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SIMULTANEOUS STABILIZATION AND CONSTANT REFERENCE TRACKING OF LTI, MIMO SYSTEMS

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ABSTRACT

It is shown that any finite number of plants that belong to certain classes of multi-input multi-output systems with no zeros in the region of instability can be simultaneously stabilized using linear, time-invariant integral-action controllers. These plants may be stable or unstable and their poles are not restricted; they may also have any number of zeros in the stable region of the complex plane. The classes of systems under consideration include plants with blocking or transmission zeros at infinity. The common controller achieves asymptotic tracking of step-input references with zero steady-state error and has a low order transfer-function. Systematic synthesis methods are presented, and a parametrization of all simultaneously stabilizing controllers with integral-action is also provided.

Key Words: Simultaneous stabilization, integral action, asymptotic tracking.

I. INTRODUCTION

Simultaneous stabilization of a finite family of (three or more) plants using linear, time-invariant (LTI) controllers is a challenging and important control problem. The issue of simultaneous stabilization of a set of models arises in various applications. Linearization of nonlinear process models at various operating points may necessitate the design of a common controller that stabilizes the linear system at any of the operating points. Maintaining stability under sensor or actuator failures for reliable operation also leads to a finite number of distinct dynamic models corresponding to failure modes to be controlled using the same controller [14]. While the robust control problem deals with controller design in the face of an infinite number of plant models representing perturbations all within a neighborhood of a nominal model, the distinct plants considered in simultaneous control need not be ‘close’ to a nominal model.

Simultaneous stabilization is a hard open problem in linear systems theory. Conditions for existence of simultaneously stabilizing controllers have been explored extensively [15, 4]. The well established result that the simultaneous stabilization of n plants is equivalent to strong stabilization of $(n-1)$ plants leads to explicit conditions for the existence of simultaneously stabilizing controllers for $n = 2$: Two plants are simultaneously stabilizable if and only if a related system (which can be derived from these two plants) is strongly stabilizable, *i.e.*, can be stabilized using a stable controller. Based on another well-known result, strong stabilizability of this single related system can be checked via the parity interlacing property of the positive real poles and (blocking) zeros [15, 4]. However, for simultaneous stabilizability of three or more plants, there are no necessary and sufficient conditions available in the most general case without any constraints on the set of plants to be simultaneously stabilized (*e.g.*, [2–5]). Alternative strategies such as time-varying or sampled-data controllers have also been developed to overcome the limitations of LTI controllers (*e.g.*, [12]). Simultaneous stabilization of nonlinear single input systems has also been investigated (*e.g.*, [16]).

The problem considered in this work is the simultaneous stabilization of a finite set of LTI, multi-input multi-output (MIMO) plants using linear, time-invariant output-feedback controllers. Single-input single-output (SISO) plants are also included as a special case. The results here deal with the problem using only time-invariant controllers. The synthesis methods result in low order controllers, which are desirable to avoid complexity issues for computation and implementation. An additional design goal here is the asymptotic tracking of constant reference inputs. Based on the well-known internal model principle [6], asymptotic tracking is achieved by duplicating the dynamic structure of the exogenous signals that the regulator has to process. Therefore, the controllers are designed to have integral-action by adding poles at the origin for the constant reference signal tracking objective. We explore sufficient conditions for simultaneous stabilizability of three or more plants, since explicit existence conditions for the completely general case of three or more arbitrary plants are not possible to obtain [4]. Sufficient conditions, such as those explored in [5] for three plants, lead to

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identifying classes of practically relevant plants for which simultaneous stabilization is achievable.

An interesting class of SISO plants was considered in [1], where it was shown that minimum-phase, strictly proper scalar plants, that all have the same high-frequency gain sign, can be simultaneously stabilized by stable and strictly proper controllers; this result is restricted to SISO plants only, and the stabilizing controllers do not provide integral-action. While all plants in [1] are strictly-proper, the design proposed here includes finite sets of plants that contain both relative degree one and relative degree zero plants; furthermore, it achieves stability and integral-action for asymptotic tracking of constant reference inputs. Also for only SISO plants, an algorithm for simultaneous stabilization of up to four plants in groups was given in [11]. Conditions for simultaneous stabilization of plants with restrictions on state-space and input-space dimensions were obtained in [7]. For stable MIMO plants, a simultaneous controller synthesis with integral-action was presented in [8]; the design is based on sufficient conditions for existence of proportional–integral–derivative (PID) controllers and applies only to stable plants. The plants considered here need not be stable and the controllers are not restricted to be PID. Existence of simultaneous controllers for MIMO plants was explored in [13] under much more conservative sufficient conditions on the plant class. The plant class in [13] does not allow plants without blocking-zeros at infinity, and it does not consider transmission zeros at infinity. Furthermore, explicit construction of the controller parameters are not provided for the more restricted plant class in [13]. Preliminary results on simultaneous controller synthesis methods that apply to MIMO/SISO unstable plants were given in [9]; the results in the present paper expand and improve these synthesis methods.

