



Brief paper

Strong stabilization of high order plants[☆]A. Nazli Gündeş^a, Hitay Özbay^{b,*}^a Department of Electrical and Computer Engineering, University of California, Davis, CA 95616, USA^b Department of Electrical and Electronics Engineering, Bilkent University, Ankara 06800, Turkey

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ABSTRACT

Designing stable feedback controllers that stabilize a given plant, known as strong stabilization, has been explored in the literature by using several algorithmic construction procedures. Many of these methods rely on step-by-step interpolation or solving an auxiliary \mathcal{H}_∞ control problem, or a set of LMIs. This paper gives an explicit construction of simple strongly stabilizing controllers for plants that have restrictive number of zeros in the extended right half plane, without any restrictions on the number or location of poles. A similar construction is also developed for the case of plants with restrictions on the poles. The order of the proposed stable controllers is at most one less than that of the plant, and they are computed by selecting just a few positive parameters determined from the \mathcal{H}_∞ norms of certain transfer functions.

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1. Introduction

Finding stable stabilizing feedback controllers, known as strong stabilization, has been extensively studied over the last half-century. There are many practical reasons why stable controllers are desired, see Youla et al. (1974) and Özbay and Garg (1995). In the literature, various design techniques have been proposed for the construction of a strongly stabilizing controller for a given plant that satisfies the parity interlacing property (PIP), i.e., the number of poles between any pair of zeros in the extended positive real axis is even. The PIP is a necessary and sufficient condition for the existence of a strongly stabilizing controller (Youla et al., 1974). This condition is also extended to a class of infinite dimensional systems by Ünal and İftar (2020) and the references therein. There are various construction methods both for finite and infinite dimensional systems. Most of these techniques require some sequential interpolation, see e.g., Doyle et al. (1992), Vidyasagar (1985), Youla et al. (1974). Also, the order of strongly stabilizing controller can be extremely large depending on the right half plane pole-zero pattern of the plant (Smith & Sodergeld, 1986).

Strong stabilization has also been studied with side constraints, such as (i) bounds on the order of the controller and

its \mathcal{H}_∞ norm, and, (ii) closed-loop \mathcal{H}_∞ performance and robustness. Reduced-order stable controllers for a robotic system are investigated by Xin and Liu (2013). Stable controllers of the same order as the plant and satisfying a pre-specified \mathcal{H}_∞ norm bound can be obtained under certain sufficient conditions by solving a one-block \mathcal{H}_∞ control problem, (Zeren & Özbay, 2000). In case (ii), where \mathcal{H}_∞ sensitivity minimization and/or robust stability are desired with stable controllers, the design is based on ARE or LMI solutions, see e.g., Campos-Delgado and Zhou (2003), Cheng et al. (2008, 2009), Gümüşsoy and Özbay (2005). Note again that these techniques lead to controllers whose orders are bounded by a factor of the order of the plant; hence, in light of Smith and Sodergeld (1986), they only work under certain sufficient conditions, for a subset of all plants satisfying the PIP.

In this paper, the focus is on restricted sets of single-input single output systems: when the plant is allowed to have arbitrary number of unstable poles, the restriction is on the number of non-minimum phase zeros; conversely, when the plant has arbitrary number of zeros in the right half plane, the restriction is on the unstable poles. This includes strictly proper plants satisfying the PIP with a single finite zero in the right half plane, with no restriction on the number of poles or their locations. For this class of plants, Smith and Sodergeld (1986) provide an upper bound on the minimum order of strongly stabilizing controllers. The construction procedure provided in the present paper is based on the computation of a few simple parameters from the \mathcal{H}_∞ norm of certain transfer functions. The significance of the design proposed here is that the stable controller is given explicitly in terms of these parameters. The order of the stable controller obtained here is at most one less than the order of the plant; therefore, it is less than or equal to the minimum order upper

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bound given by Smith and Sodergeld (1986). The paper also considers stable controller design for some other special classes of plants, such as a chain of integrators cascaded to a stable system, plants with a double positive zero on the real axis, and plants with unrestricted zeros but with restricted unstable poles.

The class of high order systems considered in the paper may be obtained from finite dimensional approximations of various distributed parameter system models appearing in wide range of engineering applications. Just to give an example, consider a physical process with stable cascade transfer functions $G(s)$, $E(s)$, and assume that around G there is an internal delayed feedback, $k(s)e^{-T_d s}$ with $T_d > 0$, and a low-pass stable $k(s)$. This leads to the overall plant transfer function

$$P(s) = \frac{E(s)G(s)}{1 + k(s)e^{-T_d s}G(s)}. \quad (1)$$

Depending on the values of $k(s)$, T_d and $G(s)$ the plant can be unstable. Moreover $E(s)$ can be non-minimum phase, for example, $E(s) = e^{-hs}$, $h > 0$, or a stable transfer function with a number of zeros in the right-half plane (RHP) due to a non-collocated actuator-sensor pair. See Morgans and Annaswamy (2008) for a model of combustion instabilities, and Rowley et al. (2002) for a model of cavity flow oscillations. Plants in the form of $P(s)$ as (1) appear in applications where there is acoustic feedback, or the underlying process is a spatially distributed system, e.g., vibrating beam with a non-collocated actuator sensor pair, (Curtain & Morris, 2009; Halevi & Wagner-Nachshoni, 2006). Another example for the unstable non-minimum phase high order plants of the form (1) comes from repetitive controllers, see e.g., Nagahara and Yamamoto (2016), Yao et al. (2013) and their references. In this case, internal delayed feedback is part of the controller (internal model principle is used to reject periodic disturbances, or track various reference signals of fixed period); the remaining freedom in the controller is used to stabilize the feedback system. In many practical applications delay terms in (1) are approximated and controllers are designed for the resulting high order rational plant model, by making sure that the rational controller also stabilizes the original infinite dimensional plant, see Özbay (2021) for additional references.

The paper is organized as follows: The main results are collected in Section 3. The stable controller designs for plants that have no (RHP) zeros other than one or two on the positive real axis in addition to zeros at infinity are in Section 3.1. Plants with restrictions on the RHP poles are studied in Section 3.2. These sections also present two useful lemmas, which provide norm equalities that may be of independent interest. Several numerical examples are given in Section 4, with concluding remarks in Section 5. The following standard notation is used:

Notation: Let \mathbb{C} denote complex numbers. The closed RHP is $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \Re(s) \geq 0\}$, and the open left-half-plane (LHP) is $\mathbb{C}_- = \{s \in \mathbb{C} \mid \Re(s) < 0\}$. The region of instability is the extended closed RHP, i.e., $\mathbb{C}_{+e} = \mathbb{C}_+ \cup \{\infty\}$. Real and positive real numbers are denoted by \mathbb{R} and \mathbb{R}_+ , respectively; \mathcal{R}_p denotes real proper rational functions of s ; $\mathcal{S} \subset \mathcal{R}_p$ is the stable subset with no poles in \mathbb{C}_{+e} . The space \mathcal{H}_∞ is the set of all bounded analytic functions in \mathbb{C}_+ . For a function $f \in \mathcal{H}_\infty$, the norm $\|\cdot\|$ is defined as $\|f\| := \text{ess sup}_{s \in \mathbb{C}_+} |f(s)|$. The degree of the polynomial d is denoted by $\deg(d)$. For simplicity, we drop (s) in transfer functions such as $P(s)$.

