $h < 0.1$. Therefore, for the 13th order controller obtained in this method 0.17 is not the delay margin. On the other hand, with the 3rd order approx-

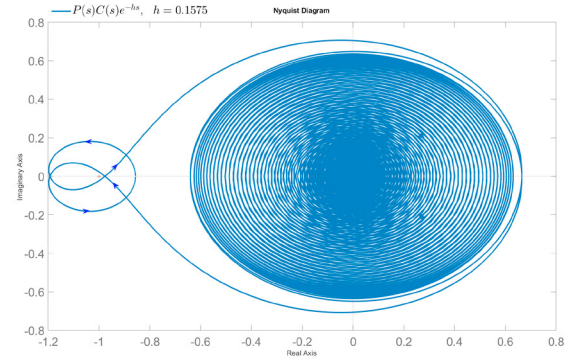


Fig. 6. Nyquist plot with the controller defined by (14) for $h = 0.1575$ sec.

imation (14), the feedback system remains stable for all $h \in [0, 0.16)$. Thus, the controller (14) has $DM = 0.16$. The Bode plots of the high order and low order controllers are shown in Fig. 7.

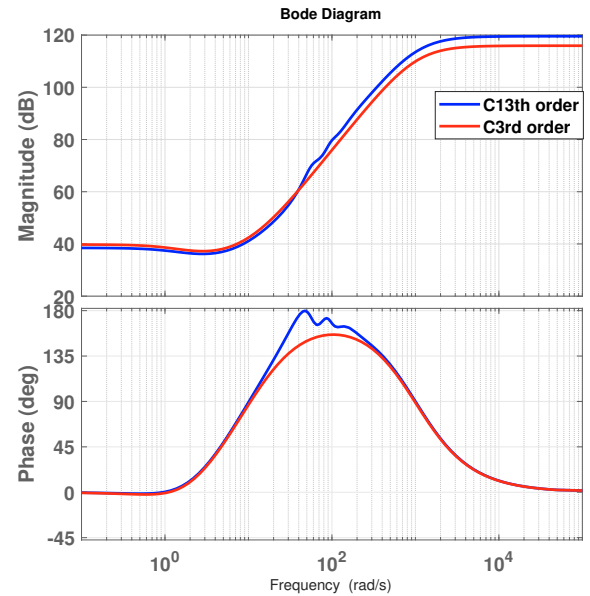


Fig. 7. Bode plots of the 13th order controller and the 3rd order controller.

In summary, the newly designed third order stable controller improves the delay margin of the system, with $z = 1.281$, by at least 60% (brings it to 0.16) compared to the delay margins (less than 0.1) of the second and third order stable controllers designed earlier in Xin and Liu (2013) and Gundes and Ozbay (2022).

Alternative delay margin optimization methods exist, see e.g. Xu et al. (2024) and Zhu et al. (2018). The resulting controllers from these methods may or may not be *stable*, because stability of the controller is not imposed as a requirement. The approach described in Section 5.1.2 of Özbay et al. (2018) leads to a 5th order *stable* controller with a delay margin of 0.13 sec, which is lower than the delay margin obtained here with the 3rd order stable controller (14).

Note that, here, the only design objective was to maximize the allowable feedback delay; that turned out to give a large delay margin. However, this may lead to very high sensitivity. The inverse of the peak sensitivity, defined as $\|S\|_\infty^{-1}$, where $S = (1 + PC)^{-1}$, is a measure of stability robustness against combined gain and phase perturbations in the plant, see e.g. Ozbay (1999). In the light of this fact, the gain of the controller $C_3(s)$, will be further optimized to minimize the peak sensitivity, while keeping the delay margin higher than an acceptable minimum. For example, if we replace $C_3(s)$, given by (14), with $\kappa C_3(s)$, where $\kappa \geq 1$, then the delay margin changes as shown in Fig. 8. The highest delay margin, 0.16 sec., is obtained with $\kappa = 1$. On the other hand, the peak sensitivity $\|S_\kappa\|_\infty$, where

$$S_\kappa = (1 + \kappa P(s)C_3(s))^{-1},$$

is extremely high for $\kappa = 1$, see Fig. 9. It is clear that choosing $\kappa \approx 1.09$ leads to a relatively small sensitivity, while keeping the delay margin in the range of 0.147 sec. to 0.15 sec.