The main results of this work are the developments of simultaneous integral-action controller synthesis methods that apply to unstable as well as stable MIMO or SISO plants, which have no zeros in the open right-half complex plane. The poles of the plants are completely unrestricted; they may be anywhere in the complex plane. There may also be any number of (blocking or transmission) zeros in the region of stability (open left-half plane), and there may be zeros at infinity as follows: The plant classes considered in Section 3.1 have blocking-zeros at infinity but otherwise have no right-half plane zeros. Theorem 1 gives a sufficient condition for simultaneous stabilizability of such plants based on their high frequency gain matrices. For SISO plants, this condition becomes necessary and sufficient for existence of integral-action controllers that achieve simultaneous stabilization. Proposition 1 develops a synthesis method for simultaneously stabilizing integral-action controllers whose transfer-functions are the same order as the number of blocking zeros at infinity. Proposition 2 in Section 3.2 extends the synthesis to plants that may have other transmission zeros at infinity in addition to the blocking zeros that factor out of every entry. The simultaneous stabilizers have integral-action and hence, achieve asymptotic tracking (and equivalently output disturbance rejection) of constant reference inputs with zero steady-state error in addition to closed-loop stability. The synthesis approaches are illustrated with numerical examples. Although we discuss continuous-time systems here, all results also apply to discrete-time systems with appropriate modifications.

Notation. The region of instability $\mathcal{U}$ is the extended closed right-half plane, *i.e.*, $\mathcal{U} = \mathbb{C}_+ \bigcup \{\infty\} = \{s \in \mathbb{C} \mid \mathcal{R}e(s) \geq 0\} \bigcup \{\infty\}$. Real and positive real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$. The set of matrices with entries in $\mathbf{S}$ is denoted by $\mathcal{M}(\mathbf{S})$; $M \in \mathcal{M}(\mathbf{S})$ is called unimodular if $M^{-1} \in \mathcal{M}(\mathbf{S})$. The $m \times m$ identity matrix is I_m; we use I when the dimension is unambiguous. The H_∞-norm of $M(s) \in \mathcal{M}(\mathbf{S})$ is denoted by $\|M(s)\|$, *i.e.*, the norm $\|\cdot\|$ is defined as $\|M\| := \sup_{s \in \partial U} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$. We use δn to denote the degree of the polynomial n. For simplicity, we drop (s) in transfer matrices such as $G(s)$ where this causes no confusion. We use coprime factorizations over $\mathbf{S}$; *i.e.*, for $G \in \mathbf{R}_\mathbf{p}^{m\times m}$, $C \in \mathbf{R}_\mathbf{p}^{m\times m}$, $G = Y^{-1}X$ denotes a left-coprime-factorization (LCF), $C = ND^{-1}$ denotes a right-coprime-factorization (RCF), where X, Y, N, $D \in \mathbf{S}^{m\times m}$, and $\det Y(\infty) \neq 0$, $\det D(\infty) \neq 0$. Let $\mathrm{rank}G(s) = r \leq m$; then $z \in \mathcal{U}$ is a transmission-zero of G if $\mathrm{rank}X(z) < r$ and it is a blocking-zero of G if $X(z) = 0$. We refer to poles and zeros in the region of instability $\mathcal{U}$ as $\mathcal{U}$-poles and $\mathcal{U}$-zeros.

II. PROBLEM DESCRIPTION

Consider the standard LTI, MIMO unity-feedback system $Sys(G, C)$ shown in Fig. 1, where $G \in \mathbf{R}_\mathbf{p}^{m\times m}$, and $C \in \mathbf{R}_\mathbf{p}^{m\times m}$ denote the plant's and the controller's transfer-functions, and rank $G = m$. The objective is to design a simple simultaneously stabilizing controller C that achieves asymptotic tracking of step-input references with zero steady-state error for a finite set of plants.

