Controller Design for Diagonal Decoupling and Integral Action

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Abstract—In the standard linear, time-invariant feedback system, controllers that achieve diagonal decoupling and closed-loop stability exist if and only if the plant satisfies the diagonal denominator condition or has no coinciding poles and zeros in the region of instability. A simple and systematic design method is presented under each of these conditions. The closed-loop poles can be placed at any desired points, and free parameters are included for satisfying additional design objectives. The designs are also extended to provide integral action in order to track step inputs with zero steady-state error.

Index Terms—Control design, decoupling control, integral-action, tracking controller.

I. INTRODUCTION

The removal of the interaction between inputs and outputs while achieving internal stabilization of the closed-loop system is a very important controller design objective for linear time-invariant (LTI), multi-input multi-output (MIMO) systems. The complete elimination of this coupling results in a diagonal and nonsingular complementary sensitivity transfer function. Research on diagonal decoupling has a very long history. This problem has been studied under different solvability conditions using various state-feedback and output-feedback approaches, (e.g., [4], [8], [13], [17]), and in some cases with configurations that may include a precompensator in the feedback loop [3].

Decoupling controller designs that also achieve internal stability were proposed using two-parameter compensation schemes (e.g., [1], [9]). The problem becomes challenging when the output being decoupled is the one used in feedback, and the controller is expected to achieve both internal stability and diagonal decoupling. In this one-degree-of-freedom configuration, diagonally decoupling controllers exist for LTI, MIMO plants with no right-half plane (RHP) pole-zero coincidences [7], [10], [14]. Various conditions were also explored for plants that do not satisfy this well-known sufficient condition [6], [11], [13], [16].

This article establishes that the decoupling problem can be solved if and only if at least one of two conditions holds: if the plant RHP pole-zero coincidences, then it can be decoupled if and only if it satisfies the diagonal denominator condition. The significance of this article is the simple and systematic design methods that provide explicit controller parameterizations for diagonal decoupling with internal stabilization. Additionally, integral action can be included in the designed controllers. There is no need to obtain coprime factorizations or to solve additional Bezout identities for the proposed controller designs. The closed-loop poles are assigned as desired, and additional free parameters are included to be used for satisfying other design objectives.

The problem is formally stated in Section II. The two decoupling conditions and the main Theorem 1 are stated in Section III. The constructive proof of existence of decoupling controllers is given by Propositions 1 and 2, which develop the new decoupling controller design method without using coprime factorization computations. In Section IV, these design procedures are extended to include integral action in addition to internal stability and decoupling, so that the steady-state errors for constant reference inputs go to zero asymptotically. Several examples of the proposed decoupling and integral-action designs are given in Section V.

The following standard notation is used: The region of instability is the extended closed RHP, \( C_+ \cup \{ \infty \} = \{ s \in C \, | \, \text{Re}(s) \geq 0 \} \cup \{ \infty \} \); the open left-half plane (OLHP) is \( C_- \). Real and positive real numbers are \( \mathbb{R} \) and \( \mathbb{R}_+ \); \( \mathbb{R}_p \) denotes real proper rational functions; \( S \subset \mathbb{R}_p \) is the stable subset with no poles in \( \mu \). The set of matrices with entries in \( S \) is \( \mathcal{M}(S) \). The identity matrix is \( I \). For simplicity, we drop \( s \) in transfer matrices such as \( P(s) \). Let \( M := \{ 1, \ldots , m \} \). A diagonal matrix, with diagonal entries \( D_j, j \in M \), is denoted by \( \begin{bmatrix} D_j \end{bmatrix} \) or \( [D_1, \ldots , D_m] \). Transmission zeros and blocking zeros of an MIMO plant are simply called zeros here; we are only interested in the zeros in \( C_+ \cup \{ \infty \} \). The degree of the polynomial \( p(s) \) is denoted by \( \deg(p) \). A polynomial that has all roots in \( C_- \) is called a Hurwitz polynomial (HP).

II. PROBLEM DESCRIPTION

The standard LTI, MIMO unity-feedback system \( \mathcal{H}(P, C) \) is in Fig. 1: \( P \in \mathbb{R}_p^{m \times m} \), and \( C \in \mathbb{R}_p^{m \times m} \) are the plant’s and the controller’s transfer functions, and rank \( P = m \). The objective is to design simple stabilizing controllers that achieve diagonal decoupling, and asymptotic tracking of step-input references with zero steady-state error. The closed-loop transfer function \( H \) from \( u \) to \( y \) is:

\[
H = \begin{bmatrix}
C(I + PC)^{-1} & -C(I + PC)^{-1}P \\
PC(I + PC)^{-1} & (I + PC)^{-1}P
\end{bmatrix}
\]

(1)

Let \( H_{vu} \) denote the input-output (complementary sensitivity) transfer function from \( u \) to \( y \). Let \( H_{ye} \) denote the input-error (sensitivity) transfer function from \( u \) to \( e \).