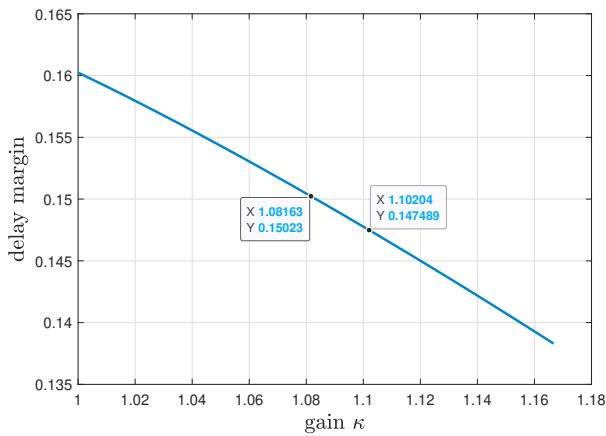


Fig. 8. Delay Margin versus the gain κ .

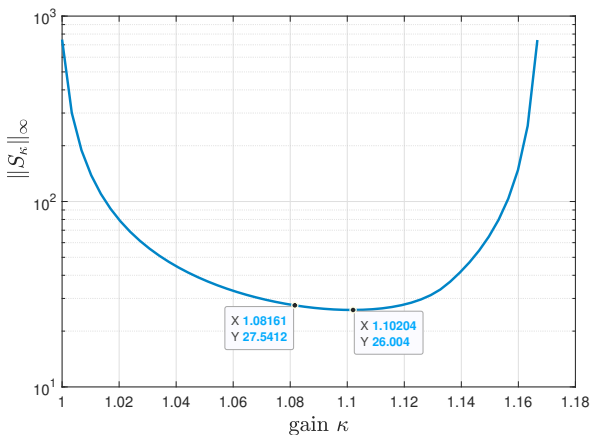


Fig. 9. Sensitivity versus the gain κ .

We should note that the optimal sensitivity minimizing controller is also stable:

$$C_1^o(s) = \frac{-0.81728(s - 10.73)(s + p_1)(s + p_2)}{k_o(s + z)(10^{-6}(s/2.378)^2 + 10^{-3}(s/2.805) + 1)}.$$

This leads to $\|(1 + PC_1^o)^{-1}\|_\infty = 5.6231$, which is the smallest achievable sensitivity level among all stabilizing

controllers. This sensitivity level is about 5 times smaller than the sensitivity level obtained with the controller $1.09 C_3(s)$, where C_3 is given by (14). On the other hand, the resulting delay margin is very small, 0.0021 sec, which is approximately 70 times smaller than the delay margin obtained by $1.09 C_3(s)$. In conclusion, the delay margin and sensitivity should be blended in the design of strongly stabilizing controllers for such systems.

5. CONCLUSIONS

We have examined the largest tolerable feedback delay by using the strongly stabilizing controllers obtained from a small gain condition (4). From Figures 3, 4 and 5 we conclude that the best value of l_r is slightly less than 1 (mid-point on the second link), that corresponds to large values of z . Note that the location of z (and hence achievable maximal delay) varies very significantly in the small interval $l_r \in (0.9987, 1.0664)$. In this interval, there is a huge difference between $l_r = 0.99$ and $l_r = 1.01$ choices. This sensitivity is much less around $l_r = 0.53$, which also guarantees a relatively large maximal delay using a third order stable controller.

There are many other strongly stabilizing controller design methods in the literature; the one presented here is amenable to managing systems with time delays and optimizing the maximal delay (within the restricted set of controllers). There is still room for further optimization: our extensive trials led us to a first order W_o in the form (10), but there is no proof that this choice is the best for the system at hand.

Clearly, maximizing the allowable delay forces us to sacrifice from other typical design objectives, such as the gain margin maximization, and sensitivity minimization. A complete design should consider these issues in a blended fashion. This point is illustrated with the numerical example considered.

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