Let $G = Y^{-1}X$ be an LCF and $C = ND^{-1}$ be an RCF. Let H_{eu} denote the (input-error) transfer-function from u to e, and

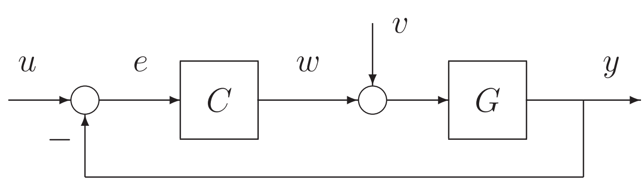

Fig. 1. Unity-Feedback System $Sys(G, C)$.

let H_{yu} denote the (input-output) transfer-function from u to y; then $H_{eu} = (I + GC)^{-1} = I - GC(I + GC)^{-1} = I - H_{yu}$.

Definition 1. (i) The system $Sys(G, C)$ is stable and has integral-action if the closed-loop transfer-function from (u, v) to (y, w) is stable, and the (input-error) transfer-function H_{eu} has blocking-zeros at $s = 0$. (ii) The controller C is said to stabilize G if C is proper and the system $Sys(G, C)$ is stable. (iii) The controller C is said to be an integral-action controller if C stabilizes G and $D(0) = 0$ for any RCF $C = ND^{-1}$. □

The controller C stabilizes $G \in \mathcal{M}(\mathbf{R_p})$ if and only if

$$M := YD + XN \tag{1}$$

is unimodular [15]. Suppose that the system $Sys(G, C)$ is stable and that step input references are applied at $u(t)$. The steady-state error $e(t)$ due to all step input vectors at $u(t)$ goes to zero as $t \to \infty$ if and only if $H_{eu}(0) = 0$. By Definition 1, the stable system $Sys(G, C)$ achieves asymptotic tracking of constant reference inputs with zero steady-state error if and only if it has integral-action. By (1), write $H_{eu} = (I + GC)^{-1} = DM^{-1}Y$. Then by Definition 1, $Sys(G, C)$ has integral-action if $C = ND^{-1}$ is an integral-action controller since $D(0) = 0$ implies $H_{eu}(0) = (DM^{-1}Y)(0) = 0$.

III. SIMULTANEOUS CONTROLLER SYNTHESIS

It is assumed throughout that plants to be simultaneously stabilized using integral-action controllers have no transmission-zeros at $s = 0$ since this condition is necessary for the existence of integral-action controllers. For all plant classes considered here, the transmission or blocking-zeros in the region of stability $\mathbb{C} \setminus \mathcal{U}$ are completely unrestricted.

3.1 Plants with blocking zeros at infinity

Let $\mathcal{G}$ denote a finite set of MIMO plants that all have exactly r blocking-zeros at infinity, where $r \geq 1$, but no other $\mathcal{U}$-zeros. Let $\mathcal{P}$ denote a finite set of MIMO plants that have no blocking-zeros at infinity or elsewhere in $\mathcal{U}$. There may be any number of plants in these sets; some plants may be stable and some unstable. These plants have no other (transmission and blocking) $\mathcal{U}$-zeros; they may have any number of (transmission and blocking) zeros anywhere in the stable region $\mathbb{C} \setminus \mathcal{U}$. There are no restrictions on the poles; they may be anywhere in $\mathbb{C}$. In the SISO case, the relative degree of the plants in $\mathcal{G}$ is r, and of those in $\mathcal{P}$ is 0. In the general MIMO case, the plants $G_j \in \mathcal{G}$ that have r blocking-zeros at infinity can be expressed as

$$G_j = Y_j^{-1} X = \left[\frac{1}{(s+a)^r} G_j^{-1} \right]^{-1} \left[\frac{1}{(s+a)^r} I \right]; \tag{2}$$

$Y_j = \frac{1}{(s+a)^r} G_j^{-1} \in \mathcal{M}(\mathbf{S})$, and $X = \frac{1}{(s+a)^r} I$ for each G_j, for any $a \in \mathbb{R}_+$. Since the plants $P_j \in \mathcal{P}$ have no transmission or blocking-zeros in $\mathcal{U}$ (not even at infinity), their inverses are stable, *i.e.*, $P_j^{-1} \in \mathcal{M}(\mathbf{S})$. Define

$$Y_j(\infty) := \left(\frac{1}{s^r} G_j(s)^{-1} \right)\Big|_{s\to\infty}; \quad Y_j(\infty)^{-1} = (s^r G_j(s))\big|_{s\to\infty}. \tag{3}$$