Definition I: i) The system \( \mathcal{H}(P, C) \) is stable if the closed-loop transfer function \( H \in \mathcal{M}(S) \). The controller \( C \) is called a stabilizing controller if \( C \) is proper and \( H \in \mathcal{M}(S) \). ii) The stable \( \mathcal{H}(P, C) \) is diagonally decoupled if \( H_{vu} \) is diagonal and nonsingular. A stabilizing controller is called a decoupling controller if \( \mathcal{H}(P, C) \) is diagonally decoupled. iii) The system \( \mathcal{H}(P, C) \) has integral action if it is stable.
and $H_{ew}(0) = 0$. A stabilizing controller $C$ is an integral-action controller if it has poles at $s = 0$. iv) The controller $C$ is a decoupling controller with integral action if $C$ stabilizes $P$, has poles at $s = 0$, and the stable system $\mathcal{A}(P, C)$ is both diagonally decoupled and has integral action.

For stable $\mathcal{A}(P, C)$, the steady-state error $e(t)$ for constant inputs applied at $u(t)$ goes to zero asymptotically if and only if the system has integral action, i.e., $H_{ew}(0) = 0$. The controller is designed with poles at zero to achieve integral action (see the internal model principle [5], [12]). A necessary condition for diagonal decoupling is that $P \in \mathbb{R}^{m \times m}_p$ is full rank $m$ since the rank of nonsingular $H_{ew} = PC(I + PC)^{-1}$ is $m$. When the design includes integral action, it is also assumed that $P$ has no zeros at $s = 0$.

**III. DECOUPLING CONTROLLER DESIGN**

Let $P \in \mathbb{R}^{m \times m}_p$, rank $P = m$. If there exist controllers such that the system $\mathcal{A}(P, C)$ is diagonally decoupled, then for some diagonal, nonsingular $\Psi \in \mathbb{S}^{m \times m}_p$, $H_{ew} = PC(I + PC)^{-1} = \Psi$ and $H \in \mathcal{M}(S)$ in (1)

$$
H = \begin{bmatrix}
    P^{-1} \Psi & P^{-1} \Psi P \\
    \Psi & (I - \Psi) P \\
\end{bmatrix}.
$$

All decoupling controllers $C \in \mathbb{R}^{m \times m}_p$ are expressed as

$$
C = P^{-1} \Psi (I - \Psi)^{-1}
$$

Since $\Psi$ is designed so that $P^{-1} \Psi \in \mathcal{M}(S)$, the controllers in (3) are proper if $(I - \Psi)^{-1} \in \mathcal{M}(\mathbb{R}^{m}_p)$. The closed-loop poles are assigned by choosing the denominators freely in the diagonal matrix $\Psi$. Write the entries of $P \in \mathbb{R}^{m \times m}_p$ in polynomial factored form

$$
P = \begin{bmatrix}
    x_{ij} \\
    y_{ij} \Psi_{ij} \\
\end{bmatrix}_{i,j \in M}
$$

$x_{ij}$ is the numerator, $(y_{ij} \Psi_{ij})$ is the monic denominator polynomial of the $ij$th entry of $P$. The roots of $y_{ij}$ are the $C_+$-poles of $P$. All roots of $y_{ij}$ are in $\mathbb{C}_+$. For $i \in M$, define $y_i$ as the monic least-common-multiple of all $y_{ij}$ in the $i$th row. Let $\varphi_i$ be any monic HP such that $\deg(\varphi_i) = \deg(y_i)$. Define $Y_i \in \mathbb{S}$ and the diagonal $Y \in \mathbb{S}^{m \times m}_p$ as (5)

$$
y_i = \lcm y_{ij}, j \in M, \quad Y_i = \begin{bmatrix}
    \varphi_i \\
    \end{bmatrix}, \quad Y = \begin{bmatrix}
    Y_i \\
\end{bmatrix}_{i \in M}.
$$

Therefore, $(y_{ij})^{-1}y_i \in \mathbb{S}, \frac{x_{ij}}{y_i} \in \mathbb{S} \implies YP \in \mathbb{S}^{m \times m}_p$. The terms $Y_i(\infty) = 1$. If all $m$ entries in the $i$th row of $Y$ are stable, then $Y_i = 1$. Write the entries of $P^{-1}$ in polynomial factored form

$$
P^{-1} = \begin{bmatrix}
    n_{ij} \\
    d_{ij} d_{ij} \\
\end{bmatrix}_{i,j \in M}
$$

