2013 American Control Conference (ACC)
Washington, DC, USA, June 17-19, 2013

Low-order simultaneous stabilization of linear bicycle models at different forward speeds

A. N. Gündeş[1] and A. Nanjangud[2]

Abstract— **Linear models of bicycles with rigidly attached riders, operating at different forward speeds, are considered as a challenging platform for the simultaneous stabilization problem. It is shown that any number of such models obtained at reasonable speeds can be simultaneously stabilized using simple, low-order controllers with only the steering torque as input. Stabilizing controllers for individual systems modeled at extremely low speeds are also proposed.**

I. INTRODUCTION

Single-track vehicles with human riders, such as bicycles, present challenging problems of modeling and control. Based on general curiosity about bicycle balance and to contribute to improved designs of specialized bicycles with better handling capabilities, a great deal of research has been devoted to the issues of bicycle stability. The linearized equations of a model based on the Whipple bicycle in [7], developed further in [4] into the form used here, have become the basis for a benchmark bicycle. The linearized equations, with the benchmark parameter values of [4], define a different linear bicycle model for each constant forward speed. The problem considered in this paper becomes the synthesis of a common feedback controller that simultaneously stabilizes this finite set of systems generated from these linear models at specific forward speeds. Bicycle-rider models and control of varying complexity have been reported and control algorithms capable of stabilizing a bicycle (both theoretically and in practice) have been developed (see, e.g., [1], [2], [3], [5], [6] and the references therein). The objective of this study is not to develop a new or refined model; discussions of the model dynamics are beyond the scope of this work. Our interest in the bicycle stability problem is due to the challenging control problem it poses as the simultaneous stabilization of linear models at different speeds of different bicycle parameters. Although bicycle stability at a fixed constant speed has been considered, the problem has never been explored from a simultaneous stabilization perspective. The simultaneous stabilization results and the systematic design procedures proposed here are completely novel approaches. Our study is based on the model with the benchmark parameters of [4]. The same four-state model is used in [3], with parameters for six different bicycles. The class of systems considered in our investigation of simultaneous stabilizability may include any finite number of plants generated by this model resulting from different constant speeds using the parameters of [4], *plus* any number of the six other bicycle models at different speeds in [3]. In Section II-A, we consider conceptual simultaneously stabilizing controller design using two control inputs: If there was an actuator input of a torque applied about a line connecting the wheel contact points, then any number of linear bicycle models operating at any forward speeds could be simultaneously stabilized. Although this second input is not realistic since the model assumes the rider to be rigidly attached to the bicycle frames, this study provides important simultaneously stabilizing controller design results for an interesting plant class. In Section II-B, the problem is much harder from a control design perspective since only the steering torque is available as input. In Section II-B.1, the problem is solved for a reasonable range of speeds (larger than 0.58 meters/second for the parameters in [4] and similar speeds ranging from 0.4185 m/s to 0.7351 m/s for the parameters of the six bicycle models in [3]). For low speeds below this range, individual controllers for each model are proposed in Section II-B.2. The benchmark parameters given in [4] are used in Section III for the numerical computations to illustrate the proposed designs.

Notation: The extended closed right-half plane $\mathcal{U} = \mathbb{C}_+ \cup \{\infty\} = \{\, s \in \mathbb{C} \,|\, \mathcal{R}e(s) \geq 0 \,\} \cup \{\infty\}$ is the region of instability. Real and positive real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. The set of real proper rational functions of s is denoted by $\mathbf{R_p}$; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$. The set of matrices with entries in $\mathbf{S}$ is $\mathcal{M}(\mathbf{S})$. A matrix $M \in \mathcal{M}(\mathbf{S})$ is called unimodular if $M^{-1} \in \mathcal{M}(\mathbf{S})$. The H_∞-norm of $M(s) \in \mathcal{M}(\mathbf{S})$ is denoted by $\|M(s)\|$, i.e., the norm $\|\cdot\|$ is defined as $\|M\| := \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$. Where this causes no confusion, we drop (s) in transfer-functions and matrices such as $G(s)$. The $m \times m$ identity matrix is I_m ; we use I when the dimension is unambiguous. The 2×2 zero-matrix is 0_2 .

II. MAIN RESULTS

Consider the linearized bicycle model

$$M\ddot{q} + v_j K_1 \dot{q} + (\, g\, K_o + v_j^2 K_2\,)q = f \ , \qquad (1)$$

where $q = [\,\phi \;\; \delta\,]^T$, $f = [\,T_\phi \;\; T_\delta\,]^T$, and ϕ is the bicycle rear-frame roll angle, δ is the handlebar steering angle, T_ϕ is the externally applied torque about the line connecting the wheel contact points, and T_δ is the resultant torque of all rider-applied handlebar forces [4], [3]. At each different constant forward speed v_j , the model (1) becomes a different plant to be stabilized. A finite class of plants is generated

[1]A. N. Gündeş is with the Electrical and Computer Engineering Department, University of California, Davis, CA 95616 angundes@ucdavis.edu

[2]A. Nanjangud is with the Mechanical and Aerospace Engineering Department, University of California, Davis, CA 95616 nanjangud@ucdavis.edu

by choosing a set of speed values. The goal is to design a controller that simultaneously stabilizes all plants in this set. Since the model in (1) is based on the assumption that the rider is fixed to the bicycle, the rider lean torque T_ϕ is not available as control input. From a conceptual design perspective, the system description in (1) can be viewed as a two-input four-output system called P_j, with T_ϕ also available; this case is studied in Section II-A. The system controlled only by the steering torque T_δ is a one-input four-output system called G_j studied in Section II-B. In (1), the numerical values for the constant matrices M, K_o, K_1, K_2 are the benchmark values given in [4], and g is the acceleration constant due to gravity. Different values for these constant matrices can also be used such as those given for six different bicycles in [3].

A. Linear bicycle model with two inputs

We explore simultaneous stabilizability of the set of two-input four-output systems obtained at constant forward speeds from the linear bicycle model in (1). Since the system has two inputs T_ϕ and T_δ, the results of this section are of theoretical interest. Let the input be $f = [T_\phi \;\; T_\delta\,]^T$; let the output be $y := [q \quad \dot{q}]^T = \begin{bmatrix}\phi & \delta & \dot{\phi} & \dot{\delta}\end{bmatrix}^T$. For any arbitrary $a \in \mathbb{R}_+$, define $Y_j \in \mathbf{S}^{2\times 2}$ as

$$Y_j := (s+a)^{-2}\left[Ms^2 + v_j K_1 s + (gK_o + v_j^2 K_2)\right] \quad (2)$$

where the entries of $Y_j =: \begin{bmatrix} Y_a & Y_{bj} \\ Y_{cj} & Y_{dj}\end{bmatrix}$ are in $\mathbf{S}$, and Y_a does not depend on the forward speed v_j. Let

$$W_j := \det Y_j = [\,Y_a Y_{dj} - Y_{bj} Y_{cj}\,] \ . \quad (3)$$

By (2), $W_j(\infty) = \det M$. The 4×2 transfer-matrix of the plant P_j from the model (1) is given as

$$P_j = X Y_j^{-1} = \begin{bmatrix}(s+a)^{-2} I_2 \\ s\,(s+a)^{-2} I_2\end{bmatrix} Y_j^{-1} \ . \quad (4)$$