Designate an arbitrary member $G_o \in \mathcal{G}$ as the nominal plant. By (3), $Y_o(\infty)^{-1} = (s^r G_o(s))|_{s\to\infty}$. For $G_j \in \mathcal{G}$, define F_j as

$$F_j := Y_j(\infty) Y_o(\infty)^{-1} = (G_j^{-1} G_o)(\infty). \tag{4}$$

Lemma 1. (Necessary existence condition for simultaneous integral-action controllers**).** Let G_o be an arbitrary member of the class $\mathcal{G}$. If all plants in $\mathcal{G}$ can be simultaneously stabilized using an integral-action controller, then for all $G_j \in \mathcal{G}$,

$$\det F_j = \det(G_j^{-1} G_o)(\infty) > 0. \tag{5}$$

Theorem 1. (Sufficient existence condition for simultaneous integral-action controllers**).** Let the constant matrices $F_j = (G_j^{-1} G_o)(\infty)$ be positive definite for all $G_j \in \mathcal{G}$. Under this assumption,

(i) all plants in $\mathcal{G}$ can be simultaneously stabilized using an integral-action controller;
(ii) if $r = 1$, then all plants in $\mathcal{G} \cup \mathcal{P}$ can be simultaneously stabilized using an integral-action controller. □

The sufficient condition for the existence of simultaneous integral-action controllers obviously holds if the class contains plants with no $\mathcal{U}$-zeros, with the exception of one strictly-proper plant $G_o \in \mathcal{G}$ since $F_o = (G_o^{-1} G_o)(\infty) = I$. We formally state this case in the following corollary:

Corollary 1. If $r = 1$ and $\mathcal{G}$ contains at most one plant $G_o \in \mathcal{G}$, then all plants $\mathcal{P} \cup \{G_o\}$ can be simultaneously stabilized using an integral-action controller. □

For SISO plants, the sufficient condition in Theorem 1 is equivalent to F_j being positive, *i.e.*, the strictly-proper plants having the same high frequency gain sign. It follows from Lemma 1 that this condition is necessary and sufficient for the existence of integral-action controllers that simultaneously stabilize all SISO plants in the class $\mathcal{G}$. This important conclusion is also stated formally as a corollary.

Corollary 2. In the SISO case, where $m = 1$, all SISO plants $G \in \mathbf{R_p}$ in $\mathcal{G}$ can be simultaneously stabilized using an integral-action controller *if and only if* $F_j > 0$. □

Remarks 1. (i) The necessary condition (5) in Lemma 1 is only for those plants that have a blocking-zero at infinity, *i.e.*, it is not required for $P_j \in \mathcal{P}$. For example, let $G_o = \frac{-1}{s-1}$, $P_1 = \frac{s+1}{s-1}$ and $P_2 = -2P_1$. Although $(sP^{-1}G_o)(\infty) = -1 \not> 0$, and $(P_1^{-1}P_2)(\infty) = -2 \not> 0$, the simple integral-action controller $C = \frac{-4(s+2)}{s}$ stabilizes every plant in the set $\{G_o, P_1, P_2\}$.
(ii) Condition (5) is not necessary for the existence of simultaneously stabilizing controllers for the plants in $\mathcal{G}$ unless integral-action is required. For example, $G_o = \frac{-1}{s-1}$ and $G_1 = \frac{1}{s+4}$, which violate (5), are simultaneously stabilizable by the constant $C = -2$, but by Lemma 1, they are not simultaneously stabilizable using any integral-action controllers.
(iii) By Corollary 1, any finite set of plants that have no $\mathcal{U}$-zeros (including infinity), with the exception of one plant with any number of blocking-zeros at infinity, are simultaneously stabilizable using an integral-action controller with no additional conditions on these plants.

Proposition 1 provides a synthesis procedure that explicitly constructs simultaneously stabilizing integral-action controllers for the set $\mathcal{G}$ (and for the set $\mathcal{G} \cup \mathcal{P}$ when $r = 1$) of MIMO plants. This construction is the proof of existence under the sufficient condition of Theorem 1. We assume that the constant matrices $F_j = (G_j^{-1}G_o)(\infty)$ are positive definite for all $G_j \in \mathcal{G}$.