$n_{ij}$ is the numerator, $(d_{ij} d_{ij})$ is the monic denominator polynomial of the (possibly improper) $ij$th entry of $P^{-1}$. All roots of $d_{ij}$ are in $C_+$; all roots of $d_{ij}$ are in $\mathbb{C}_+$. Since $\text{rank } P = m$, $(YP)^{-1}$ exists, and it may be proper or improper. Write the entries of $(YP)^{-1}$ in factored form

$$
(YP)^{-1} = \begin{bmatrix}
    e_{ij} \\
    \frac{f_{ij}}{f_{ij} f_{ij}} \\
\end{bmatrix}_{i,j \in M} = \begin{bmatrix}
    n_{ij} \\
    d_{ij} d_{ij} y_i \\
\end{bmatrix}_{i,j \in M}
$$

$e_{ij}$ is the numerator, $(f_{ij} f_{ij})$ is the monic denominator polynomial of the $ij$th entry of $(YP)^{-1}$. All roots of $f_{ij}$ are in $C_+$ and all roots of $f_{ij}$ are in $\mathbb{C}_+$. For $j \in M$, define $f_j$ as the monic least-common-multiple of all $f_{ij}$ in the $j$th column; therefore, $(f_{ij})^{-1} f_j \in \mathbb{S}$

$$
f_j = \lcm f_{ij}, \quad i \in M.
$$

The expression for $P^{-1}$ in (6) is a special case of (7) when $P \in \mathcal{M}(S)$, since $Y = I$ in that case.

An important sufficient condition for existence of decoupling controllers is obtained in terms of the $C_+$-zeros of $f_j$ as stated in Condition 1 below: The system $\mathcal{A}(P, C)$ can be decoupled if $y_j$ has no zeros coinciding with the $C_+$-zeros of the corresponding $f_j$. This sufficient existence condition is equivalent to each $(y_j, f_j)$ pair being coprime. Furthermore, if $P$ does not satisfy Condition 1, then the only way it can be decoupled is if $P$ has no coinciding RHP poles and zeros, which is stated as Condition 2.

**Condition 1. Diagonal denominator condition:** For $\ell = 1, \ldots, \mu$, let $z_{\ell j} \in C_+$ be the $\mu_j$ distinct roots of $f_j$, where $0 \leq \mu_j$. The multiplicity of $z_{\ell j}$ is $m_{\ell j}$. For $j \in M$, if $y_j(z_{\ell j}) \neq 0$ for all $z_{\ell 1}, z_{\ell 2}, \ldots, z_{\ell \mu_j} \in C_+$, then $P$ satisfies the diagonal denominator condition.

If $P$ does not satisfy the diagonal denominator condition, i.e., for some $j \in M$, $y_j(z_{\ell j}) = 0$ for any of the $z_{\ell 1}, z_{\ell 2}, \ldots, z_{\ell \mu_j} \in C_+$, then this $z_{\ell j} \in C_+$ is also a pole of $P$. Define $\gamma \in S$ as the monic least-common-multiple of all $y_{ij}$ in all entries of $P$ in (4)

$$
\gamma := \lcm y_{ij}, \quad i, j \in M.
$$

Define $\lambda$ as the monic least-common-multiple of all $d_{ij}$ in all entries of $P^{-1}$ in (6)

$$
\lambda = \lcm d_{ij}, \quad i, j \in M.
$$

Therefore, $(d_{ij})^{-1} \lambda \in S$. If $P^{-1}$ has no finite $C_+$-poles, then all $d_{ij} = 1$, which implies $\lambda = 1$.

**Condition 2. No RHP pole-zero coincidence condition:** For $\ell = 1, \ldots, \mu$, let $z_{\ell 0} \in C_+$ be the $\mu_0$ distinct roots of $\lambda$ where $0 < \mu_0$. The multiplicity of $z_{\ell 0}$ is $m_{\ell 0}$. If $\gamma(z_{\ell 0}) \neq 0$ for $z_{\ell 1}, z_{\ell 2}, \ldots, z_{\ell \mu_0} \in C_+$, then $P$ satisfies the no RHP pole-zero coincidence condition.

**Theorem 1:** Necessary and sufficient conditions for decoupling: Let $P \in \mathbb{R}^{m \times m}_p$, rank $P = m$. There exist decoupling controllers if and only if $P$ satisfies at least one of the two conditions: Condition 1 (diagonal denominator condition) or Condition 2 (no RHP pole-zero coincidence condition).

**Proof of Theorem 1:** Necessity: Suppose that Conditions 1 and 2 both fail. Let $x_i \in C_+$ be a common zero of $y_i$ and $f_j$ for some $j$. Then, $P$ has a pole at $s = x$ that appears in the $j$th row, and $s = x$ is also a pole that appears in some column of $P^{-1}$. By (2), for closed-loop stability, $P^{-1} \Psi \in \mathcal{M}(S)$ and $(I - \Psi) P \in \mathcal{M}(S)$ imply either $\psi_i(z) = 0$ or that every entry of $\Psi$ is the same, which requires that $P$ has no coinciding RHP poles and zeros. But also by (2), $(I - \Psi) P \in \mathcal{M}(S)$ implies $(1 - \psi_i(s))$ must have a zero at $s = x$ to cancel the pole of $P$ in the $j$th row. This is a contradiction since $\psi_i(z) = 0$. Therefore, the transfer function $H_{ew} = \Psi$ of the stable closed-loop system cannot be diagonal when both conditions fail.