2. Problem definition

Consider a plant P to be stabilized by a controller C via standard unity negative feedback as in Fig. 1.

We say that the feedback system denoted by $\mathcal{S}(C, P)$ is stable, i.e., C stabilizes P , if $S := (1 + PC)^{-1}$, PS , CS are in \mathcal{H}_∞ .

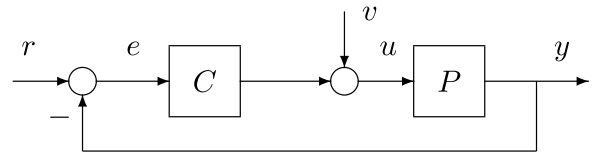


Fig. 1. Unity-feedback system $\mathcal{S}(C, P)$.

Assume that the plant given in (1) is approximated and a rational model $P \in \mathcal{R}_p$ is obtained. Then it can be factored into numerator and denominator polynomials as

$$P = \frac{n}{d}, \quad (2)$$

where $\deg(n) \leq \deg(d)$. Without loss of generality, the polynomials n and d are monic, and hence, we may assume that $(xP)(\infty) = 1$ for any monic polynomial $x(s)$, where $\deg(x) = \deg(d) - \deg(n)$, because for any real $\lambda \neq 0$, if C stabilizes P , then $\lambda^{-1}C$ stabilizes λP . Let θ be a monic Hurwitz polynomial, where $\deg(\theta) = \deg(d)$. Choose the roots of θ where it may be desirable to place the closed-loop poles in \mathbb{C}_- based on the performance specifications. The \mathbb{C}_- -poles of P can be included in the set of closed-loop poles. Define $P = D^{-1}N$, where the factors $D, N \in \mathcal{S}$ are

$$D := \frac{d}{\theta}, \quad N := \frac{n}{\theta}. \quad (3)$$

Since $\deg(d) = \deg(\theta)$ and the polynomials d and θ are both monic, $D(\infty) = 1$.

In Section 3, stable controllers that stabilize various classes of proper plants P are proposed. Once a stable controller C_s that stabilizes P is obtained, all controllers that stabilize P are then obtained from C_s as

$$C = \frac{C_s + QD}{1 - QN} \quad (4)$$

for any $Q \in \mathcal{S}$ such that $Q(\infty) \neq P(\infty)^{-1}$. The controllers in (4) are stable if and only if $(1 - QN)$ is a unit in \mathcal{S} . A sufficient condition is $\|Q\| < \|N\|^{-1}$. \square

3. Main results

The plants considered for strong stabilization in Section 3.1 have restrictions on the zeros in the extended closed RHP \mathbb{C}_{+e} . These plants have any number of poles anywhere in \mathbb{C} . In Section 3.2, restrictions are imposed on the RHP poles, and the zeros are completely free.

3.1. Designs for plants with restricted RHP zeros

Five classes of plants with restrictions on the RHP zeros are considered in the following five propositions in the order of increasing level of difficulty as far as strong stabilization is concerned. Lemma 3.1 presents a crucial norm equality used in the proposed designs for plants that have multiple zeros at infinity:

Lemma 3.1. Let $\rho_i \in \mathbb{R}_+$. Define the r th order Hurwitz polynomial φ as

$$\varphi = \prod_{i=1}^r (s + \rho_i). \quad (5)$$

(a) Then

$$\left\| \frac{1}{s} \left(1 - \prod_{i=1}^r \frac{\rho_i}{(s + \rho_i)} \right) \right\| = \sum_{i=1}^r \frac{1}{\rho_i}. \quad (6)$$

In particular, if $\rho_1 = \dots = \rho_r = \rho$, $\varphi = (s + \rho)^r$, then

$$\left\| \frac{1}{s} \left(1 - \frac{\rho^r}{(s + \rho)^r} \right) \right\| = \frac{r}{\rho}. \quad (7)$$

(b) Furthermore, the r th order polynomial Φ is Hurwitz, where Φ is defined as in (8):

$$\Phi := s^{-1}((s + \rho)^{(r+1)} - \rho^{(r+1)}). \quad (8)$$

□

Proof of Lemma 3.1. (a) For $r = 1$, (6) obviously holds since $\left\| \frac{1}{s} \left(1 - \frac{\rho_1}{s + \rho_1} \right) \right\| = \frac{1}{\rho_1}$. Similarly, for $r = 2$,

$$\begin{aligned} \left\| \frac{1}{s} \left(1 - \frac{\rho_1 \rho_2}{(s + \rho_1)(s + \rho_2)} \right) \right\| &= \left\| \frac{(s + \rho_1 + \rho_2)}{(s + \rho_1)(s + \rho_2)} \right\| \\ &\leq \left\| \frac{1}{s + \rho_1} \right\| \left\| \frac{(s + \rho_1 + \rho_2)}{s + \rho_2} \right\| \\ &= \frac{1}{\rho_1} + \frac{1}{\rho_2}, \end{aligned} \quad (9)$$

where the upper bound is achieved since the norm equality holds at $s = 0$. Suppose (6) holds for r and prove it for $(r + 1)$. With φ as in (5),

$$\begin{aligned} &\left\| \frac{(s + \rho_{(r+1)})\varphi - \rho_{(r+1)} \prod_{i=1}^r \rho_i}{s(s + \rho_{(r+1)})\varphi} \right\| \\ &= \left\| \frac{s\varphi}{s(s + \rho_{(r+1)})\varphi} + \frac{\rho_{(r+1)}}{s(s + \rho_{(r+1)})} \left(1 - \frac{\prod_{i=1}^r \rho_i}{\varphi} \right) \right\| \\ &\leq \left\| \frac{1}{s + \rho_{(r+1)}} \right\| + \left\| \frac{\rho_{(r+1)}}{s + \rho_{(r+1)}} \right\| \left\| \frac{1}{s} \left(1 - \frac{\prod_{i=1}^r \rho_i}{\varphi} \right) \right\| \\ &= \frac{1}{\rho_{(r+1)}} + \sum_{i=1}^r \frac{1}{\rho_i} = \sum_{i=1}^{(r+1)} \frac{1}{\rho_i}. \end{aligned} \quad (10)$$

Equality holds at $s = 0$. By induction, (6) holds.

(b) Now to prove that all roots of Φ given in (8) are in \mathbb{C}_- . Define $G := -\rho^{(r+1)}(s + \rho)^{-(r+1)}$. Then, since $G(0) = 1$ and $|G(j\omega)| < 1$ for all $\omega \neq 0$, the Nyquist plot of G does not encircle the point $(-1, 0)$ but goes through it. Therefore,

$$(1 + G)^{-1} = \frac{(s + \rho)^{(r+1)}}{s\Phi} \quad (11)$$

has no \mathbb{C}_+ -poles except for one pole at $s = 0$, which implies that Φ is Hurwitz. □

3.1.1. Plants with no extended RHP zeros

Proposition 1 presents a simple stable controller design for plants that have stable inverses, i.e., $P^{-1} \in \mathcal{S}$: This rather trivial case is presented for the sake of completeness and establishing building blocks for the other cases.