Proposition 1 (Simultaneous integral-action controller synthesis). Let $\mathcal{G}$ be a finite set of MIMO plants that have r blocking-zeros at infinity, but no other $\mathcal{U}$-zeros. Let $\mathcal{P}$ be a finite set of MIMO plants that have no $\mathcal{U}$-zeros.
(i) Suppose that $r \geq 1$. Choose an arbitrary plant $G_o \in \mathcal{G}$, with $Y_o(\infty)^{-1} = (s^r G_o(s))|_{s\to\infty}$. Let $g \in \mathbb{R}_+$. Let $\varphi(s)$ be any monic $(r-1)$-th order Hurwitz polynomial (*i.e.*, the roots of $\varphi(s)$ are all in $\mathbb{C}\setminus\mathcal{U}$); if $r = 1$, then $\varphi(s) = 1$. If all $F_j = (G_j^{-1}G_o)(\infty)$ defined as in (4) are (symmetric) positive definite for all $G_j \in \mathcal{G}$, define Ψ_j as

$$\Psi_j := s\left[\frac{1}{(s+g)\varphi(s)} G_j^{-1}(s)Y_o(\infty)^{-1} - F_j\right]. \tag{6}$$

If $r > 1$, define Φ_j as

$$\Phi_j := s\left[\left(I + \frac{s}{\alpha(s+g)\varphi(s)} G_j^{-1}(s)Y_o(\infty)^{-1}\right)^{-1} \left(\frac{g}{\alpha(s+g)\varphi(s)} G_j^{-1}(s)Y_o(\infty)^{-1} - I\right)\right]. \tag{7}$$

Let

$$C_r = \alpha\beta^{r-1} \frac{(s+g)\varphi(s)}{\beta^{r-1}s + (s+g)[(s+\beta)^{r-1} - \beta^{r-1}]} Y_o(\infty). \tag{8}$$

Under these assumptions,
(a) C_r is an integral-action controller that simultaneously stabilizes all plants $G_j \in \mathcal{G}$ for $\alpha, \beta \in \mathbb{R}_+$ satisfying

$$\alpha > \max_{G_j \in \mathcal{G}} \|\Psi_j\|, \tag{9}$$

$$\beta > (r-1) \max_{G_j \in \mathcal{G}} \|\Phi_j\|. \tag{10}$$

(b) If $r = 1$, then C_r in (8) becomes

$$C_1 = \alpha \frac{(s+g)}{s} Y_o(\infty). \tag{11}$$

Then C_1 is an integral-action controller that simultaneously stabilizes all plants $G_j \in \mathcal{G}$ for $\alpha \in \mathbb{R}_+$ satisfying (9); C_1 is an integral-action controller that simultaneously stabilizes all plants G_j, P_j in $\mathcal{G} \cup \mathcal{P}$ for $\alpha \in \mathbb{R}_+$ satisfying (9) and

$$\alpha > \max_{P_j \in \mathcal{P}} \left\| \frac{s}{s+g} P_j^{-1}(s)Y_o(\infty)^{-1} \right\|. \tag{12}$$

(ii) Suppose that $r = 0$, *i.e.*, $\mathcal{G} = \emptyset$. Let $K \in \mathbb{R}^{m\times m}$ be any nonsingular matrix, and $g \in \mathbb{R}_+$. Let

$$C_0 = \alpha \frac{(s+g)}{s} K. \tag{13}$$

Then C_0 is an integral-action controller that simultaneously stabilizes all plants $P_j \in \mathcal{P}$ for $\alpha \in \mathbb{R}_+$ satisfying

$$\alpha > \max_{P_j \in \mathcal{P}} \left\| \frac{s}{s+g} P_j^{-1}(s)K^{-1} \right\|. \tag{14}$$

□

Remarks 2. The controller C_r in (8) that simultaneously stabilizes the plants in the set $\mathcal{G}$, and the controller C_1 in (11) that simultaneously stabilizes the plants in the set $\mathcal{G} \cup \mathcal{P}$ when $r = 1$, are simple and low order integral-action controllers, with r poles in each non-zero entry of the controller transfer matrix. The transfer-function (matrix) of C_r is bi-proper and has a stable inverse since $g > 0$ and $\varphi(s)$

is a strictly Hurwitz polynomial. The r poles of C_r that appear in every non-zero entry are the roots of $d(s) := (s+g)[(s+\beta)^{r-1} - \beta^{r-1}] + \beta^{r-1}s$. Clearly, one pole is at $s = 0$ (provides integral-action); the remaining $(r-1)$ poles are all in the open left-half plane $\mathbb{C}\setminus\mathcal{U}$. The zeros of C_r can be chosen completely arbitrarily in the open left-half complex plane since the choice of $g > 0$ and $\varphi(s)$ is free.