Consider a controller $C_p \in \mathbf{R_p}^{2\times 4}$,

$$C_p = D^{-1} N \begin{bmatrix} aI_2 & I_2\end{bmatrix} \ , \quad (5)$$

where D, $N \in \mathbf{S}^{2\times 2}$. Using P and C_p given in (4) and (5), the controller C_p stabilizes each plant P_j if and only if (D, N) are such that $N\begin{bmatrix} aI_2 & I_2\end{bmatrix} X + D\,Y_j$ is unimodular, equivalently, F_j is unimodular, where

$$F_j \; := \; (s+a)^{-1} N + D\,Y_j \ . \quad (6)$$

Let the (input-error) transfer-function from u to e be denoted by $H_{eu} \in \mathbf{R_p}^{4\times 4}$ and let the (input-output) transfer-function from u to y be denoted by $H_{yu} \in \mathbf{R_p}^{4\times 4}$. Then $H_{eu} = (I + P_j C_p)^{-1} = I - P_j C_p (I + P_j C_p)^{-1} = I - H_{yu}$. Using the representations of P_j and C_p given in (4) and (5), the closed-loop transfer-function H_{yu} can be written as $H_{yu} = P_j (I_2 + C_p P_j)^{-1} C_p = X\,F_j^{-1} N \begin{bmatrix} aI_2 & I_2\end{bmatrix} = \begin{bmatrix} \frac{a}{(s+a)^2} F_j^{-1} N & \frac{1}{(s+a)^2} F_j^{-1} N \\ \frac{a\,s}{(s+a)^2} F_j^{-1} N & \frac{s}{(s+a)^2} F_j^{-1} N\end{bmatrix}$. Let $\mathcal{P}$ be a finite set of plants, where $P_j \in \mathcal{P}$ is described as in (4) Controllers that simultaneously stabilize any number of plants in $\mathcal{P}$ exist for this class and can be designed using the simple synthesis procedure. Proposition 1-(a) gives a constant controller design. The design has freedom in the choice of the positive real constant a, and the resulting α satisfying a norm bound. In Proposition 1-(b), the controller has integral-action due to the pole at $s = 0$. The design freedom is in the choice of the Hurwitz polynomials $n(s)$, $d(s)$, and the resulting α satisfying a norm bound. For simple implementation, the order of these polynomials should be low; if $n(s)$ has degree one, then the transfer-matrix C_p is in the form of a proportional-plus-integral (PI) controller. Although the only objective here is to show synthesis for simultaneously stabilizing controllers, the freedom in the parameters in this design method can be used to achieve additional performance requirements.

Proposition 1: (*Simultaneous controller design for* $\mathcal{P}$):
Consider finitely many plant models $P_j \in \mathbf{R_p}^{4\times 2}$, described as in (4), with Y_j as in (2).

a) Choose any $\alpha \in \mathbb{R}_+$ satisfying

$$\alpha \; > \; \max_{Y_j} \; \| \, (s+a) Y_j M^{-1} - s\,I \, \| \ . \quad (7)$$

Then a controller $C_p \in \mathbf{R_p}^{2\times 4}$ that strongly stabilizes all P_j is given by

$$C_p = \alpha\, M \begin{bmatrix} aI_2 & I_2\end{bmatrix} \ . \quad (8)$$

b) Choose any two monic, Hurwitz polynomials $n(s), d(s)$, where $\deg n(s) \geq 1$, $\deg d(s) = (\deg n(s) - 1)$. Choose any $\alpha \in \mathbb{R}_+$ satisfying

$$\alpha \; > \; \max_{Y_j} \; \| \; s\,(\frac{(s+a)d(s)}{n(s)} Y_j\, M^{-1} - I_2) \; \| \ . \quad (9)$$

Then an integral-action controller $C_p \in \mathbf{R_p}^{2\times 4}$ that stabilizes all P_j is given by

$$C_p = \frac{\alpha\, n(s)}{s\, d(s)} M \begin{bmatrix} aI_2 & I_2\end{bmatrix} \ . \quad (10)$$

Remarks: The controller in (10) of Proposition 1 is in the form of (5), where $N = \alpha M$, $D = \frac{s\,d(s)}{n(s)} I$. If constant inputs are applied in the first two components of the input vector u (with zero inputs applied in the last two components), with $D(0) = \frac{s\,d(s)}{n(s)} I|_{s=0} = 0_2$, the input-error transfer-function at $s = 0$ becomes $H_{eu}(0) = \begin{bmatrix} 0_2 & \frac{-1}{a} I_2 \\ 0_2 & I_2\end{bmatrix}$. Therefore, the steady-state error due to constant input references (with zeros in the third and fourth components) goes to zero asymptotically. Hence, C_p in (10) is an integral-action controller. □

B. Linear bicycle model with one input

In this section, it is assumed that the system has only one input, T_δ. The externally applied torque T_ϕ about the line connecting the wheel contact points is zero. Since the bicycle model in (1) presumes to contain a rider rigidly attached to its main frame, only the second input T_δ is available as an actuator input to the plant. Under this assumption, we change the plant description of P_j in Section II-A to define a one-input four-output plant transfer-matrix $G_j \in \mathbf{R_p}^{4\times 1}$ as

$$G_j = P_j \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix} \frac{1}{(s+a)^2} I_2 \\ \frac{s}{(s+a)^2} I_2\end{bmatrix} Y_j^{-1} \begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{(s+a)^2 W_j} \begin{bmatrix} -Y_{bj} \\ Y_a \\ -sY_{bj} \\ sY_a\end{bmatrix} ; \quad (11)$$

$$G_j = \tilde{Y}_j^{-1}\tilde{X}_g = (\begin{bmatrix} Y_j & 0_2 \\ 0_2 & I_2 \end{bmatrix} R)^{-1} \begin{bmatrix} \frac{1}{s+a} I_2 \\ 0_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \quad (12)$$

where $R = \begin{bmatrix} aI_2 & I_2 \\ \frac{-a\,s}{s+a} I_2 & \frac{a}{s+a} I_2 \end{bmatrix} \in \mathbf{S}^{4\times 4}$ is unimodular. Consider a controller $C_g = \begin{bmatrix} aC_1 & aC_2 & C_1 & C_2 \end{bmatrix} \in \mathbf{R_p}^{1\times 4}$,

$$C_g = \begin{bmatrix} aN_1D_1^{-1} & aN_2D_2^{-1} & N_1D_1^{-1} & N_2D_2^{-1} \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a^{-1}D_g & -a^{-1}I \\ 0_2 & I_2 \end{bmatrix}^{-1} , \quad (13)$$

where $N_1, D_1, N_2, D_2 \in \mathbf{S}$, $C_1 = N_1 D_1^{-1}$, $C_2 = N_2 D_2^{-1}$, $D_g := \mathrm{diag}\,[\, D_1 \quad D_2 \,] \in \mathbf{S}^{2\times 2}$. The controller C_g in (13) stabilizes each G_j if and only if

$$\begin{bmatrix} Y_j & 0_2 \\ 0_2 & I_2 \end{bmatrix} R \begin{bmatrix} a^{-1}D_g & a^{-1}I \\ 0_2 & I_2 \end{bmatrix} + \begin{bmatrix} 0 \\ (s+a)^{-1} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} N_1 & N_2 & 0 & 0 \end{bmatrix}$$

is unimodular, which is satisfied if and only if $\begin{bmatrix} Y_a D_1 & Y_{bj} D_2 \\ Y_{cj}D_1 + \frac{1}{s+a}N_1 & Y_{dj}D_2 + \frac{1}{s+a}N_2 \end{bmatrix}$ is unimodular, equivalently, E_j in (14) is a unit in $\mathbf{S}$, i.e., $E_j^{-1} \in \mathbf{S}$, where

$$E_j := W_j D_2 D_1 - (s+a)^{-1} Y_{bj} N_1 D_2 + (s+a)^{-1} Y_a N_2 D_1 , \quad (14)$$

$$E_j^{-1} = (s+a)\,[\,(s+a)\,W_j + Y_a\,C_2 - Y_{bj}C_1]^{-1}\,D_2^{-1}D_1^{-1}.$$