The sufficiency of each condition in Theorem 1 is proved by explicit construction. Propositions 1 and 2 give detailed construction of a complete set of decoupling controllers for $P$ satisfying the diagonal denominator condition and the no RHP pole-zero coincidence condition, respectively. These designs also allow placing the closed-loop poles at any desired locations in the OLHP.

**Proposition 1:** Decoupling controller design procedure for $P$ satisfying the diagonal denominator condition.

**Step 1:** For each entry of $(YP)^{-1}$, define the integers $\rho_{ij} \geq 0$, and $\rho_j$ as the largest of all $\rho_{ij}$ in the $j$th column as (11). If the $ij$th entry
The corresponding diagonal input-output transfer function is \( H_{uu} = \Psi = \text{diag} \{ \psi_j \}_{j \in M} \), where \( \psi_j \in \mathbb{S} \) satisfies \( \psi_j \neq 0 \), and \( \psi_j(\infty) \neq F_j(\infty)^{-1} \) to ensure properness of \( C \). The closed-loop poles are the roots of the polynomial \( \xi_j \) of \( F_j \) and the poles of \( q_j \).

**Remarks 3:** Special Case: Decoupling design for invertible plants: If \( P \notin \mathbb{M}(\mathbb{S}) \), but \( P^{-1} \in \mathbb{S}^{m \times m} \), then \( (YP)^{-1} \in \mathbb{M}(\mathbb{S}) \).

Therefore, \( F_j = 1 \), and \( F = I \) in (12); hence, the diagonal denominator condition holds for invertible-stable plants. The decoupling controller design in Proposition 1 is simplified by choosing \( \psi_j = (1 - q_j Y_j) \) as in (15) of part (ii) as follows: Let \( Q = \text{diag} \{ q_j \}_{j \in M} \in \mathbb{S}^{m \times m} \) be any stable, nonsingular, diagonal matrix, with \( q_j \in \mathbb{S} \) satisfying \( \psi_j(\infty) \neq 0 \). Let \( \Psi = \text{diag} \{ (1 - q_j Y_j) \}_{j \in M} \). The decoupling controllers in (17) become

\[
C = P^{-1}(I - QY)(QY)^{-1} = P^{-1}(Y^{-1}Q^{-1} - I).
\]

The diagonal input-output transfer function is \( H_{uu} = \Psi = \text{diag} \{ I - QY \} \), where \( q_j(\infty) \neq 0 \) so that \( C \) is proper. The closed-loop poles are the roots of the polynomial \( \varphi_j \) of \( Y_j \) and the poles of \( q_j \).

The diagonal decoupling controller design in Proposition 1 only applies to plants satisfying the assumption \( q_j(z_j) \neq 0 \) for the \( C_\ast \)-zeros \( z_j \in C_\ast + \) of \( f_j \), for \( \ell = 1, \ldots, m \). If at least one \( \psi_j \) has zeros that coincide with \( z_j \) of the corresponding \( f_j \) it is still possible to decouple the system \( \mathcal{F}(P, C) \) if the plant’s \( C_\ast \)-zeros do not coincide with its poles. Proposition 2 gives a procedure similar to the method in Proposition 1. Although decoupling for the case of plants with noncoincident \( C_\ast \)-zeros and poles has been studied extensively (e.g., [6], [7], [10]), the procedure given here provides a much simpler and complete design without computing coprime factorizations.

**Proposition 2:** Decoupling controller design procedure for \( P \) satisfying the no RHP pole-zero coincidence condition.

Let \( P \in \mathbb{R}^{m \times m}_+ \), \( P \notin \mathbb{M}(\mathbb{S}) \), \( P^{-1} \notin \mathbb{M}(\mathbb{S}) \), rank \( P = m \). With \( Y_j \in \mathbb{S} \) as in (5), suppose that, for at least one \( y_j, j \in M, y_j(z_j) = 0 \) for at least one of the \( C_\ast \)-zeros \( z_j \in C_\ast + \) of \( f_j \), \( \ell = 1, \ldots, m \). If \( z_j(\infty) \neq 0 \) at each \( C_\ast \)-zero \( z_\ell \) of \( \lambda \), then:

**Step 1:** Let \( \varphi \) be a monic HP such that \( \text{deg}(\varphi) = \text{deg}(\gamma) \); define \( \Gamma \in \mathbb{S} \)

\[
\Gamma := \frac{\gamma}{\varphi}.
\]

**Step 2:** Define the integers \( \tilde{\rho}_i \geq 0 \) for each entry of \( P^{-1} \), and \( \rho \) as the largest of all \( \tilde{\rho}_i \). If \( P^{-1} \) is proper, then \( \rho = 0 \).

\[
\tilde{\rho}_i := \max \{ 0, \text{deg}(s_{i,j}) - \text{deg}(d_{j,2}d_i^* + i) \}, \quad \rho := \max_{i,j} \tilde{\rho}_i, \quad i, j \in M.
\]

Let \( \xi \) be any monic HP such that \( \text{deg}(\xi) = \rho + \text{deg}(\lambda) \). Define \( \Lambda \in \mathbb{S} \) as in (22); then \( \Lambda P^{-1} \in \mathbb{S}^{m \times m} \).