Proposition 1 (Stable Controllers for Inverse-Stable Plants). Let $P \in \mathcal{R}_p$, $P^{-1} \in \mathcal{S}$, i.e., $P(s) \neq 0$ for all $s \in \mathbb{C}_+ \cup \{\infty\}$. For $Q \in \mathcal{S}$, $Q(\infty) \neq 0$, all stable stabilizing controllers $C \in \mathcal{S}$ are:

$$C = (1 - P^{-1}Q)Q^{-1} = Q^{-1} - P^{-1}, \quad (12)$$

where $Q^{-1} \in \mathcal{S}$, i.e., Q is a unit in \mathcal{S} . The condition $Q(\infty) \neq 0$ holds for all units in \mathcal{S} .

The controller C in (12) is constant if and only if

$$Q = (K + P^{-1})^{-1} \in \mathcal{S} \quad (13)$$

for some nonzero constant $K \in \mathbb{R}$, equivalently, $(1 + K^{-1}P^{-1})$ is a unit in \mathcal{S} . A sufficient condition for $K \in \mathbb{R}$ that gives a constant controller is $|K| > \|P^{-1}\|$. □

The above result implies that if a plant is in the form $[P_1 \quad 1]^T P_2$ with P_1 and P_2 satisfying PIP and $P_2^{-1} \in \mathcal{S}$, then a two-step design can be applied to obtain a strongly stabilizing controller $[C_1 \quad C_2]$ where $C_2 = (1 - P_2^{-1}Q)Q^{-1}$ with Q as in Proposition 1, and C_1 is a strongly stabilizing controller for P_1Q .

3.1.2. Plants with zeros at infinity

Consider plants with no finite \mathbb{C}_+ -zeros. These plants have \mathbb{C}_- -zeros, and $(r + 1)$ zeros at infinity, $r \geq 0$, i.e., $(s^r P)(\infty) = 0$, $(s^{(r+1)} P)(\infty) \neq 0$. Although this might also appear as an easy class to handle, as discussed in Smith and Sodergeld (1986) the number of zeros at infinity plays an important role in the order of strongly stabilizing controllers.

Proposition 2 (Stable Controllers for Strictly Proper P). Let $P \in \mathcal{R}_s$ and let the relative degree of P be $(r + 1)$; P has no finite \mathbb{C}_+ -zeros, i.e., $[(s + a)^{(r+1)} P]^{-1} \in \mathcal{S}$ for any $a > 0$.

(a) If $r = 0$, then a stable controller C_s that stabilizes P is:

$$C_s = \frac{1}{N(s)} [1 - D(s)]. \quad (14)$$

(b) If $r \geq 1$, then choose $\rho_i \in \mathbb{R}_+$, $i = 1, \dots, r$, satisfying (15):

$$\sum_{i=1}^r \frac{1}{\rho_i} < \|s[1 - D(s)]\|^{-1}. \quad (15)$$

A stable controller C_s that stabilizes P is:

$$C_s = \left(\prod_{i=1}^r \frac{\rho_i}{s + \rho_i} \right) \frac{1}{N(s)} [1 - D(s)]. \quad (16)$$

If all ρ_i are chosen the same, i.e., $\rho_i = \rho \in \mathbb{R}_+$, then (15) and (16) are simplified as

$$\rho > r \|s[1 - D(s)]\|, \quad (17)$$

$$C_s = \frac{\rho^r}{(s + \rho)^r N(s)} [1 - D(s)]. \quad (18)$$

The stable controllers C_s in (14), and in (16), (18) all have order one less than the plant's order. □

Proof of Proposition 2. (a) For $r = 0$, the plant has one zero at infinity. Since $D(\infty) = 1$, it follows that $N^{-1}[1 - D] \in \mathcal{S}$. Therefore, C_s in (14) is stable. The closed-loop system is stable since $(C_s N + D) = 1$.

(b) For $r \geq 1$, with the polynomial φ defined as in (5), $(\varphi N)^{-1}[1 - D] \in \mathcal{S}$ implies that C_s in (16) is stable. Then, write $(C_s N + D)$ as:

$$\begin{aligned} C_s N + D &= \frac{\prod_{i=1}^r \rho_i}{\varphi} + \left(1 - \frac{\prod_{i=1}^r \rho_i}{\varphi} \right) D \\ &= 1 - \frac{1}{s} \left(1 - \frac{\prod_{i=1}^r \rho_i}{\varphi} \right) s[1 - D]. \end{aligned} \quad (19)$$

Since $\rho_i > 0$ satisfy (15) for $i = 1, \dots, r$, by Lemma 3.1,

$$\left\| \frac{1}{s} \left(1 - \frac{\prod_{i=1}^r \rho_i}{\varphi} \right) s[1 - D] \right\| \leq \sum_{i=1}^r \frac{1}{\rho_i} \|s[1 - D]\| < 1. \quad (20)$$

Therefore, $(C_s N + D)$ is a unit in \mathcal{S} ; hence, C_s is a stabilizing controller for P . □

For strictly proper plants with no finite \mathbb{C}_+ -zeros, it was shown in Smith and Sodergeld (1986) that it is possible to find strongly stabilizing controllers with order less than or equal to r , i.e., one less than the plant's relative degree. The controllers designed in Proposition 2 have order one less than the plant's order, which is

typically higher than r , and they may have zeros in \mathbb{C}_+ . **Proposition 3** gives another strongly stabilizing controller design with no zeros in \mathbb{C}_+ and has order r .

Proposition 3 (Unit Controllers for Strictly Proper P). Let $P \in \mathcal{R}_s$ satisfy the assumptions of **Proposition 2**.

(a) If $r = 0$, then P can be stabilized by a constant controller

$$C_s = \rho, \quad (21)$$

where $\rho \in \mathbb{R}_+$ satisfies condition (22):

$$\rho > \|P^{-1} - s\|. \quad (22)$$

(b) If $r \geq 1$, choose any r th order monic Hurwitz polynomial χ . Then P can be stabilized by a unit controller given by (23):

$$C_s = \frac{\rho^{(r+1)} \chi(s)}{s^{-1}((s + \rho)^{(r+1)} - \rho^{(r+1)})}, \quad (23)$$

where $\rho \in \mathbb{R}_+$ satisfies condition (24):

$$\rho > (r + 1)\|(\chi(s)P(s))^{-1} - s\|. \quad (24)$$

The stable controller C_s in (23) has order r , one less than the plant's relative degree. \square

Proof of Proposition 3. (a) For $r = 0$, $(s + \rho)N$ is a unit in \mathcal{S} . Then $[P^{-1} - s] = [n^{-1}d - s] = n^{-1}(d - sn)$ is bi-proper since $\deg(d) = \deg(n) + 1$. Write $(C_s N + D)$ as

$$\begin{aligned} (C_s N + D) &= (s + \rho)N \left(\frac{\rho}{(s + \rho)} + \frac{1}{(s + \rho)} N^{-1} D \right) \\ &= (s + \rho)N \left(1 + \frac{1}{(s + \rho)} [P^{-1} - s] \right). \end{aligned} \quad (25)$$

Since ρ satisfies (22), it follows that $(C_s N + D)$ is a unit; hence, C_s in (21) stabilizes P .