With the plant G_j as in (11), $H_{eu} = (I + G_jC_g)^{-1} = I - G_jC_g(I + G_jC_g)^{-1} = I - H_{yu}$. With G_j given in (11) and C_g given in (13), using (14), $H_{yu} = \begin{bmatrix} \frac{1}{(s+a)^2} I_2 \\ \frac{s}{(s+a)^2} I_2 \end{bmatrix} \begin{bmatrix} -Y_{bj} \\ Y_a \end{bmatrix} E_j^{-1} \begin{bmatrix} aN_1D_2 & aN_2D_1 & N_1D_2 & N_2D_1 \end{bmatrix}$.

Let $\mathcal{G}$ be a finite set of plants, where $G_j \in \mathcal{G}$ is described as in (11), or equivalently (12). The problem of controller design that simultaneously stabilizes finitely many plant models $G_j \in \mathcal{G}$ is more challenging than the simultaneous controller synthesis given in Proposition 1 for the two-input systems P_j. By Proposition 1, there exist controllers that simultaneously stabilize any number of plants P_j. However, simultaneous stabilization of the one-input plants $G_j \in \mathcal{G}$ depends on the speeds v_j in the model (1). Using the numerical values given in [4], Y_{bj} in (2) is

$$Y_{bj} = (s+a)^{-2}[\,2.319s^2 + v_j 33.866 s - g2.599 + v_j^2 76.597\,]. \quad (15)$$

Clearly, for Y_{bj} given in (15), $Y_{bj}^{-1} \in \mathbf{S}$ for $v_j > v_\star$, where

$$v_\star = \sqrt{g2.599/76.597} \approx 0.5769 \text{ m/s}. \quad (16)$$

For the numerical values given in [3], $Y_{bj}^{-1} \in \mathbf{S}$ for all six bicycle models; the values of $v_\star$ for these six bicycle models are $\{\,0.4726, 0.48, 0.4577, 0.4972, 0.4185, 0.7351\,\}$. Let $\mathcal{G}_\star \subset \mathcal{G}$ be the subset of the set of plants $\mathcal{G}$ that contains the plant models G_j modeled at forward speeds $v_j > v_\star$. Any number of models with parameters of [4] and [3] can be combined in the set $\mathcal{G}_\star$ in the speed range $v_j > v_\star$ of each particular model. In Section II-B.1, it is shown that simultaneous stabilization of any number of plants $G_j \in \mathcal{G}_\star$ modeled at $v_j > v_\star$ is achievable using simple, low-order controllers. For speeds $v_j \le v_\star$, $Y_{bj}^{-1} \notin \mathbf{S}$; hence, simultaneous stabilization of plants modeled at these low speeds may or may not be achievable. Although simultaneous stabilization is not resolved for $G_j \in \mathcal{G} \setminus \mathcal{G}_\star$, a controller design procedure for each individual plant G_j modeled at individual speeds $v_j \le v_\star$ is given in Section II-B.2.

1) Simultaneous controllers for normal and high speeds: Any finite number of plants $G_j \in \mathcal{G}_\star$ modeled at speeds $v_j > v_\star$ can be simultaneously stabilized using simple controllers as in Proposition 2. The speed range is determined by the parameters given in [4] as (16) or as in [3].

Proposition 2: (*Simultaneous controller design for $\mathcal{G}_\star$*): Consider finitely many plant models $G_j \in \mathbf{R_p}^{4\times 1}$, described as in (12), with Y_j as in (2). Let $v_j > v_\star$ and hence, $Y_{bj}^{-1} \in \mathbf{S}$. With $M_2 := [1 \;\; 0]\, M \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, define $\Phi \in \mathbb{R}_+$ as

$$\Phi \;:=\; (\,\det M\,)\, M_2^{-1} \;. \quad (17)$$

a) Let $C_1 = -\beta$ for any $\beta \in \mathbb{R}_+$ satisfying

$$\beta \;>\; \max_{Y_j} \|\,(s+a)Y_{bj}^{-1}W_j - s\,\Phi\,\| \;. \quad (18)$$

Let $C_2 = N_2$ for any $N_2 \in \mathbf{S}$ satisfying

$$\|N_2\| < \min_{Y_j} \| \frac{1}{(s+a)} Y_a (W_j + \frac{\beta}{(s+a)} Y_{bj}\,)^{-1} \|^{-1}. \quad (19)$$

With $C_1 = -\beta$, $C_2 = N_2$, a controller $C_g \in \mathbf{R_p}^{1\times 4}$ that strongly stabilizes all G_j is given by

$$C_g = \begin{bmatrix} -a\beta & aN_2 & -\beta & N_2 \end{bmatrix} \;. \quad (20)$$

b) Let $C_1 = -\beta$, where $\beta \in \mathbb{R}_+$ satisfies (18). Choose polynomials $n(s), d(s)$; $n(0) > 0$, $d(s)$ is monic and Hurwitz, $\deg d(s) \ge \{0, (\deg n(s) - 1)\,\}$. Define $\Psi_j \in \mathbb{R}_+$ as

$$\Psi_j \;:=\; \frac{n(0)}{a\,d(0)} (\,W_j(0) + \frac{\beta}{a} Y_{bj}(0)\,)^{-1} \;. \quad (21)$$

Let C_2 be

$$C_2 = \frac{\varepsilon\, n(s)}{s\, d(s)} Y_a(0)^{-1} \;, \quad (22)$$

for any $\varepsilon \in \mathbb{R}_+$ satisfying $\varepsilon <$

$$\min_{Y_j} \| \frac{1}{s} [\, \frac{n}{(s+a)d} Y_a Y_a(0)^{-1} (W_j + \frac{\beta}{s+a} Y_{bj})^{-1} - \Psi_j] \|^{-1}. \quad (23)$$