**Step 3:** Define \( \theta \in \mathbb{S} \) as in (23); \( \theta(\infty) = 1 \) \( \Rightarrow \theta(\infty) = \prod_{\ell=1}^m (1 - \Gamma(z_\ell)^{-1})^{m \nu} \) if \( \rho = 0 \); \( \theta(\infty) = 0 \) if \( \rho \neq 0 \).

\[
\theta := (1 - \Gamma)^{\nu} \prod_{\ell=1}^m (1 - \Gamma(z_\ell)^{-1})^{m \nu}.
\]

**Step 4:** Define \( \psi \in \mathbb{S} \) as

\[
\psi = (\theta + \Gamma \Lambda)^{-1}, \quad q_j \in \mathbb{S}, \quad \psi(\infty) = (1 - \theta(\infty))\Lambda(\infty)^{-1}.
\]

Then, \( C \) in (25) is a decoupling controller

\[
C = \psi(I - \psi)^{-1} P^{-1} = (\theta + \Gamma \Lambda)^{-1}(1 - \theta - \Gamma \Lambda)^{-1} P^{-1}.
\]
The corresponding diagonal transfer function is $H_{yu} = \psi I$. The closed-loop poles are the roots of the polynomials $\varphi$, $\xi$, and the poles of $\eta$.

**Remarks 4: Justification of Proposition 2:** With $\psi$ as in (24), and $C$ given by (25), $\Delta P^{-1} \in M(S)$ implies $\psi P^{-1} \in M(S)$ since the $C_r$-zeros of $\theta$ and of $\Lambda$ are the same. Furthermore, $(-1 - \psi) P \in M(S)$, and in $H$ of (1), $H_{uu} = \psi P^{-1} \in M(S)$, $H_{w} = -H_{uu} P = -\psi I \in M(S)$, $H_{yu} = P H_{uu} = \psi I \in M(S)$, $H_{yu} = (1 - \psi) P \in M(S)$. In (24), the constraint on $\theta$ ensures that $(-1 - \psi) P \in \mathbb{R}_+$, and hence, the decoupling controller $C$ in (17) is proper.

**Remarks 5: Plants that satisfy both sufficient conditions:** Some plants with no coinciding $C_r$-poles and zeros also satisfy the assumptions of Proposition 1. The controller design procedure first checks if Condition 1 holds, and applies Proposition 1 to such plants. The plant in Example 3 is an example of a plant satisfying both conditions.

**IV. INTEGRAL-ACTION DECOUPLING CONTROLLERS**

The necessary condition for existence of integral-action controllers is that $P$ has no zeros at $s = 0$. Under this assumption, the proposed decoupling controller designs are extended to include integral action in Propositions 3 and 4.

**Proposition 3:** Integral-action decoupling controllers for $P$ satisfying the diagonal denominator condition.

Under the assumption of Proposition 1, let $P$ have no zeros at $s = 0$. The decoupling controller in (17) becomes also an integral-action controller $C_I = P^{-1} \psi(I - \psi) P^{-1}$ if the entries $\psi_j \in S$ of $\psi_I = \text{diag} \{\psi_j\}_{j \in M} \in S^{m \times m}$ has one of three possible values

i) If $Y_j = 1$, then for any $\alpha_r \in \mathbb{R}_+$

$$\tilde{\psi}_j = F_j \left( F_j(0)^{-1} + \frac{s}{s + \alpha_r} \tilde{\psi}_j \right)$$  (26)

ii) If $Y_j = 1$, and $F_j = 1$, then for any $\alpha_r \in \mathbb{R}_+$

$$\tilde{\psi}_j = \left( 1 - \frac{s}{s + \alpha_r} \tilde{\psi}_j Y_j \right), \quad \tilde{\psi}_j \in S, \quad \tilde{\psi}_j(\infty) \neq 0 \quad (27) \text{for all entries in (26) and (27).}$

iii) If $Y_j = 1$, and $F_j = 1$, then for any $\alpha_r \in \mathbb{R}_+$

$$\tilde{\psi}_j = \theta_j + F_j(0)^{-1}(1 - \theta_j) + \frac{s}{s + \alpha_r} \tilde{\psi}_j F_j Y_j$$  (28)

The decoupled closed-loop transfer function with integral action is $H_{yuI} = \psi I$, and the de-gain of the input-error transfer function is $H_{yuI}(0) = I - \psi(0) = 0$.