Example 1. Consider the following plants $G_j \in \mathcal{G}$, with $r = 4$: $G_o = \dfrac{k_o}{(s-p_1)(s-p_2)(s^2+p_3^2)}$, $G_j = \dfrac{k_j(s+z_j)^{v_j}}{(s-p_1)(s-p_2)(s^2+p_3^2)(s-h_j)^{v_j}}$. Suppose that $k_o = k_1 = 3$, $k_2 = 1$, $k_3 = 2.5$, $k_4 = 2$; $p_1 = 1$, $p_2 = 2$, $p_3 = 3$; $z_1 = 4$, $z_2 = 5$, $z_3 = 1$, $z_4 = 6$; $h_1 = 5$, $h_2 = -7$, $h_3 = 0.5$, $h_4 = -4$; $v_1 = 1$, $v_2 = 3$, $v_3 = 4$, $v_4 = 1$. With $Y_o(\infty)^{-1} = k_o = 3$, $F_j = k_o/k_j > 0$, design a fourth order controller for the five plants in $\mathcal{G}$ following Proposition 1a. Let $g = 3$ and $\varphi(s) = (s+2)^3$. By (9), we choose $\alpha = 25 > \max\{12, 21, 23.915, 21.60, 21\}$ and $\beta = 120 > 3\ \max\{22, 37.4037, 20.2744, 31.7641, 22.7445\}$ and obtain the controller $C_r = \dfrac{144\times 10^5(s+3)(s+2)^3}{s(s+152.6)(s^2+210.4s+12170)}$ as in (8), which has integral-action due to the pole at $s = 0$; C_r has three other poles in the open left-half plane. The plant with the highest order in the set $\mathcal{G}$ is G_3 (with an order of 8); the order of the designed controller is only $r = 4$.

3.2 Plants with transmission zeros at infinity

In this section, we consider the set $\mathcal{G}^t$ of MIMO plants with transmission-zeros at infinity that may not appear in every entry of the transfer-matrix with the same multiplicity. Let $G_j \in \mathcal{G}^t \subset R_p^{m\times m}$ have an LCF $G_j = Y_j^{-1}X$ such that $\operatorname{rank}X(\infty) < m$ but $\operatorname{rank}X(s) = m$ for all other $s \in \mathbb{C}_+$. Write

$$X = \frac{1}{(s+a)^r}X_t, \tag{15}$$

where $a \in \mathbb{R}_+$, $r \geq 0$ is the number of blocking-zeros at infinity for each $G_j \in \mathcal{G}^t$, and $\operatorname{rank}X_t(\infty) < m$ but $X_t(\infty) \neq 0$; *i.e.*, X_t book-keeps the transmission-zeros at infinity that $G_j \in \mathcal{G}^t$ may have in addition to the r blocking-zeros at infinity. With $n_{k\ell}$ and $d_{k\ell}$ as polynomials, write

$$X_t^{-1} = \left[\frac{n_{k\ell}(s)}{d_{k\ell}(s)}\right]_{k,\ell\in\{1,\dots,m\}}. \tag{16}$$

Since each G_j has no $\mathcal{U}$-zeros other than at infinity, X_t^{-1} has no poles in the closed right-half complex plane $\mathbb{C}_+$ (*i.e.*, the polynomials $d_{k\ell}$ are Hurwitz) but may have poles at infinity. Define the integers $\rho_{k\ell}$ as

$$\rho_{k\ell} := \delta n_{k\ell} - \delta d_{k\ell},\quad \text{if } \delta n_{k\ell} > \delta d_{k\ell};\quad \rho_{k\ell} := 0,\quad \text{if } \delta n_{k\ell} \leq \delta d_{k\ell}, \tag{17}$$

and for $\ell = 1, \dots, m$, define ρ_ℓ and r_ℓ as

$$\rho_\ell := \max_{1\leq k\leq m}\rho_{k\ell},\quad r_\ell := \begin{cases} r+\rho_\ell, & \text{if } r+\rho_\ell > 0 \\ 1, & \text{if } r+\rho_\ell = 0 \end{cases}. \tag{18}$$