(b) For $r \geq 1$, $(s + \rho)^{(r+1)}N$ is a unit in \mathcal{S} . By **Lemma 3.1**-(b), Φ given by (8) is Hurwitz, and hence, $C_s = \Phi^{-1} \rho^{(r+1)} \chi$ given in (23) is also a unit in \mathcal{S} . Then $[(\chi P)^{-1} - s] = [(\chi n)^{-1} d - s] = \chi^{-1} n^{-1} (d - s \chi n)$ is bi-proper since $\deg(d) = \deg(n) + \deg \chi + 1$. Write $(C_s N + D)$ as

$$\begin{aligned} (C_s N + D) &= \frac{\chi}{\Phi} (s + \rho)^{(r+1)} N \left(\frac{\rho^{(r+1)}}{(s + \rho)^{(r+1)}} + \frac{\Phi}{(s + \rho)^{(r+1)} \chi} P^{-1} \right) \\ &= \frac{\chi}{\Phi} (s + \rho)^{(r+1)} N \left(1 + \frac{\Phi}{(s + \rho)^{(r+1)} [(\chi P)^{-1} - s]} \right). \end{aligned} \quad (26)$$

By **Lemma 3.1**-(b), $\|\frac{\Phi}{(s + \rho)^r}\| = \frac{(r+1)}{\rho}$, and since ρ satisfies (24), it follows that $(C_s N + D)$ is a unit; hence, C_s in (21) stabilizes P . \square

3.1.3. Plants with one positive real zero

Bi-proper plants that have a single finite non-negative zero $z \geq 0$ obviously satisfy the PIP. **Proposition 4** presents simple stable controller design for plants whose zeros are all in \mathbb{C}_- except for one finite zero at $z \in \mathbb{C}_+$. The case of a single zero at infinity was covered in **Proposition 2** in the previous section.

Proposition 4 (Stable Controllers for P with One Positive Zero). Let $P \in \mathcal{R}_p$ have only one \mathbb{C}_+ -zero $z \geq 0$, i.e., $(s - z)[(s + a)P]^{-1} \in \mathcal{S}$ for any $a > 0$; P may have other \mathbb{C}_- -zeros. A stable stabilizing controller $C_s \in \mathcal{S}$ is:

$$C_s = \frac{1}{N(s)} [D(z) - D(s)]. \quad (27)$$

The order of the stable controller C in (27) is $(\deg(n) - 1)$, i.e., one less than the order of P . \square

Proof of Proposition 4. Since $[D(z) - D]$ has a zero at $s = z$ that cancels the pole of N^{-1} , the controller C_s in (27) is stable. The closed-loop system is stable since $(C_s N + D) = D(z)$ is a unit in \mathcal{S} . \square

3.1.4. Strictly proper plants with a positive zero

Plants with a single finite non-negative zero $z \geq 0$ in addition to zeros at infinity satisfy the PIP if and only if $D(z) > 0$. For P satisfying the PIP, the assumptions d and θ are monic imply $d(z) > 0$ since $\theta(z) > 0$. **Proposition 5** presents simple stable controller design for plants with zeros all in \mathbb{C}_- except for one finite zero at $z \in \mathbb{C}_+$ and $(r + 1)$ zeros at infinity, $r \geq 0$.

Proposition 5 (Stable Controllers for Strictly Proper Plants with One Positive Zero). Let $P \in \mathcal{R}_s$ and let the relative degree of P be $(r + 1)$; P has one finite \mathbb{C}_+ -zero at $z \geq 0$, i.e., $(s - z)[(s + a)^{(r+2)}P]^{-1} \in \mathcal{S}$ for any $a > 0$. Choose $b \in \mathbb{R}_+$ so that $-b$ is in the desirable region for closed-loop poles, and satisfying

$$b > z [D(z) - 1]. \quad (28)$$

For $b > 0$ satisfying (28), define $\beta \in \mathbb{R}_+$ and $U \in \mathcal{S}$ as

$$\beta = \frac{b - z [D(z) - 1]}{D(z)}, \quad U = \frac{(s + b)}{(s + \beta)}. \quad (29)$$

(a) If $r = 0$, then a stable controller C_s that stabilizes P is:

$$C_s = \frac{1}{N(s)} [U - D(s)]. \quad (30)$$

(b) If $r \geq 1$, then choose $\rho_i \in \mathbb{R}_+$, $i = 1, \dots, r$, satisfying

$$\sum_{i=1}^r \frac{1}{\rho_i} < \|s [1 - U^{-1} D(s)]\|^{-1}. \quad (31)$$

If all ρ_i are chosen the same, i.e., $\rho_i = \rho \in \mathbb{R}$, $i = 1, \dots, r$, then (31) is simplified as

$$\rho > r \|s [1 - U^{-1} D(s)]\|. \quad (32)$$

A stable controller C_s that stabilizes P is:

$$C_s = \left(\prod_{i=1}^r \frac{\rho_i}{(s + \rho_i)} \right) \frac{1}{N(s)} [U - D(s)]. \quad (33)$$

If all $\rho_i = \rho$, then the controller C_s in (33) becomes

$$C_s = \frac{\rho^r}{(s + \rho)^r N(s)} [U - D(s)]. \quad (34)$$

The order of the stable controllers C_s in (30) and (33)-(34) is one less than the plant's order. \square

Remark 1. The roots of $\theta(s)$ and $-b < 0$ are the closed-loop poles; therefore, the choice of $b \in \mathbb{R}_+$ satisfying the lower bound in (28) should be based on the desirable region for the closed-loop pole locations. The choice of $b > 0$ also determines the value of $\beta > 0$ in (29). The poles of the stable controller C_s proposed in (30) are at $-\beta < 0$ and at the plant's LHP zeros.

Proof of Proposition 5. Since b satisfies (28), it follows that $\beta > 0$. Define $\tilde{D} \in \mathcal{S}$ as

$$\tilde{D}(s) := U^{-1} D(s). \quad (35)$$

Then $\tilde{D}(z) = \tilde{D}(\infty) = 1$ implies $[1 - \tilde{D}]$ is strictly proper and has a zero at $z \in \mathbb{C}_+$.

(a) If $r = 0$, then P has one zero at infinity in addition to the zero at $z \in \mathbb{C}_+$, $N^{-1}[1 - \tilde{D}] \in \mathcal{S}$. Write the controller C_s in (30) as $C_s = UN^{-1}[1 - \tilde{D}]$. Therefore, C_s is stable. Furthermore, the closed-loop system is stable since $(C_s N + D)$ is a unit in \mathcal{S} , where

$$C_s N + D = U(NN^{-1}[1 - \tilde{D}] + \tilde{D}) = U. \quad (36)$$

(b) For $r \geq 1$, $\prod_{i=1}^r (s + \rho_i)^{-1} N^{-1} [1 - \tilde{D}] \in \mathcal{S}$ and $\beta, \rho_i \in \mathbb{R}_+$ imply that the controller C_s in (33) is stable. With the polynomial φ defined as in (5), write $(C_s N + D)$ as:

$$\begin{aligned} C_s N + D &= U \left(\prod_{i=1}^r \frac{\rho_i}{(s + \rho_i)} N N^{-1} [1 - \tilde{D}] + \tilde{D} \right) \\ &= U \left(\frac{\prod_{i=1}^r \rho_i}{\varphi} + \left(1 - \frac{\prod_{i=1}^r \rho_i}{\varphi} \right) \tilde{D} \right) \\ &= U \left(1 - \frac{1}{s} \left(1 - \frac{\prod_{i=1}^r \rho_i}{\varphi} \right) s [1 - \tilde{D}] \right). \end{aligned} \quad (37)$$

Since $\rho_i > 0$ satisfy (31), it follows by Lemma 3.1 that

$$\left\| \frac{1}{s} \left(1 - \frac{\prod_{i=1}^r \rho_i}{\varphi} \right) s [1 - \tilde{D}] \right\| \leq \sum_{i=1}^r \frac{1}{\rho_i} \|s [1 - \tilde{D}]\| < 1. \quad (38)$$