**Remarks 6:** Justification of integral action for Proposition 3: For $\psi_{\tilde{\psi}}$ as in (26)–(28), $\Psi_I(0) = I$. Therefore, $C_I$ in (29) has poles at $s = 0$. In (26)–(28), the constraints on $\tilde{\psi}_j(\infty)$ ensure that $(-1 - \psi_{\tilde{\psi}})^{-1} \in \mathbb{R}_+$; $C$ in (17) is a proper decoupling controller with integral action, and $H_{yuI}(0) = I - \Psi(0) = 0$, i.e., the system $\mathcal{J}(P, C)$ is decoupled and has integral action.

**Remarks 7:** Special Case: Integral-action decoupling controllers for stable plants: If $P \in S^{m \times m}$ has no zeros at $s = 0$, i.e., $\text{rank} P(0) = m$, then the integral-action decoupling controller design in Proposition 3 is simplified, and the decoupling $C$ in (18) is also an integral-action controller if the diagonal $Q = \text{diag} \{\tilde{\psi}_j\}_{j \in M} \in S^{m \times m}$ satisfies (31) for any $\alpha_r \in \mathbb{R}_+$ and any diagonal $Q_j = \text{diag} \{\tilde{\psi}_j\}_{j \in M} \in S^{m \times m}$

$$Q = \text{diag} \left[ F_j(0)^{-1} + \frac{s}{s + \alpha_s} \tilde{\psi}_j \right]_{j \in M}$$  (31)

$\tilde{\psi}_j \in S, \quad \tilde{\psi}_j(\infty) \neq F_j(\infty)^{-1} - F_j(0)^{-1}$, for $j \in M$. With $Q \in S^{m \times m}$ as in (31), the integral-action decoupling controller $C_I$ is given by (29), with $\psi_{\tilde{\psi}}$ as in (26). For $\tilde{\psi}_j = 0$ in (31), the "nominal" integral-action decoupling controller $C_I^* = \text{diag} \left[ F_j(0)^{-1} - F_j(0)^{-1} \right]_{j \in M}$.

The diagonal input-output transfer function $H_{yuI}$ is

$$H_{yuI} = \text{diag} \left[ F_j(0)^{-1} + \frac{s}{s + \alpha_s} \tilde{\psi}_j \right]_{j \in M}$$  (33)

$\tilde{\psi}_j \in S$ satisfy $\tilde{\psi}_j(\infty) \neq F_j(\infty)^{-1} - F_j(0)^{-1}$.

**Remarks 8:** Special Case: Integral-action decoupling design for inverse-stable plants: If $P \notin M(S)$, but $P^{-1} \in S^{m \times m}$, then $P$ has no zeros at $s = 0$, i.e., $\text{rank} P(0) = m$. The integral-action decoupling controller design in Proposition 3 is simplified, and the decoupling $C$ in (19) is also an integral-action controller if the diagonal $Q \in S^{m \times m}$ satisfies (34) for any $\alpha_r \in \mathbb{R}_+$

$$\frac{s}{s + \alpha_r} Q = \frac{s}{s + \alpha_s} \text{diag} \{\tilde{\psi}_j\}_{j \in M}, \quad \tilde{\psi}_j(\infty) \neq 0 \quad (34)$$

For diagonal $Q \in S^{m \times m}$ as in (34), the integral-action decoupling controller $C_I$ and the diagonal input-output transfer function $H_{yuI}$ are

$$C_I = P^{-1} \left[ \frac{s + \alpha_s}{s}(Q Y)^{-1} - I \right], \quad H_{yuI} = I - \frac{s}{s + \alpha_s} Q Y$$  (35)

**Proposition 4:** Integral-action decoupling controllers for $P$ satisfying the no RHP pole-zero coincidence condition.

Under the assumption of Proposition 2, let $P$ have no zeros at $s = 0$. For any $\alpha_r \in \mathbb{R}_+$, let $\psi_I \in S$ be

$$\psi_I = \theta + \Lambda \psi(0)^{-1}(1 - \theta) + \frac{s}{s + \alpha_r} \tilde{\psi}_j$$  (36)

$\tilde{\psi}_j \in S, \quad \tilde{\psi}_j(\infty) \neq (1 - \theta(\infty))(\Lambda(\infty)^{-1} - \Lambda(0)^{-1})$. Then, with $\psi_I$ as in (36), $C_I$ in (37) is a decoupling integral-action controller

$$C_I = \psi_I(1 - \psi_I)^{-1} P^{-1}$$  (37)

$$\tilde{\psi}_j = 0$$

The integral-action decoupling controller becomes

$$C_I^* = \left[ (1 - \theta)^{-1} - (1 - \Lambda(0)^{-1}) \right] P^{-1}.$$  (38)

Then, $H_{yuI} = \psi I$, and $H_{yuI} = (1 - \psi I) = [(1 - \theta)(1 - \Lambda(0)^{-1}) - \frac{s}{s + \alpha_r} \tilde{\psi}_j \Lambda \Gamma I)$.