Let $a \in \mathbb{R}_+$; for $\ell = 1, \dots, m$, define $\lambda_\ell := (s+a)^{\rho_\ell}$ and

$$\begin{aligned}\Lambda &:= \operatorname{diag}\begin{bmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_m\end{bmatrix} \\ &= \operatorname{diag}\begin{bmatrix}(s+a)^{\rho_1} & (s+a)^{\rho_2} & \cdots & (s+a)^{\rho_m}\end{bmatrix}.\end{aligned} \tag{19}$$

Although X_t^{-1} may be improper, $X_t^{-1}\Lambda^{-1}$ is stable since $\dfrac{n_{k\ell}}{d_{k\ell}(s+a)^{\rho_\ell}} \in S$. Define $Y_j(\infty)$ as

$$Y_j(\infty) := (X(s)G_j(s)^{-1})\big|_{s\to\infty} = \left(\frac{1}{s^r}X_t(s)G_j(s)^{-1}\right)\Bigg|_{s\to\infty}, \tag{20}$$

i.e., $Y_j(\infty)^{-1} = (s^rG_j(s)X_t^{-1}(s))\big|_{s\to\infty}$. Designate an arbitrary member $G_o \in \mathcal{G}^t$ as the nominal plant. By (20), $Y_o(\infty)^{-1} = \left(s^rG_o(s)X_t^{-1}(s)\right)\big|_{s\to\infty}$. Define F_j as

$$F_j := Y_j(\infty)Y_o(\infty)^{-1} = (X_tG_j^{-1}G_oX_t^{-1})(\infty). \tag{21}$$

We assume that the constant matrices F_j are positive definite for all $G_j \in \mathcal{G}^t$.

Proposition 2 (Simultaneous integral-action controller synthesis for plants with transmission-zeros at infinity)**.** Consider the finite set $\mathcal{G}^t$ of MIMO plants. Choose any arbitrary plant $G_o \in \mathcal{G}^t$, with $Y_o(\infty)^{-1} = \left(s^rG_o(s)X_t^{-1}(s)\right)\big|_{s\to\infty}$. For $\ell = 1, \dots, m$, let $\varphi_\ell(s)$ be any monic $(r_\ell - 1)$-th order Hurwitz polynomial (*i.e.*, $\varphi_\ell(s)$ has $(r_\ell - 1)$ roots in $\mathbb{C}\setminus\mathcal{U}$); $\varphi_\ell(s) = 1$ if $r_\ell = 1$. For $\ell = 1, \dots, m$, define φ_ℓ and χ_ℓ as

$$\phi_\ell(s) := \begin{cases}(s+g)\varphi_\ell(s), & \text{if } r+\rho_\ell > 0 \\ 1, & \text{if } r+\rho_\ell = 0\end{cases}, \tag{22}$$

$$\chi_\ell(s) := (s+\beta)^{r_\ell-1} - \beta^{r_\ell-1}. \tag{23}$$

If $F_j = Y_j(\infty)Y_o(\infty)^{-1}$ defined as in (21) are positive definite for all $G_j \in \mathcal{G}^t$, define Ψ_{tj}, and Φ_{tj} as

$$\begin{aligned}\Psi_{tj} := s\Bigg(&Y_jY_o(\infty)^{-1}\operatorname{diag}\left[\frac{1}{\phi_\ell(s)}\right]_{\ell=1}^m \\ &\operatorname{diag}\left[\frac{(s+a)^{r_\ell}}{(s+a)^{r_\ell-(r+\rho_\ell)}}\right]_{\ell=1}^m - F_j\Bigg),\end{aligned} \tag{24}$$

$$\Phi_{tj}(s) := s\left[(s+a)(F_j s + \alpha I + \Psi_{tj})^{-1} Y_j Y_o(\infty)^{-1} \operatorname{diag}\left[\frac{(s+a)^{r_\ell - 1}}{\varphi_\ell(s)}\right]_{\ell=1}^{m} - I\right]. \quad (25)$$

Let

$$C_t = \alpha X_t^{-1} \Lambda^{-1} \operatorname{diag}\left[\frac{\beta^{r_\ell - 1}\varphi_\ell(s)}{\beta^{r_\ell - 1}s + (s+g)\chi_\ell(s)}\right]_{\ell=1}^{m} Y_o(\infty). \quad (26)$$