Therefore, $(C_s N + D)$ is a unit in \mathcal{S} ; hence, $C_s \in \mathcal{S}$ is a stabilizing controller for P . \square

Remark 2. When $r = 0$, the controller (30) is in the form $N^{-1}(U - D)$, where U as in (29) is a unit in \mathcal{S} satisfying $U(z) = D(z)$ and $U(\infty) = D(\infty) = 1$. The procedure of Doyle et al. (1992) (which is essentially based on Youla et al., 1974), when applied to this class of systems, gives a U , which is in the form $U(s) = (1 + aY(s))^\ell D(z)$ where $Y \in \mathcal{S}$ such that $Y(z) = 0$, $\|Y\| \leq 1$, and $\ell \geq 1$ is the smallest integer such that there exists $a \in (-1, 1)$ leading to $(1 + aY(\infty))^\ell = D(z)$. Since Y is not specified, typically one takes the easiest form $Y(s) = \frac{s-z}{s+z}$, but then if $D(z) < 0.5$ we have $\ell \geq 2$; in fact ℓ gets very large if $D(z)$ gets small (i.e. \mathbb{C}_+ zero gets close to a pole), leading to very high order controllers. Obviously, for a different choice of Y , our first order U can be captured, but this is not known *a priori*. Moreover, when $r \geq 1$, multiple interpolation conditions at $s = \infty$ must be taken into account, Proposition 5(b) formally shows that, in this case, $(\epsilon s + 1)^{-r} N^{-1}(U - D)$ strongly stabilizes P for some small enough ϵ (which is ρ^{-1} in our case). This fact was mentioned without proof in Xin and Liu (2013).

3.1.5. Plants with a double positive zero

Plants that have a double finite non-negative real zero at $z \in \mathbb{C}_-$ but no other zeros in the extended RHP satisfy the PIP. Proposition 6 presents a stable controller design for this class of plants.

Define $F \in \mathcal{S}$, which has a zero at $z \in \mathbb{C}_+$ as

$$F(s) := [1 - D(z)^{-1} D(s)]. \quad (39)$$

By construction, $N^{-1} F^2 \in \mathcal{S}$, and $D^{-1}(1 - F^2) \in \mathcal{S}$; equivalently, $P^{-1} F^2 \in \mathcal{S}$, and $P(1 - F^2) \in \mathcal{S}$.

Lemma 3.2 is used in the proof of Proposition 6:

Lemma 3.2. Let $F \in \mathcal{S}$. Then for any integer $k \geq 1$, (40) holds for some $G \in \mathcal{S}$:

$$\left(1 + \frac{F}{k} \right)^k = 1 + F + GF^2. \quad (40)$$

\square

Proof of Lemma 3.2. It is obvious that (40) holds for $k = 1$ since $(1 + F)^1 = 1 + F + 0F^2$. The value of G can be obtained for each k using the binomial expansion; for example, $G = 1/4$ for $k = 2$, $G = \frac{1}{3}(1 + \frac{1}{9}F)$ for $k = 3$. The general case can be proven by induction. Suppose (40) holds for k . Define

$$\hat{F} = \frac{k}{k+1} F. \quad (41)$$

By assumption, $(1 + k^{-1} \hat{F})^k = 1 + \hat{F} + \hat{G} \hat{F}^2$ for some $\hat{G} \in \mathcal{S}$. Then for $(k + 1)$ we have

$$\begin{aligned} \left(1 + \frac{F}{(k+1)} \right)^{(k+1)} &= \left(1 + \frac{1}{k} \frac{k}{(k+1)} F \right)^k \left(1 + \frac{F}{(k+1)} \right) \\ &= \left(1 + \frac{\hat{F}}{k} \right)^k \left(1 + \frac{F}{(k+1)} \right) = (1 + \hat{F} + \hat{G} \hat{F}^2) \left(1 + \frac{F}{(k+1)} \right) \\ &= 1 + \hat{F} + \hat{G} \hat{F}^2 + \frac{F}{(k+1)} + \hat{F} \frac{F}{(k+1)} + \hat{G} \hat{F}^2 \frac{F}{(k+1)} \\ &= 1 + F + \frac{k}{(k+1)^2} \left(1 + k \hat{G} \left(1 + \frac{F}{(k+1)} \right) \right) F^2. \end{aligned} \quad (42)$$

It follows by induction that (40) holds. \square

Proposition 6 (Stable Controllers for P with Double Positive Zeros). Let $P \in \mathcal{R}_p$ have only two zeros in \mathbb{C}_+ both at $z \geq 0$, i.e., $(s - z)^2 [(s + a)^2 P]^{-1} \in \mathcal{S}$ for any $a > 0$; P may have other \mathbb{C}_- zeros and it may have any number of poles in \mathbb{C}_+ . Let F be as in (39) and k be the smallest integer such that

$$k > \|F\| \quad (43)$$

A stable stabilizing controller $C_s \in \mathcal{S}$ is:

$$C_s(s) = \frac{D(z) - D(s) \left(1 + \frac{F(s)}{k} \right)^k}{N(s) \left(1 + \frac{F(s)}{k} \right)^k}. \quad (44)$$

\square

Proof of Proposition 6. If $k = 1$, i.e., $\|F\| = \|1 - D(z)^{-1} D\| < 1$, then $(1 + F)$ is a unit in \mathcal{S} . Writing $D(z)^{-1}(1 + F) = D^{-1}(1 - F^2)$, the controller (44) becomes

$$C_s = \frac{1 - D(z)^{-1} D(1 + F)}{ND(z)^{-1}(1 + F)} = \frac{N^{-1} F^2}{D(z)^{-1}(1 + F)}. \quad (45)$$

It follows that $C_s \in \mathcal{S}$ in (45) since $N^{-1} F^2 \in \mathcal{S}$ and $(1 + F)^{-1} \in \mathcal{S}$. If $k \geq 2$, using $D(z)^{-1}(1 + F) = D^{-1}(1 - F^2)$ again, write the numerator of C_s in (44) as a multiple of F^2 : $D(z)[1 - D(z)^{-1} D(1 + \frac{F}{k})^k] = D(z)[F^2 + D(z)^{-1} D(1 + F - (1 + \frac{F}{k})^k)] = D(z)[F^2 - D(z)^{-1} D F^2 G]$ for some $G \in \mathcal{S}$. For example, using the binomial coefficients, for $k = 2$, $G = 1/4$; for $k = 3$, $G = \frac{1}{3}(1 + \frac{1}{9}F)$. Therefore, C_s in (44) can be expressed as

$$C_s = \frac{N^{-1} F^2 [1 - D(z)^{-1} D G]}{D(z)^{-1} (1 + \frac{F}{k})^k}. \quad (46)$$

Since $N^{-1} F^2 \in \mathcal{S}$ and $\frac{1}{k} \|F\| < 1$ implies that $(1 + \frac{1}{k} F)^k$ is a unit in \mathcal{S} , it follows that C_s in (44) is stable. Write $(C_s N + D)$ as

$$\frac{N^{-1} [D(z) - D(1 + \frac{F}{k})^k]}{(1 + \frac{F}{k})^k} N + D = \frac{D(z)}{(1 + \frac{F}{k})^k}. \quad (47)$$

The closed-loop system is stable since $(C_s N + D)$ is a unit in \mathcal{S} . \square