With $C_1 = -\beta$, and C_2 as in (22), a controller $C_g \in \mathbf{R_p}^{1\times 4}$ that stabilizes all G_j is given by

$$C_g = \begin{bmatrix} -a\beta & \frac{a\varepsilon n(s)}{sd(s)} Y_a(0)^{-1} & -\beta & \frac{\varepsilon n(s)}{sd(s)} Y_a(0)^{-1} \end{bmatrix}. \quad (24)$$

c) Choose any monic Hurwitz polynomials $\tilde{n}(s), \tilde{d}(s)$, where $\deg \tilde{n}(s) \ge 1$, and $\deg \tilde{d}(s) = (\deg \tilde{n}(s) - 1)$. Choose any $\tilde{\beta} \in \mathbb{R}_+$ satisfying

$$\tilde{\beta} \;>\; \max_{Y_j} \|\, s\,(\, \frac{(s+a)\tilde{d}(s)}{\tilde{n}(s)} Y_{bj}^{-1} W_j - \Phi\,)\,\| \;. \quad (25)$$

Let $C_2 = N_2$ for any $N_2 \in \mathbf{S}$ satisfying

$$\|\,N_2\,\| \;<\; \min_{Y_j} \|\, \frac{1}{s+a} Y_a\,(\,W_j + \frac{\tilde{\beta}\,\tilde{n}(s)}{s\,(s+a)\,\tilde{d}(s)} Y_{bj}\,)^{-1}\,\|^{-1} \;. \quad (26)$$

With $C_1 = \dfrac{-\tilde{\beta}\,\tilde{n}(s)}{s\,\tilde{d}(s)}$, and $C_2 = N_2$, a controller $C_g \in \mathbf{R_p}^{1\times 4}$ that stabilizes all G_j is given by

$$C_g = \begin{bmatrix} \frac{-a\,\tilde{\beta}\,\tilde{n}(s)}{s\,\tilde{d}(s)} & a\,N_2 & \frac{-\tilde{\beta}\,\tilde{n}(s)}{s\,\tilde{d}(s)} & N_2 \end{bmatrix} \;. \quad (27)$$

Remarks: 1) The controller C_g in (20) that simultaneously stabilizes all plants $G_j \in \mathcal{G}_\star$ is stable for all choices of $\beta \in \mathbb{R}_+$ satisfying (18) and of $N_2 \in \mathbf{S}$ satisfying (19); therefore, any number of plants in $\mathcal{G}_\star$ are strongly simultaneously stabilizable. If the stable parameter is chosen as a constant that satisfies (19), then C_g becomes a constant controller. There are infinitely many choices for the controller in (20) but to keep the design simple, $N_2 \in \mathbf{S}$ should be chosen as a low-order stable transfer-function. **2)** The controllers C_g in (24) and in (27) have no poles in $\mathcal{U}$ except at $s = 0$. These controllers can be made simple and low-order by choosing low-order polynomials $n, d, \tilde{n}, \tilde{d}$ for the design parameters out of the infinitely many possibilities. If $n(s)$ is a first order polynomial and $d(s) = 1$, then C_g in (24) contains only proportional and PI terms. Similarly, a first order $\tilde{n}(s)$ and $\tilde{d}(s) = 1$ gives proportional and PI terms for C_g in (27). **3)** Let the input-error transfer-function of the error between the first two input and output components be denoted by $H_{\phi\delta}$. By (14), $H_{\phi\delta}$ is $H_{\phi\delta} = \begin{bmatrix} 1 + \frac{a}{(s+a)^2} Y_{bj} E_j^{-1} N_1 D_2 & \frac{a}{(s+a)^2} Y_{bj} E_j^{-1} N_2 D_1 \\ \frac{-a}{(s+a)^2} Y_a E_j^{-1} N_1 D_2 & 1 - \frac{a}{(s+a)^2} Y_a E_j^{-1} N_2 D_1 \end{bmatrix}$. Suppose that constant inputs are applied in the first two components u_1, u_2 of the input vector u (with zero inputs applied in the last two components). In Proposition 2-(b), the controller C_g in (24) has integral-action in $C_2 = N_2 D_2^{-1}$, i.e., $D_2(0) = 0$. In this case, $H_{\phi\delta}(0)$ becomes $H_{\phi\delta}(0) = \begin{bmatrix} (a^{-1} E_j^{-1} Y_a N_2 D_1)(0) & (a^{-1} Y_{bj} E_j^{-1} N_2 D_1)(0) \\ 0 & 0 \end{bmatrix}$. Therefore, the steady-state error in the second output due to constant input references (with zeros in the third and fourth components) goes to zero asymptotically. Hence, C_g in (24) is a partial integral-action controller. In Proposition 2-(c), the controller C_g in (27) has integral-action in $C_1 = N_1 D_1^{-1}$, i.e., $D_1(0) = 0$. In this case, $H_{\phi\delta}(0)$ becomes $H_{\phi\delta}(0) = \begin{bmatrix} 0 & 0 \\ (\frac{-1}{a} Y_a E_j^{-1} N_1 D_2)(0) & (\frac{1}{a} E_j^{-1} Y_{bj} N_1 D_2)(0) \end{bmatrix}$. Therefore, the steady-state error in the first output due to constant input references (with zeros in the third and fourth components) goes to zero asymptotically. Hence, C_g in (27) is a partial integral-action controller. **4)** In Proposition 2-(b) and (c), only one of the controllers C_1 or C_2 is designed to have integral-action. If $C_1 = N_1 D_1^{-1}$, $C_2 = N_2 D_2^{-1}$ have $D_1(0) = D_2(0) = 0$, then $E_j(0) = 0$ by (14), which contradicts $E_j \in \mathbf{S}$ being a unit. Therefore, for stabilizing controllers C_g as in (13), C_1 and C_2 cannot both have integral-action together. □

2) Controllers for individual systems for very low speeds: In Section II-B.1, we proposed a simultaneous stabilization method in the speed range $v_j > v_\star$, based on Y_{bj} being a unit in $\mathbf{S}$. For $v_j \leq v_\star \approx 0.5769$, each Y_{bj} given in (15) has an open right-half plane zero at $\zeta \in \mathbb{R}_+ \cup \{0\}$,

$$Y_{bj} = \hat{Y}_{bj} \frac{(s-\zeta)}{(s+a)} = 2.319 \frac{(s+p_j)}{(s+a)} \frac{(s-\zeta)}{(s+a)}, \qquad (28)$$

where $p_j > 0$ for all forward speeds v_j; hence, $\hat{Y}_{bj}$ is a unit in $\mathbf{S}$. The zero at $\zeta \geq 0$ belongs to one of the four entries of Y_j and is not a transmission-zero of the plant $G_j \in \mathcal{G} \setminus \mathcal{G}_\star$. From the description (12), the only transmission-zero of G_j in the region of instability is at infinity.

Controller design for $G_j \in \mathcal{G}$ is based on finding $N_1, D_1, N_2, D_2 \in \mathbf{S}$ such that E_j in (14) is a unit in $\mathbf{S}$. In Section II-B.1, this design is achieved under the assumption that $Y_{bj}^{-1} \in \mathbf{S}$. In this section, we propose a stabilizing controller design for individual plants G_j under the condition that $Y_{bj}^{-1} \notin \mathbf{S}$. This case implies that the bicycle is moving forward at an extremely slow speed, which makes simultaneous stabilization more challenging. This study does not provide a general result to determine simultaneously stabilizability of models in this speed range.