**Remarks 9:** Justification of integral action for Proposition 4: For $\psi_I$ as in (36), $\psi(0)^{-1}$ is in $C_I$ in (37) has poles at $s = 0$, and $H_{yuI}(0) = (1 - \psi(0)) I = 0$. The constraint on $\tilde{\psi}_j(\infty)$ ensures that $(-1 - \psi_{\tilde{\psi}})^{-1} \in \mathbb{R}_+$; therefore, $C$ in (37) is proper.
Example 1: $P$ satisfies Condition 1; has one coinciding RHP pole-zero: The plant $P = \frac{s^2 + \frac{9}{2} s + \frac{1}{2}}{(s + 2)(s + 4)}$ has $\mathbb{C}_+\text{-}p$oles at 1, 2, and $\mathbb{C}_-$zeros at 1, 5, and infinity. Then, $y_1 = (s - 2)$, $y_2 = (s - 1)$. Define $h_1 = (s + 4)$, $h_2 = (s - 5)$. Choose $\varphi_1 = (s + 3)$, $\varphi_2 = (s + 5)$, $Y = \text{diag} \left[ \frac{\varphi_1}{\gamma}, \frac{\varphi_2}{\gamma} \right]$. The the of $f_1 = y_2 h_2$ are $z_{1,1} = z_{2,1} = 5$; the zero of $f_2 = h_2$ is $z_{2,2} = 5$. With $p_1 = 2 = 1$, $\xi_1$ is a third order and $\xi_2$ is a second order HP; $F_1 = \frac{1}{\varphi_1}$ and $F_2 = \frac{1}{\varphi_2}$. By (13), $\theta_1 = (1 - Y_1)/(1 - Y_1(1 - Y_1)/(1 - Y_1(1 - Y_1(1 - Y_1)))$ and $\theta_2 = (1 - Y_2)/(1 - Y_2(1 - Y_2(1 - Y_2(1 - Y_2)))$. By (16), $\psi_j(\infty) = 0$ for $j = 1, 2$ implies $\psi_1 = (\theta_1 + q_j F_j Y_j)$ for any $q_j \in \mathbb{C}$. Let $g_1 := (5s^2 + 18s + 297)$. Choose $q_1 = 2 = 0$; $H_{uv} = \text{diag} \left[ \theta_1, \theta_2 \right]$. By (17), $C = \text{diag} \left[ \frac{\psi_1}{\varphi_1}, \frac{\psi_2}{\varphi_2} \right]$. $P-$diagram is an integral-action controller.

Example 2: $P$ satisfies Condition 1, has two coinciding RHP pole zeros: Define $h_1 = (s + 1)$, $h_2 = (s - 2)$, $h_3 = (s + 5)$, $h_4 = (s + 7)$. The plant $P = \frac{(s + 2)(s + 4)}{(s + 2)(s + 4)}$ has coinciding $\mathbb{C}_+$-poles and zeros at $s = 1$ and $s = 2$. With $y_1 = 1$, $y_2 = 2$, $y_3 = h_1$, $y_4 = (s + 2)$, $\varphi_2 = (s + 5)$; then $Y = \text{diag} \left[ \frac{\varphi_1}{\gamma}, \frac{\varphi_2}{\gamma} \right]$. By Proposition 1, $f_1 = h_1 f_2 = h_3 f_3 = h_4 f_4$. The $\mathbb{C}_+$-zero of $f_3$ is $z_{1,1} = 2$; $Y_3(\xi_1) \neq 0$. By (13), $\theta_3 = (1 - Y_3)/(1 - Y_3(1 - Y_3(1 - Y_3)))$. From (14)-(16), $\psi_1 = F_1 q_1 = \frac{1}{\varphi_1} q_1$, $\psi_2 = F_2 q_2 = (1 - \frac{1}{\varphi_2} q_2)$, $\psi_3 = (\frac{1}{\varphi_3} + \frac{1}{\varphi_4} q_1) q_1$, $\psi_4 = 0$, $\theta_4 = 0$, $\theta_5 = 0$; $\xi_1, \xi_3$ are second-order HPS

b) Choose $\xi_1 = (s + 5), q_1 = 0.5, q_2 = 1, q_3 = 0$. Define $g_1 := (s + 19)$, $g_2 := (s + 34)$, $g_3 := (33s^2 + 464s + 1388)$. By (17), $C = \text{diag} \left[ \frac{\psi_1}{\varphi_1}, \frac{\psi_2}{\varphi_2} \right]$. $P-$diagram is an integral-action controller.

b) In (26)-(28), choose $\xi_1 = \varphi_3$, $\xi_2 = 0$, $\xi_3 = 1$, $\varphi_4 = 4$. Then, $\psi_1 = \frac{10h_3 h_4}{h_3 h_4}$, $\psi_2 = 0$, $\psi_3 = 0$, $\psi_4 = \frac{10h_3 h_4}{h_3 h_4}$. $C = \text{diag} \left[ \frac{\psi_1}{\varphi_1}, \frac{\psi_2}{\varphi_2} \right]$ is an integral-action controller.