3.2. Designs for plants with restricted RHP poles

In this section, there are no restrictions on the zeros of $P \in \mathcal{R}_p$, but the RHP poles are at the origin or on the imaginary axis. There may be any number of poles in the stable region \mathbb{C}_- . Two classes of plants with restrictions on the RHP poles are considered in Propositions 7 and 8. Lemma 3.3 presents a dual result of Lemma 3.1 for a norm equality used in the proposed designs for plants that have multiple poles at $s = 0$:

Lemma 3.3. Let $\mu_i \in \mathbb{R}_+$. Define the m th order Hurwitz polynomial ϕ as

$$\phi := \prod_{i=1}^m (s + \mu_i). \tag{48}$$

Then

$$\|s \left(1 - \frac{s^m}{\phi} \right)\| = \sum_{i=1}^m \mu_i. \tag{49}$$

In particular, if $\mu_1 = \dots = \mu_m = \mu$, $\phi = (s + \mu)^m$, then

$$\|s \left(1 - \frac{s^m}{(s + \mu)^m} \right)\| = m\mu. \tag{50}$$

□

Proof of Lemma 3.3. For $m = 1$, (49) obviously holds since $\|s(1 - \frac{s}{s+\mu_1})\| = \mu_1$. Similarly, for $m = 2$,

$$\begin{aligned} \|s(1 - \frac{s^2}{(s + \mu_1)(s + \mu_2)})\| &= \left\| \frac{s[(\mu_1 + \mu_2)s + \mu_1\mu_2]}{(s + \mu_1)(s + \mu_2)} \right\| \\ &\leq \left\| \frac{s}{s + \mu_1} \right\| \left\| \frac{(\mu_1 + \mu_2)s + \mu_1\mu_2}{s + \mu_2} \right\| \\ &= \mu_1 + \mu_2, \end{aligned} \tag{51}$$

where the upper bound of the inequality is achieved since the norm equality holds at infinity. Suppose that (6) holds for m and prove it for $(m + 1)$. With ϕ as in (48),

$$\begin{aligned} &\left\| \frac{s[(s + \mu_{(m+1)})\phi - s^{(m+1)}]}{(s + \mu_{(m+1)})\phi} \right\| \\ &= \left\| \frac{\mu_{(m+1)}s\phi}{(s + \mu_{(m+1)})\phi} + \frac{s^2}{(s + \mu_{(m+1)})} \left(1 - \frac{s^m}{\phi} \right) \right\| \\ &\leq \left\| \frac{\mu_{(m+1)}s}{s + \mu_{(m+1)}} \right\| + \left\| \frac{s}{s + \mu_{(m+1)}} \right\| \left\| s \left(1 - \frac{s^m}{\phi} \right) \right\| \\ &= \mu_{(m+1)} + \sum_{i=1}^m \mu_i = \sum_{i=1}^{(m+1)} \mu_i. \end{aligned} \tag{52}$$

Equality holds at infinity. By induction, (49) holds. □

3.2.1. Plants with a chain of integrators

Plants formed by a chain of integrators that have no \mathbb{C}_+ -poles other than m poles at $s = 0$ satisfy the PIP. Clearly, such class of systems are of practical interest and in the literature various stabilization techniques have been proposed, see e.g., [Niculescu and Michiels \(2004\)](#) where delayed proportional controllers are considered (these are very special *stable* controllers). Here we propose a simple design method to obtain strongly stabilizing controllers for plants in the form

$$P(s) = \frac{n_E(s)}{d_E(s)} \left(\frac{e^{-hs}G(s)}{1 + k(s)e^{-T_d s}G(s)} \right) \tag{53}$$

where $h \geq 0$, $T_d \geq 0$, $k, G \in \mathcal{H}_\infty$ with $\|kG\| < 1$, and n_E, d_E are polynomials such that $n_E(0) \neq 0$, $d_E(s) = s^m d_s(s)$ with a Hurwitz polynomial d_s . When $h = 0$ and $T_d = 0$, the plant is finite dimensional, but the next result is presented for the general case. For such plants, $\hat{P}_m := (s^m P)$ has no poles in the RHP but may be improper.

Proposition 7 (Stable Controllers for P with Poles at Zero). Let P be in the form (53). Choose Z to be a meromorphic function in \mathcal{H}_∞ with $Z(0) = \hat{P}_m(0)^{-1}$ such that the relative degree of Z is at least $(m - 1)$, i.e., $s^{(m-1)}Z \in \mathcal{H}_\infty$. For $i = 1, \dots, m$, let $\mu_i \in \mathbb{R}_+$ satisfy

$$\sum_{i=1}^m \mu_i < \left\| \frac{s^m P(s)Z(s) - 1}{s} \right\|^{-1}. \tag{54}$$

With the polynomial $\phi(s) = \prod_{i=1}^m (s + \mu_i)$, as in (48), a stable controller $C_s \in \mathcal{H}_\infty$ that stabilizes P is:

$$C_s(s) = [\phi(s) - s^m]Z(s). \tag{55}$$

If all μ_i are chosen the same, i.e., $\mu_i = \mu$, $i = 1, \dots, m$, then (54) is simplified as $\mu < (m \|\frac{s^m PZ - 1}{s}\|)^{-1}$, and $\phi = (s + \mu)^m$. In the rational case, where $h = 0$ and $T_d = 0$, the order of the controller in (55) is the same as the order of Z , which can be chosen as $(m - 1)$, one less than the number of poles at $s = 0$. □

Proof of Proposition 7. Since the relative degree of Z is at least $(m - 1)$, the controller C_s in (55) is stable. The choice of $Z(0)$ is so that $[s^m PZ - 1]$ has a zero at $s = 0$. A coprime factorization for P is

$$P = \left(\frac{s^m}{\phi} \right)^{-1} \left(\frac{s^m}{\phi} P \right) = D^{-1}N. \tag{56}$$

Write $(C_s N + D)$ as

$$\begin{aligned} C_s N + D &= \frac{s^m}{\phi} + C_s \frac{s^m}{\phi} P = \frac{s^m}{\phi} + \frac{(\phi - s^m)}{\phi} s^m PZ \\ &= 1 + \frac{s(\phi - s^m)}{\phi} \frac{[s^m PZ - 1]}{s}. \end{aligned} \tag{57}$$

Using the norm equality (49) in Lemma 3.3, if the small gain condition (54) holds, it follows that

$$\left\| \frac{s(\phi - s^m)}{\phi} \frac{[s^m PZ - 1]}{s} \right\| \leq \sum_{i=1}^m \mu_i \left\| \frac{s^m PZ - 1}{s} \right\| < 1. \tag{58}$$

Therefore, $(C_s N + D)$ is a unit in \mathcal{H}_∞ ; hence, C_s is a stabilizing controller for P . □

3.2.2. Plants with one pair of imaginary poles

In this section we consider plants P given in the form (53), where d_E is a Hurwitz polynomial, n_E is an arbitrary polynomial (in complete generality it can also be a quasi-polynomial) with $\deg(n_E) \leq \deg(d_E)$, and $k, G \in \mathcal{H}_\infty$, $T_d > 0$ are such that the equation

$$1 + k(s)e^{-T_d s}G(s) = 0 \tag{59}$$

has one pair of roots on the Im -axis, say $\pm j\omega$, with $\omega > 0$, and all other roots are in \mathbb{C}_- . Moreover $n_E(j\omega) \neq 0$. For such plants the PIP is satisfied automatically. Also note that $\hat{P} := [(s^2 + \omega^2)P]$ has no poles in the RHP but may be improper. By choosing $\psi(s)$ as any second order monic Hurwitz polynomial, a coprime factorization for P is

$$P = \left(\frac{s^2 + \omega^2}{\psi} \right)^{-1} \left(\frac{(s^2 + \omega^2)P}{\psi} \right) = D^{-1}N. \tag{60}$$