Proposition 3: (*Controller design for $\mathcal{G} \setminus \mathcal{G}_\star$*): Consider a fixed plant model $G_j \in \mathbf{R_p}^{4\times 1}$ described as in (12) with $v_j \leq v_\star$; hence, Y_{bj} is as in (28). **a)** Let C_1 be

$$C_1 = \hat{\beta}\, Y_{bj}^{-1} [\, W_j(s) - W_j(\zeta) \,] [\, \frac{\hat{\beta}}{s+a} + (\det M)^{-1} W_j(\zeta) \,]^{-1} \qquad (29)$$

for any $\hat{\beta} \in \mathbb{R}_+$ satisfying

$$\hat{\beta} \;>\; \|\, (s+a) W_j(s) (\det M)^{-1} - s\, I \,\| \,. \qquad (30)$$

Let $C_2 = N_2$ for any $N_2 \in \mathbf{S}$ satisfying

$$\|\, N_2 \,\| \;<\; \|\, \frac{1}{(s+a)} Y_a \,(\, W_j - \frac{1}{s+a} Y_{bj} C_1 \,)^{-1} \,\|^{-1} \,. \qquad (31)$$

With C_1 as in (29) and $C_2 = N_2$ satisfying (31), a controller $C_g \in \mathbf{R_p}^{1\times 4}$ that stabilizes the system G_j is given by

$$C_g = \begin{bmatrix} aC_1 & aC_2 & C_1 & C_2 \end{bmatrix} \,. \qquad (32)$$

b) Let C_1 be as in (29) for any $\hat{\beta} \in \mathbb{R}_+$ satisfying (30). Define N_1, D_1 as

$$N_1 = \hat{\beta}\, Y_{bj}^{-1} [\, W_j(s) - W_j(\zeta) \,] \,, \qquad (33)$$

$$D_1 = [\, \frac{\hat{\beta}}{s+a} + (\det M)^{-1} W_j(\zeta) \,] \,. \qquad (34)$$

Choose polynomials $\hat{n}(s), \hat{d}(s)$; $\hat{n}(0) = \hat{d}(0)$, $\hat{d}(s)$ is Hurwitz, $\deg \hat{d}(s) \geq \{\, 0, (\deg \hat{n}(s) - 1) \,\}$. Define $\hat{\Psi}_j \in \mathbb{R}_+$ as

$$\hat{\Psi}_j \;:=\; [\, \frac{\hat{\beta}}{a} + W_j(0) (\det M)^{-1} \,]\, W_j(\zeta) \,. \qquad (35)$$

Let C_2 be

$$C_2 = \frac{\hat{\varepsilon}\, \hat{n}(s)}{s\, \hat{d}(s)} \,\hat{\Psi}_j\, Y_a(0)^{-1} \,, \qquad (36)$$

for any $\hat{\varepsilon} \in \mathbb{R}_+$ satisfying

$$\hat{\varepsilon} \;<\; \|\, \frac{1}{s} [\, \frac{\hat{n}(s)}{(s+a)\hat{d}(s)} Y_a D_1 \hat{U}_j^{-1} \hat{\Psi}_j Y_a(0)^{-1} - 1 \,] \,\|^{-1} \,. \qquad (37)$$

With C_1 as in (29) and C_2 as in (36), a controller $C_g \in \mathbf{R_p}^{1\times 4}$ that stabilizes all G_j is given by (32). □

Remarks: The term C_1 in (29) of the controller C_g in (32) is third order and biproper. It's poles are at $\{-a, -p_j, -b_j\}$; $a > 0$ is the arbitrarily chosen design parameter, $-p_j < 0$ is the negative zero of Y_{bj} as defined in (28), and $b_j = (a + \hat{\beta} \det M / W_j(\zeta))$. Since $W_j(\zeta) = \det Y_j(\zeta) = Y_a(\zeta) Y_{dj}(\zeta)$ may be negative for some forward speeds, b_j may be negative, implying C_1 may have one pole in the unstable region. If a stable controller design is desired, a large enough $a > 0$ can be chosen that ensures a positive value for b_j for all forward speeds v_j. In Proposition 3-(a), C_2 of (32) is always stable; it can be made simple by choosing a constant or low-order N_2. The controller C_g given by (32) in Proposition 3-(b) only adds integral-action to the term C_2, which has poles in the region of stability except for one pole at $s = 0$. This term can be a simple PI controller by choosing a first order $\hat{n}(s)$ and $\hat{d}(s) = 1$. $\square$

III. Application

We apply the proposed controller synthesis procedures of Propositions 1, 2, 3, with the values from [4] as benchmark parameters for the linearized bicycle model in (1):

$$M = \begin{bmatrix} 80.81722 & 2.31941332208709 \\ 2.31941332208709 & 0.29784188199686 \end{bmatrix},$$

$$K_o = \begin{bmatrix} -80.95 & -2.59951685249872 \\ -2.59951685249872 & -0.80329488458618 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0 & 33.86641391492494 \\ -0.85035641456978 & 1.68540397397560 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0 & 76.59734589573222 \\ 0 & 2.65431523794604 \end{bmatrix}.$$

The entries of Y_j in (2) are then calculated as $Y_a = (s+a)^{-2}[\,80.817s^2 - g80.95\,]$, $Y_{bj} = (s+a)^{-2}[\,2.319s^2 + v_j 33.866s - g2.599 + v_j^2 76.597\,]$, $Y_{cj} = (s+a)^{-2}[\,2.319s^2 - v_j 0.850s - g2.599\,]$, $Y_{dj} = (s+a)^{-2}[\,0.297s^2 + v_j 1.685s - g0.803 + v_j^2 2.654\,]$. Propositions 1, 2, 3 present systematic controller design procedures with infinitely many choices for the parameters within the specified constraints. In these numerical examples, we choose these parameters so that the resulting controllers are simple and low order. Within the design freedom, we also choose controllers that result in closed-loop poles that are not too close to the imaginary-axis. The simultaneously stabilizing controller designs apply to any number of plants in the classes $\mathcal{P}$ and $\mathcal{G}_\star$ in Proposition 1 and Proposition 2. We choose the following forward speeds (in meters/second) to illustrate the simultaneous stabilization results: $\mathcal{V}_1 = \{0, 0.05, 0.15, 0.25, 0.4\}$, $\mathcal{V}_2 = \{0.58, 1.5, 2.5, 3.6, 5, 7.5, 8, 10\}$. The set $\mathcal{V}_2$ includes the three speeds considered in [3]. Appropriate modifications are made to the entries of Y_j in (2) if plant models of the six bicycles in [3] are included in the sets $\mathcal{P}$ and $\mathcal{G}_\star$.

Application of Proposition 1: Consider a set of plants $\mathcal{P}$ as in Section II-A; the 13 plants $P_j \in \mathcal{P}$ are modeled at the speeds in the set $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$. Choose $a = 10$. **a)** The norm in (7) grows as the speed in $\mathcal{V}$ increases, and is satisfied for $\alpha > 3314$. Choosing $\alpha = 3500$, the controller C_p in (8) is $C_p = 3500\, M \begin{bmatrix} 10I_2 & I_2 \end{bmatrix}$. The four closed-loop poles of all 13 systems have sufficient damping; the pole closest to the imaginary-axis at -9.961 for the speed $v = 10$ m/s. **b)** For a simple design, choose $n = (s+6)$, $d = 1$. The norm in (9) grows as the speed increases, and is satisfied for $\alpha > 2203$. Choosing $\alpha = 2500$, $C_p = 2500 \frac{(s+6)}{s} M \begin{bmatrix} 10I_2 & I_2 \end{bmatrix}$ in (10). The pole closest to the imaginary-axis of the 13 systems simultaneously stabilized is at -5.919 for $v = 10$ m/s.