Example 3: $P$ satisfies both Conditions 1 and 2: The plant $P = \frac{(s + 1)(s + 2)}{(s + 1)(s + 4)}$ has $\mathbb{C}_+$-poles at 2, 4, and none coinciding with the $\mathbb{C}_-$zeros at 1, 11, and infinity. Define $\varphi_1 = (s + 3), \varphi_2 = (s + 5), Y = \text{diag} \left[ \frac{\varphi_1}{\gamma}, \frac{\varphi_2}{\gamma} \right]$. The zeros of $f_1 = h_1 f_2$ are $z_{1,1} = z_{2,1} = 11$; the zero of $f_2 = h_2$ is $z_{2,2} = 11$. With $p_1 = 2 = 1$, choosing $\xi_1 = \varphi_1$ and $\xi_2 = \varphi_2$, $F_1 = \frac{1}{\varphi_1}$, $F_2 = \frac{1}{\varphi_2}$. By (13), $\theta_1 = (1 - Y_1)/(1 - Y_1(1 - Y_1(1 - Y_1(1 - Y_1))))$, $\theta_2 = (1 - Y_2)/(1 - Y_2(1 - Y_2(1 - Y_2(1 - Y_2))))$. By (16), $F_j(\infty) = 0$ implies $\psi_j = (\theta_j + q_j F_j Y_j)$ for any $q_j \in \mathbb{C}$. Let $g_1 = (s + 8) = 0 = g_2 = (7s + 179), g_3 = (11s^2 + 72s + 621)$; by (17), $C = \text{diag} \left[ \frac{\psi_1}{\varphi_1}, \frac{\psi_2}{\varphi_2} \right]$. $P-$diagram is an integral-action controller.

Example 4: $P$ satisfies Condition 2 but not Condition 1: Define $h_1 = (s + 1)$, $h_2 = (s - 1)$, $h_3 = (s^2 - 6s + 3)$, $h_4 = (s - 3)$. $g_1 = (s - 4), g_2 = (s + 4)$. The plant $P = \frac{5(s - 4)}{5(s + 4)}$ has no coincident $\mathbb{C}_+$-poles and zeros. Then, $y_1 = h_2, y_2 = y_3 = h_4$. Choose $\varphi_1 = (s + 2), \varphi_2 = \varphi_3 g_2$. Then $Y = \text{diag} \left[ \frac{\psi_1}{\varphi_1}, \frac{\psi_2}{\varphi_2} \right]$. $(YP)^{-1} = \frac{h_1 h_2}{h_1 h_2}$ is an integral-action controller.

With $\gamma = \frac{\varphi_2}{\varphi_2}$, $P$ satisfies the no RHP pole-zero coincidence condition since $\gamma(4) \neq 0$. Let $\varphi = \varphi_2$, and since $\varphi = 1$, choose $\xi$ as any second-order HP; for example, $\xi = \varphi$. Then, $\Gamma = \varphi, \Lambda = \varphi$. By (23), $\theta = (1 - \Gamma)/(1 - (\Gamma - 1)^{-1})$. Following Proposition 2, choosing $q = 0$, by (25), $C = \theta(1 - \theta)^{-1} P^{-1}$, and $H_{uv} = \psi = 0 = \frac{1}{\psi}$. $P-$diagram is an integral-action controller. Since $P$ has no zeros at $s = 0$, integral-action decoupling controllers are obtained by Proposition 4. For $\dot{q} = 0$, by (37), $C_0$ becomes $C_0 = \frac{1}{\psi(4 + 1)^{-1} \psi(4 + 1)^{-1}}$, and $H_{uv} = \psi = 0 = \frac{1}{\psi}$. $P-$diagram is an integral-action controller. Since $P$ has no zeros at $s = 0$, integral-action decoupling controllers are obtained by Proposition 4. For $\dot{q} = 0$, by (37), $C_0$ becomes $C_0 = \frac{1}{\psi(4 + 1)^{-1} \psi(4 + 1)^{-1}}$, and $H_{uv} = \psi = 0 = \frac{1}{\psi}$. $P-$diagram is an integral-action controller.
VI. CONCLUSION

Simple decoupling controller design procedures were proposed for all plant classes that can be diagonally decoupled. The designs include the option of integral action in the controllers, which implies that steady-state errors for constant reference inputs go to zero asymptotically. The controllers are derived from the inverse of the plant’s transfer function, without using coprime factorizations. With minor modifications, the designs can be applied to non-square full row-rank plants by using right inverses.

REFERENCES