To simplify the notation, define $u, w \in \mathbb{R}$ as

$$u := \Re\{N(j\omega)\}, \quad w := |N(j\omega)|^2, \tag{61}$$

and let

$$R(s) := \frac{s}{(s^2 + \omega^2)} \left(1 - \frac{2u}{w}N(s) + \frac{1}{w}N(s)^2 \right). \tag{62}$$

Proposition 8 (Stable Controllers for Plants with Imaginary-Axis Poles). Let P have poles at $s = \pm j\omega$, and no other poles in \mathbb{C}_+ , as specified above. Define N, D as in (60). Let $\alpha \in \mathbb{R}_+$ satisfy

$$\alpha < \|R\|^{-1} \tag{63}$$

where R as defined in (62) is in \mathcal{H}_∞ . Then, a stable stabilizing controller $C_s \in \mathcal{H}_\infty$ is:

$$C_s = \frac{\alpha s}{w \psi(s)} [2u - N(s)]. \tag{64}$$

□

Proof of Proposition 8. The controller in (64) is stable since ψ is Hurwitz and $N \in \mathcal{H}_\infty$. Write $(C_s N + D)$ as

$$\begin{aligned} C_s N + D &= \frac{(s^2 + p^2)}{\psi(s)} + \frac{\alpha s}{w \psi(s)} [2u - N(s)]N(s) \\ &= \frac{s^2 + \alpha s + p^2}{\psi(s)} \left(1 - \frac{\alpha s}{s^2 + \alpha s + p^2} \left[1 - \frac{2u}{w} N + \frac{1}{w} N^2 \right] \right) \\ &= \frac{s^2 + \alpha s + p^2}{\psi(s)} \left(1 - \alpha \frac{s^2 + p^2}{s^2 + \alpha s + p^2} R(s) \right). \end{aligned} \quad (65)$$

Since $(1 - NN(jp)^{-1})(1 - NN(-jp)^{-1}) = [1 - \frac{2u}{w} N + \frac{1}{w} N^2]$ has zeros at $\pm jp$, the term $R(s) = \frac{s}{(s^2 + p^2)} [1 - \frac{2u}{w} N + \frac{1}{w} N^2] \in \mathcal{H}_\infty$. Using the norm equality

$$\left\| \frac{(s^2 + p^2)}{(s^2 + \alpha s + p^2)} \right\| = 1, \quad (66)$$

for $\alpha \in \mathbb{R}_+$ satisfying (63), it follows that

$$\left\| \frac{\alpha(s^2 + p^2)}{(s^2 + \alpha s + p^2)} R(s) \right\| \leq \alpha \|R(s)\| < 1. \quad (67)$$

Therefore, $(C_s N + D)$ is a unit, and hence, the closed-loop system is stable. \square

When N is infinite dimensional, so is C_s ; for practical implementation a finite dimensional controller

$$C_a(s) = \frac{\alpha s}{w \psi(s)} [2u - N_a(s)] \quad (68)$$

can be used. In (68) N_a is a stable rational approximation of N . Now, C_a stabilizes P if and only if $(D + C_a N)$ is a unit in \mathcal{H}_∞ . Using (65) we obtain

$$\begin{aligned} D + C_a N &= \\ &= \frac{(s^2 + \alpha s + p^2)}{\psi(s)} \left(1 - \alpha \frac{(s^2 + p^2)}{s^2 + \alpha s + p^2} R(s) + \Delta_a(s) \right) \end{aligned} \quad (69)$$

where

$$\Delta_a(s) = \frac{\alpha s}{(s^2 + \alpha s + p^2)} \frac{N(s)}{w} (N(s) - N_a(s)). \quad (70)$$

According to Proposition 8 we have $\|\alpha R\| < 1 - \varepsilon$ for some $1 > \varepsilon > 0$. Therefore, C_a stabilizes P if $\|\Delta_a\| < \varepsilon$. Since $\alpha s / (s^2 + \alpha s + p^2)$ is a band pass filter whose peak magnitude is 1, attained at $s = jp$, the values of $N_a(j\omega)$ near $\omega = p$ should be relatively close to $N(jp)$, in particular we should have

$$\left| 1 - \frac{N(jp)N_a(jp)}{|N(jp)|^2} \right| < \varepsilon.$$

Also note that as α decreases, ε increases (leading to increasing room for approximation error). On the other hand, as seen from (69), very small values of α are not desirable since that leads to closed-loop poles near $\pm jp$. This is the trade-off between the selection of α and freedom in the approximation of N .

4. Examples

Example 4.1 (Application of Propositions 2 and 3). We first design a stable controller C_s as in (16) for a plant with relative degree 3 ($r = 2$):

$$P = \frac{n_s(s)}{d_s(s)(s - 0.1)(s^2 + 1)}. \quad (71)$$

The first design is independent of the Hurwitz polynomials n_s and d_s of equal degree; their roots are the \mathbb{C}_- -zeros and \mathbb{C}_- -poles of the plant. Let $D(s) = \frac{(s - 0.1)(s^2 + 1)}{(s + 0.2)(s^2 + s + 1)}$. By (15), $\rho_1, \rho_2 \in \mathbb{R}_+$, satisfy

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} < \|s[1 - D(s)]\|^{-1} = 0.7360. \quad (72)$$

Choosing $\rho_1 = 2, \rho_2 = 5$ satisfying (72), the controller in (16) becomes

$$C_s = \frac{13 d_s(s)(s^2 + 0.1538s + 0.2308)}{n_s(s)(s + 5)(s + 2)}. \quad (73)$$

The order of C_s in (73) is one less than the plant's order. Now we follow the alternate design of Proposition 3 to design a stable controller of order $r = 2$. To compute the controller parameters, we specify the plant's stable part as $n_s(s) = (s + 0.5)^3, d_s(s) = (s + 0.25)(s + 1)^2$. There are obviously infinitely many choices for the second order Hurwitz polynomial χ ; for example, let $\chi = (s^2 + s + 1)$. Then (24) is satisfied for $\rho > (r + 1)\|(\chi P)^{-1} - s\| = 3.0011$. Choosing $\rho = 3.1$, the unit controller in (23) becomes:

$$C_s = \frac{29.791(s^2 + s + 1)}{(s^2 + 9.3s + 28.83)}. \quad (74)$$

The stable controller C_s in (74) has order 2, one less than the plant's relative degree, whereas the controller in (73) is 5-th order for third-order n_s and d_s .