Application of Proposition 2: Consider a set of plants $\mathcal{G}_\star$ as in Section II-B.1, where the 8 plants $G_j \in \mathcal{G}_\star$ are modeled at the speeds in the set $\mathcal{V}_2$, where $v_j \in \mathcal{V}_2$ satisfy $v_j > v_\star$. Choose $a = 10$. **a)** The norm in (18) is satisfied for $\beta > 2087.9$; it does not exhibit a pattern for the speeds in $\mathcal{V}_2$. Choosing $\beta = 2088$ and a constant $N_2 = -1.3$ satisfying (19), the simultaneously stabilizing controller in (20) is $C_g = \begin{bmatrix} -20880 & -13 & -2088 & -1.3 \end{bmatrix}$. The pole closest to the imaginary-axis of the 8 systems is at -0.05268 corresponding to the lowest speed $v = 0.58$ m/s in $\mathcal{V}_2$. Excluding this low speed from the set, the pole closest to the imaginary-axis of the remaining 7 systems is at -3.4218 for $v = 1.5$ m/s. **b)** For a low-order design, we choose $n(s) = d(s) = 1$. For $\beta = 2088$, which satisfies (18) as in part (a), the norm (23) is satisfied for $\varepsilon < 0.301$. Choosing $\varepsilon = 0.25$, the simultaneously stabilizing controller in (24) is $C_g = \begin{bmatrix} -20880 & \frac{-250}{794.11s} & -2088 & \frac{-25}{794.11s} \end{bmatrix}$. One of the closed-loop poles is very close to the origin for all 8 systems. Better closed-loop damping may be achieved with higher order choices for n, d. **c)** Choosing $\tilde{n} = (s+1)$, $\tilde{d} = 1$, the norm (25) is satisfied for $\tilde{\beta} > 1008.195$. Choosing $\tilde{\beta} = 1100$ and simply a constant N_2 satisfying (26) as $N_2 = 0.8$, C_g in (27) is $C_g = \begin{bmatrix} \frac{-11000(s+1)}{s} & 8 & \frac{-1100(s+1)}{s} & 0.8 \end{bmatrix}$.

Application of Proposition 3: Let the speeds for the individual models to be stabilized be $v_1 = 0.15$, $v_2 = 0.25$, $v_3 = 0.4$, $v_4 = 0.57$ m/s, which are all less than $v_\star$. Choose $a = 10$. Then $\hat{\beta} = 31$ satisfies (30) for each of these four speeds. For simplicity, choose a constant N_2 satisfying (31). The controllers C_{1j}, C_{2j} in C_{gj} of (32) corresponding to v_j are: $C_{11} = \frac{50320(s-6.25)(s+5.78)(s+3.03)}{(s+10)(s+6289)(s+4.47)}$, $C_{21} = 0.012$; $C_{12} = \frac{24729.9(s-6.82)(s+5.95)(s+2.94)}{(s+10)(s+3110)(s+5.32)}$, $C_{22} = 0.03$; $C_{13} = \frac{13616(s-7.59)(s+6.195)(s+2.78)}{(s+10)(s+1732.7)(s+6.69)}$, $C_{23} = 0.058$; $C_{14} = \frac{8361.7(s-8.32)(s+6.46)(s+2.55)}{(s+10)(s+1079)(s+8.35)}$, $C_{24} = 0.092$. Keep the same C_{1j} for each speed v_j and re-design C_2 as an integral-action controller as in Proposition 3-(b). Choose $\hat{n} = \hat{d} = 1$ for simplicity. Then $\hat{\varepsilon} = 0.125$ satisfies (37) for each of these four speeds. The new C_{2j} for each v_j are $C_{21} = \frac{0.125}{8538.8\,s}$; $C_{22} = \frac{0.125}{4201.6\,s}$; $C_{23} = \frac{0.125}{2243.2\,s}$; $C_{24} = \frac{0.125}{1451.8\,s}$.

IV. Conclusions

Under the assumptions of Propositions 1 and 2, any number of bicycles modeled at different forward speeds can be simultaneously stabilized with either two inputs or with only the steering input. The proposed controllers are simple and low-order, with freedom in the design parameters that can be used to achieve better performance. For extremely low speeds, the design given in Proposition 3 provides stabilization of individual models at fixed forward speeds with only the steering torque as input.

Appendix: Proofs

Proof of Proposition 1: **a)** With $N = \alpha M$, $D = I$, the constant C_p in (8) is as in (5); it stabilizes all P_j if and only if F_j in (6) is unimodular, equivalently, $\frac{1}{(s+a)}\alpha M + Y_j = \frac{(s+\alpha)}{(s+a)}(\frac{\alpha}{s+\alpha} I + \frac{1}{(s+\alpha)}(s+a)Y_j M^{-1})M = \frac{(s+\alpha)}{(s+a)}(I + \frac{1}{s+\alpha}[(s+a)Y_j M^{-1} - sI])M$ is unimodular. By (2), $Y_j(\infty) = M$ implies $[s(Y_j M^{-1} - I)] \in \mathcal{M}(\mathbf{S})$. For α satisfying (7), $\|\frac{1}{s+\alpha}[\,(s+a)Y_j M^{-1} - sI\,]\| \leq \frac{1}{\alpha}\|(s+a)Y_j M^{-1} - sI\| < 1$.

Therefore, (6) is unimodular for all P_j. Since it is stable, C_p in (8) is *strongly* stabilizing. **b)** With $N = \alpha M$, $D = \dfrac{s\,d(s)}{n(s)} I \in \mathcal{M}(\mathbf{S})$ since $n(s)$ is Hurwitz, C_p stabilizes all P_j if and only if F_j in (6) is unimodular, equivalently, $\frac{1}{(s+a)}\alpha M + \frac{s\,d(s)}{n(s)} Y_j = \frac{(s+\alpha)}{(s+a)}(\,I + \frac{1}{s+\alpha}[s(\frac{d(s)(s+a)}{n(s)} Y_j M^{-1} - I)])M$ is unimodular. By (2), $Y_j(\infty) = M$ and $\frac{d(s)(s+a)}{n(s)}|_{s\to\infty} = 1$ imply $[s(\frac{d(s)(s+a)}{n(s)} Y_j M^{-1} - I)] \in \mathcal{M}(\mathbf{S})$. For α satisfying (7), $\|\,\frac{1}{s+\alpha}[s(\,\frac{d(s)(s+a)}{n(s)} Y_j M^{-1} - I)]\,\| \le \frac{1}{\alpha}\,\|\,s(\,\frac{d(s)(s+a)}{n(s)} Y_j M^{-1} - I)\| < 1$. Therefore, (6) is unimodular for all P_j. Hence, C_p in (10) is an integral-action controller that stabilizes all P_j. □