Example 4.2 (Application of Proposition 5). The following plant model representing "acrobot" and "pendubot" robotic systems is considered in Xin and Liu (2013) for reduced order strong stabilization:

$$P(s) = \frac{\lambda (s^2 - z^2)}{(s^2 - p_1^2)(s^2 - p_2^2)}, \quad \lambda \neq 0, \quad p_2 > p_1 > z > 0. \quad (75)$$

A coprime factorization $P = D^{-1}N$ is obtained by defining $D(s) = \frac{(s - p_1)(s - p_2)}{(s + p_1)(s + p_2)}$. The system satisfies the PIP; it has a single zero in \mathbb{C}_+ and is strictly proper with relative degree two, i.e. $r = 1$. In particular, for the acrobot model data $\lambda = -1.3545, z = 1.281, p_1 = 2.24$ and $p_2 = 6.101$ we have $D(z) = 0.1778$. Applying Proposition 5(b) we obtain a controller in the form

$$C_s(s) = \frac{\lambda^{-1} k_c (s + \alpha)(s + p_1)(s + p_2)}{(\rho^{-1}s + 1)(s + \beta)(s + z)} \quad (76)$$

where b, β and ρ are as in (28), (29), (32), and $k_c(s + \alpha)$ is the polynomial obtained by the division

$$\frac{(s + b)(s + p_1)(s + p_2) - (s + \beta)(s - p_1)(s - p_2)}{(s - z)}. \quad (77)$$

Generically, the order of the controller (76) is 3, one less than the order of the plant. However, there is freedom in the selection of b and ρ , which may lead to $\alpha = \rho$, and hence, makes the controller second order. It can be shown that

$$\alpha = \frac{p_1 p_2 (b + z)}{p_1 p_2 - z(b + p_1 + p_2)}. \quad (78)$$

If $\alpha > 0$ and $\rho = \alpha$ satisfies (32), then controller order can be reduced to two; this depends on the choice of b . For example, for each $b \in [b_{\min}, b_{\max}]$ there is an acceptable ρ ; $b_{\min} = 0.67$ is determined from (32), and $b_{\max} = 2.3274$ comes from $\alpha > 0$. In particular, with $b = 0.8$ we have $\beta = 10.4206, \alpha = 14.535$, and it can be verified that $\rho = \alpha$ satisfies (32), which leads to

$$C_s(s) = \frac{-75.7487 (s + 6.101)(s + 2.24)}{(s + 10.4206)(s + 1.281)}. \quad (79)$$

A specific second order controller is obtained in Xin and Liu (2013), that is different than the one above. With the free choices of b and ρ in Proposition 5, we can obtain infinitely many different second order strongly stabilizing controllers for this plant.

Clearly, the controller (79) does not track step-like or sinusoidal references, neither does the controller of Xin and Liu (2013). For tracking of periodic references of period T_d , as well as step-like inputs a plug-in controller block consisting of a reference generator can be appended (a repetitive controller). In

this case we search for a strongly stabilizing controller C_v for the extended plant

$$P_v(s) = P(s) \frac{1}{1 - (1 + \varepsilon)f(s)e^{-sT_d}} \quad (80)$$

where P is as in (75), $\varepsilon > 0$ is a small number determined by the steady state error tolerance and $f \in \mathcal{S}$ is a low pass filter with $f(0) = 1$ and $|f(j\omega)| \leq 1$, for all ω . Let $P_v = D_v^{-1}N_v$ where

$$D_v(s) = \frac{(s - p_1)(s - p_2)}{(s + p_1)(s + p_2)} \left(\frac{1 - (1 + \varepsilon)f(s)e^{-sT_d}}{1 - (1 + \varepsilon)^{-1}f(s)e^{-sT_d}} \right). \quad (81)$$

Note that $D_v \in \mathcal{H}_\infty$ with (typically) a large number of zeros in \mathbb{C}_+ and $D_v(\infty) = 1$. Proposition 5(b) is still applicable for this system, hence

$$C_v(s) = \frac{\lambda^{-1}}{(\rho^{-1}s + 1)} \frac{1}{N_v(s)} \left(\frac{s + b}{s + \beta} - D_v(s) \right) \quad (82)$$

where b, β and ρ are as in (28), (29), (32) with D replaced by D_v . As a numerical example, consider $T_d = 0.25$, $f(s) = (\varepsilon s + 1)^{-1}$, $\varepsilon = 0.01$ with the acrobot parameters given above. Then, $b > -1.0665$, choosing $b = 0.8$ as before, we have $\beta = 11.1484$ and $\rho > 14.0745$. In this case the controller is infinite dimensional and has the form

$$C_v(s) = \frac{\lambda^{-1}(s + p_1)^2(s + p_2)^2(1 - (1 + \varepsilon)^{-1}f(s)e^{-sT_d})}{(\rho^{-1}s + 1)(s + z)} \times \left(\frac{b - \beta + (s + \beta)R_v(s)}{(s + \beta)(s - z)} \right) \quad (83)$$

where $R_v(s) = 1 - D_v(s)$, which is strictly proper, and by construction $b - \beta + (z + \beta)R_v(z) = 0$. Therefore, the controller can be separated into a stable rational transfer function in cascade with a stable time delay system whose internal structure contains an infinite dimensional system whose impulse response has finite support. There are good finite dimensional approximation methods for such systems for their practical implementation (Nagahara & Yamamoto, 2014).

Example 4.3 (Application of Proposition 8). Consider the plant (84) where $k(s) = 2$, $T_d = \frac{\pi}{2}$

$$P(s) = \frac{e^{-hs}G(s)}{1 + k(s)e^{-T_d s}G(s)}, \quad G(s) = \frac{1}{(s + 1)^2}, \quad (84)$$

with arbitrary $h \geq 0$. The system satisfies the assumptions of Proposition 8 with double poles at $\pm j$, i.e. $p = 1$, and for any $a > 0$ we define $\psi(s) = (s + a)^2$ that leads to

$$N(s) = \left(\frac{e^{-hs}(s^2 + 1)}{(s + 1)^2 + 2e^{-T_d s}} \right) \frac{1}{(s + a)^2}. \quad (85)$$

In particular when $a = 1$ and $h = 2T_d$ we have $N(jp) = -(2 + (2 + \pi)j)^{-1}$, which gives $w = (4 + (2 + \pi)^2)^{-1}$ and $u = -2w$. Then, the \mathcal{H}_∞ norm of R can be computed as $\|R\| = 3.6426$. Choosing $\alpha = 0.25 < 1/3.6426$ we have a strongly stabilizing controller

$$C_s(s) = -\frac{s}{(s + 1)^2} \left(1 + \frac{N(s)}{4w} \right).$$

Using a second order Padé approximation for $e^{-T_d s}$ in N , and performing approximate pole-zero cancellations, a finite dimensional N_a is obtained as

$$N_a(s) = D_{h,m}(s) \frac{(s^2 + 3.82s + 4.863)}{(s + 1)^2(s^2 + 5.678s + 14.34)}$$

where $D_{h,m}$ is the m th order Padé approximation for e^{-hs} . With $\alpha = 0.25$ we have $\varepsilon = 0.089$. In this case, with $D_{h,3}(s)$ we have $\|\Delta\| = 0.043$ and hence, the finite dimensional controller obtained by replacing N with N_a strongly stabilizes P . However, using $D_{h,2}$ in N_a the condition $\|\Delta\| < \varepsilon$ is violated.

5. Conclusions

The strong stabilization problem is solved for several classes of proper unstable plants. These classes include plants with restrictions on finite RHP zeros. The significance of the proposed solution is that explicit controller transfer functions are obtained by computing a few simple parameters. This construction is extremely straightforward, and unlike algorithmic approaches, each design requires only one or two easy steps. Furthermore, the order of the stable controllers is at most one less than the given plant's order. Extensions of the proposed method to additional plant classes are under investigation.

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