Proof of Proposition 2: **a)** With $N_1 = C_1$, $N_2 = C_2$, $D_1 = D_2 = 1$, the stable C_g in (20) stabilizes all G_j if and only if $E_j = W_j - \frac{(-\beta)}{(s+a)} Y_{bj} + \frac{1}{(s+a)} Y_a N_2 = U_j + \frac{1}{(s+a)} Y_a N_2$ in (14) is a unit in $\mathbf{S}$, where, since $Y_{bj}^{-1} \in \mathbf{S}$, $U_j := W_j + \frac{\beta}{(s+a)} Y_{bj} = \frac{(\Phi\,s+\beta)}{(s+a)} Y_{bj}(1 + \frac{1}{(\Phi\,s+\beta)}[(s+a)Y_{bj}^{-1}W_j - s\Phi])$. By (2), $W_j(\infty) = \det M$ implies $(Y_{bj}^{-1}W_j)(\infty) = \Phi$; hence $[sY_{bj}^{-1}W_j - s\Phi] \in \mathbf{S}$. For the numerical values given, $\Phi > 0$ implies $(\Phi\,s + \beta)^{-1} \in \mathbf{S}$. For β satisfying (18), $\|\frac{1}{(\Phi\,s+\beta)}[(s + a)Y_{bj}^{-1}\,W_j - \Phi s]\| \le \frac{1}{\beta}\|\,(s+a)Y_{bj}^{-1}W_j - \Phi s\| < 1$ implies $U_j^{-1} \in \mathbf{S}$; then $E_j = (1 + \frac{1}{(s+a)} Y_a N_2 U_j^{-1})U_j$, where, for any $N_2 \in \mathbf{S}$ satisfying (19), $\|\frac{1}{(s+a)} Y_a N_2 U_j^{-1}\| \le \|\frac{1}{(s+a)} Y_a U_j^{-1}\|\|N_2\| < 1$ implies $E_j^{-1} \in \mathbf{S}$ for all G_j. Since C_g in (20) is stable, it is a *strongly* stabilizing controller. **b)** With $D_1 = 1$, $N_2 = \frac{\varepsilon n(s)}{(s+e)d(s)} Y_a(0)^{-1}$, $D_2 = \frac{s}{(s+e)}$ for any $e \in \mathbb{R}_+$, C_g stabilizes all G_j if and only if (14) holds, equivalently, $E_j = W_j D_2 + \frac{\beta}{(s+a)} Y_{bj} D_2 + \frac{1}{(s+a)} Y_a N_2 = \frac{s}{(s+e)} U_j + \frac{1}{(s+a)} Y_a \frac{\varepsilon n(s)}{(s+e)d(s)} Y_a(0)^{-1}$ is a unit in $\mathbf{S}$. From part (a), $U_j^{-1} \in \mathbf{S}$, $E_j = (\frac{s}{(s+e)} + \frac{\varepsilon}{(s+a)} Y_a \frac{n(s)}{(s+e)d(s)} Y_a(0)^{-1} U_j^{-1})U_j = \frac{(s+\varepsilon\Psi_j)}{(s+e)}(1 + \frac{\varepsilon s}{(s+\varepsilon Psi_j)}\frac{1}{s}[\frac{n(s)}{(s+a)d(s)} Y_a Y_a(0)^{-1} U_j^{-1} - \Psi_j])U_j$. Now $U_j(\infty) = \det M = \det Y(\infty) > 0$. Since $U_j^{-1} \in \mathbf{S}$, $U_j(s)$ does not change sign for $s \in \mathcal{U}$; hence, $U_j(0) > 0$. By assumption, $n(0)/d(0) > 0$; hence, $\Psi_j > 0$ and $\frac{\varepsilon\,s}{(s+\varepsilon\Psi_j)} \in \mathbf{S}$. Since $[\frac{n(s)}{(s+a)d(s)} Y_a Y_a(0)^{-1} U_j^{-1}]|_{s=0} = \Psi_j$, we have $s^{-1}[\frac{n(s)}{(s+a)d(s)} Y_a Y_a(0)^{-1} U_j^{-1} - \Psi_j] \in \mathbf{S}$ and for ε satisfying (23), $\|\frac{\varepsilon\,s}{(s+\varepsilon\Psi_j)}\frac{1}{s}[\frac{n(s)}{(s+a)d(s)} Y_a Y_a(0)^{-1} U_j^{-1} - \Psi_j]\| \le \|\frac{\varepsilon\,s}{(s+\varepsilon\Psi_j)}\|\|\frac{1}{s}[\frac{n(s)}{(s+a)d(s)} Y_a Y_a(0)^{-1} U_j^{-1} - \Psi_j]\| = \varepsilon\|\frac{1}{s}[\frac{n(s)}{(s+a)d(s)} Y_a Y_a(0)^{-1} U_j^{-1} - \Psi_j]\| < 1$. Therefore, E_j is a unit in $\mathbf{S}$ for all G_j. **c)** With $N_1 = \frac{-\tilde{\beta}\,\tilde{n}(s)}{(s+e)\tilde{d}(s)}$, $D_1 = \frac{s}{(s+e)}$ for any $e \in \mathbb{R}_+$, and $D_2 = 1$, the controller in (27) is in the form of (13). Due to the assumptions, $\frac{\tilde{n}(s)}{(s+e)\tilde{d}(s)}$ is a unit in $\mathbf{S}$. Define $V_j := W_j D_1 - \frac{1}{(s+a)} Y_{bj} N_1 = W_j \frac{s}{(s+e)} + \frac{\tilde{\beta}}{(s+a)} Y_{bj} \frac{\tilde{n}(s)}{(s+e)\tilde{d}(s)} = Y_{bj} \frac{\tilde{n}(s)}{(s+e)\tilde{d}(s)} \frac{(\Phi s+\tilde{\beta})}{(s+a)}(1 + \frac{1}{(\Phi s+\tilde{\beta})}[s\frac{(s+a)\tilde{d}(s)}{\tilde{n}(s)} Y_{bj}^{-1} W_j - s\Phi])$. By (2), $(Y_{bj}^{-1}W_j)(\infty) = \Phi$ implies $[s\frac{(s+a)\tilde{d}(s)}{\tilde{n}(s)} Y_{bj}^{-1} W_j - s\Phi] \in \mathbf{S}$. Since $\Phi > 0$ implies $(\Phi s + \tilde{\beta})^{-1} \in \mathbf{S}$, for $\tilde{\beta}$ satisfying (25), $\|\frac{1}{(\Phi\,s+\tilde{\beta})}[s\frac{(s+a)\tilde{d}(s)}{\tilde{n}(s)} Y_{bj}^{-1} W_j - s\Phi]\| \le \frac{1}{\tilde{\beta}}\|s\frac{(s+a)\tilde{d}(s)}{\tilde{n}(s)} Y_{bj}^{-1} W_j - s\Phi]\| < 1$. Since $\frac{\tilde{n}(s)}{(s+e)\tilde{d}(s)} Y_{bj}^{-1} \in \mathbf{S}$, it follows that $V_j^{-1} \in \mathbf{S}$. The controller C_g stabilizes all G_j if and only if (14) holds, i.e., $E_j = (1 + \frac{s}{(s+e)(s+a)} Y_a N_2 V_j^{-1})V_j = (1 + \frac{1}{(s+a)} Y_a N_2 [W_j + \frac{\tilde{\beta}\tilde{n}(s)}{s(s+a)\tilde{d}(s)} Y_{bj}]^{-1})V_j$ is a unit in $\mathbf{S}$. For $N_2 \in \mathbf{S}$ satisfying (26), $\|\frac{s}{(s+e)(s+a)} Y_a\,N_2\,V_j^{-1}\| \le \|\frac{1}{(s+a)} Y_a [W_j + \frac{\tilde{\beta}\tilde{n}(s)}{s(s+a)\tilde{d}(s)} Y_{bj}]^{-1}\|\|N_2\| < 1$. Therefore, E_j is a unit in $\mathbf{S}$ for all G_j. □

Proof of Proposition 3: **a)** With N_1, D_1 be as in (33)-(34), C_1 in (29) is $C_1 = N_1 D_1^{-1}$, where $Y_{bj}^{-1}[W_j(s) - W_j(\zeta)] \in \mathbf{S}$ since the only $\mathcal{U}$-zero of Y_{bj} is at $s = \zeta$. Therefore, $C_g \in \mathbf{R_p}^{1\times 4}$ stabilizes G_j if and only if $E_j^{-1} \in \mathbf{S}$, equivalently, $E_j = (\,W_j D_1 - \frac{1}{(s+a)} Y_{bj} N_1\,)\,D_2 + \frac{1}{(s+a)} Y_a D_1 N_2 = \hat{U}_j D_2 + \frac{1}{(s+a)} Y_a D_1 N_2$ is a unit in $\mathbf{S}$, where $\hat{U}_j := W_j D_1 - \frac{1}{(s+a)} Y_{bj} N_1 = \frac{(s+\hat{\beta})}{(s+a)}[\,1 + \frac{1}{(s+\hat{\beta})}\,(\,(s+a)W_j(\det M)^{-1} - sI)]W_j(\zeta)$. Since $W_j(\infty) = \det M$, the term $(sW_j(\det M)^{-1} - sI) \in \mathbf{S}$. For $\hat{\beta}$ satisfying (30), $\|\frac{1}{(s+\hat{\beta})}((s+a)W_j(\det M)^{-1} - sI)\| \le \frac{1}{\hat{\beta}}\|(s+a)W_j(\det M)^{-1} - sI\| < 1$ implies $\hat{U}_j^{-1} \in \mathbf{S}$. With $D_2 = 1$, $E_j = (\,1 + \frac{1}{(s+a)} Y_a D_1 N_2 \hat{U}_j^{-1})\hat{U}_j$, where, for any $N_2 \in \mathbf{S}$ satisfying (31), $\|\frac{1}{(s+a)} Y_a D_1 N_2 \hat{U}_j^{-1}\| \le \|\frac{1}{(s+a)} Y_a\,(\,W_j - \frac{1}{s+a} Y_{bj} C_1\,)^{-1}\|\|N_2\| < 1$. Therefore, $E_j^{-1} \in \mathbf{S}$; hence, C_g stabilizes G_j. **b)** Let $N_2 = \frac{\hat{\varepsilon}\hat{n}(s)}{(s+e)\hat{d}(s)} \hat{\Psi} Y_a(0)^{-1}$, $D_2 = \frac{s}{(s+e)}$ for any $e \in \mathbb{R}_+$. From part (a), with N_1, D_1 as in (33)-(34), $\hat{U}_j^{-1} \in \mathbf{S}$ implies $E_j = \hat{U}_j D_2 + \frac{1}{(s+a)} Y_a D_1 N_2 = \frac{s}{(s+e)}\hat{U}_j + \frac{1}{(s+a)} Y_a D_1 \frac{\hat{\varepsilon}\hat{n}(s)}{(s+e)\hat{d}(s)} \hat{\Psi}_j Y_a(0)^{-1} = \frac{(s+\hat{\varepsilon})}{(s+e)}(1 + \frac{\hat{\varepsilon}\,s}{(s+\hat{\varepsilon})}\frac{1}{s}[\frac{\hat{n}(s)}{(s+a)\hat{d}(s)} Y_a D_1 \hat{U}_j^{-1}\hat{\Psi}_j Y_a(0)^{-1} - 1])\hat{U}_j$. By assumption, $\hat{n}(0)/\hat{d}(0) = 1$ and $(\hat{U}_j D_1^{-1})(0) = \Psi_j$. Therefore, $s^{-1}[\frac{\hat{n}(s)}{(s+a)\hat{d}(s)} Y_a D_1 \hat{U}_j^{-1}\hat{\Psi}_j Y_a(0)^{-1} - 1] \in \mathbf{S}$ and for $\hat{\varepsilon}$ satisfying (37), we have $\|\frac{\hat{\varepsilon}\,s}{(s+\hat{\varepsilon})}\frac{1}{s}[\frac{\hat{n}(s)}{(s+a)\hat{d}(s)} Y_a D_1 \hat{U}_j^{-1}\hat{\Psi}_j Y_a(0)^{-1} - 1]\| \le \|\frac{\hat{\varepsilon}\,s}{(s+\hat{\varepsilon})}\|\|\frac{1}{s}[\frac{\hat{n}(s)}{(s+a)\hat{d}(s)} Y_a D_1 \hat{U}_j^{-1}\hat{\Psi}_j Y_a(0)^{-1} - 1]\| = \hat{\varepsilon}\|\frac{1}{s}[\frac{\hat{n}(s)}{(s+a)\hat{d}(s)} Y_a D_1 \hat{U}_j^{-1}\hat{\Psi}_j Y_a(0)^{-1} - 1]\| < 1$. Hence, $E_j^{-1} \in \mathbf{S}$ and C_g in (32) stabilizes G_j. □

REFERENCES

[1] K. J. Åström, R. E. Klein, A. Lennartsson, "Bicycle dynamics and control, adapted bicycles for education and research," *IEEE Control Syst. Mag.* 25, pp. 2647, 2005.

[2] N. H. Getz, J. E. Marsden, "Control for an autonomous bicycle," *Proc. IEEE Conf. on Robotics and Automation*, vol. 2, pp. 1397-1402, 1995.

[3] R. Hess, J. K. Moore, M. Hubbard, "Modeling the manually controlled bicycle," *IEEE Transactions on Systems, Man and Cybernetics, Part A*, vol. 42, no. 3, pp. 545-557, 2012.

[4] J. P. Meijaard, J. M. Papadopoulos, A. Ruina, A. L. Schwab, "Linearized dynamics equations for the balance and steer of a bicycle: A benchmark and review," *Proc. Royal Society A*, pp. 1955-1982, 2007.

[5] M. Nagai, "Analysis of rider and single-track-vehicle system; its application to computer-controlled bicycles," *Automatica*, vol. 19, no. 6, pp. 737-740, 1983.

[6] R. S. Sharp, "On the stability and control of the bicycle," *Trans. ASME Applied Mech. Review*, vol. 61, no. 6, pp. 060803-1–06803-24, 2008.

[7] F. J. W. Whipple, "The stability of the motion of a bicycle," *Quart. J. Pure Appl. Math.*, vol. 30, no. 120, pp. 312348, 1899.