Under these assumptions, C_t is an integral-action controller that simultaneously stabilizes all plants $G_j \in \mathcal{G}^t$ for $\alpha, \beta \in \mathbb{R}_+$ satisfying

$$\alpha > \max_{G_j \in \mathcal{G}^t} \|\Psi_{tj}(s)\|, \quad (27)$$

$$\beta > \max_{1 \le \ell \le m}(r_\ell - 1) \max_{G_j \in \mathcal{G}^t} \|\Phi_{tj}(s)\|. \quad (28)$$

Remarks 3. (i) (Parametrization of all simultaneously stabilizing integral-action controllers). The integral-action controllers shown in Propositions 1 and 2, which simultaneously stabilize the plants in $\mathcal{G}$ (or $\mathcal{G} \cup \mathcal{P}$ when $r = 1$), or $\mathcal{G}^t$ are low order controllers (with r poles in each non-zero entry of the controller transfer matrix C_r and r_ℓ poles in each non-zero entry of the controller transfer matrix C_t). Although the synthesis methods offer flexibility in the choice of parameters, the fact that the order is low restricts achievable design objectives. Once the existence of simultaneous integral-action controllers is established through the proposed controllers, a parametrization of other simultaneous integral-action stabilizers without order restrictions can be obtained as follows: Under the assumptions of Proposition 1, suppose that C_r is the integral-action controller in (8) or in (11) if $r = 1$. Let $G_o \in \mathcal{G}$ be any member of the set chosen as the nominal plant. Then all integral-action controllers simultaneously stabilizing the plants $G_j = Y_j^{-1}X \in \mathcal{G}$ are

$$C = \left(C_r + \frac{1}{(s+a)^r} G_o^{-1} Q C_r\right)\left(I - \frac{1}{(s+a)^r} Q C_r\right)^{-1}, \quad (29)$$

where $Q \in \mathcal{M}(\mathbf{S})$ is such that

$$\left[I + \frac{1}{(s+a)^r}(I + C_r G_j)^{-1} C_r (G_j G_o^{-1} - I) Q\right] \text{ is unimodular for all } G_j \in \mathcal{G}. \quad (30)$$

As a sufficient condition to satisfy this unimodularity condition in (30), $Q \in \mathcal{M}(\mathbf{S})$ can be chosen such that

$$\|Q\| < \left\|(I + C_r G_j)^{-1} C_r (G_j G_o^{-1} - I)\right\|^{-1}.$$

The simultaneously stabilizing controllers in (29) have integral-action if and only if $Q(0) = 0$. Although the controller in (8) has transfer-function matrix with r-th order entries, the order of the controllers in (29) are unrestricted. The parametrization in (29) can be used to select controllers to achieve other design objectives that may not be achievable with the order restriction of C_r. Similarly, under the assumptions of Proposition 2, suppose that C_t is the integral-action controller in (26). Let $G_o \in \mathcal{G}^t$ be any member of the set chosen as the nominal plant. Then all integral-action controllers simultaneously stabilizing the plants $G_j = Y_j^{-1}X \in \mathcal{G}^t$ are given by

$$C = \left(C_t + Y_o Q \operatorname{diag}\left[\frac{\beta^{r_\ell - 1}\phi_\ell(s)}{\beta^{r_\ell - 1}s + \chi_\ell(s)}\right]_{\ell=1}^{m} Y_o(\infty)\right) \left(I - XQ \operatorname{diag}\left[\frac{\beta^{r_\ell - 1}\phi_\ell(s)}{\beta^{r_\ell - 1}s + \chi_\ell(s)}\right]_{\ell=1}^{m} Y_o(\infty)\right)^{-1}, \quad (31)$$

where $Q \in \mathcal{M}(\mathbf{S})$ is such that

$$\left[I + \operatorname{diag}\left[\frac{\beta^{r_\ell - 1}\phi_\ell(s)}{\beta^{r_\ell - 1}s + \chi_\ell(s)}\right]_{\ell=1}^{m} Y_o(\infty)(I + G_j C_t)^{-1} G_j (X Y_o - Y_j X) Q\right] \text{is unimodular} \quad (32)$$

for all $G_j \in \mathcal{G}^t$. To satisfy this unimodularity condition in (32), $Q \in M(S)$ can be chosen such that

$$\|Q\| < \left\|\operatorname{diag}\left[\frac{\beta^{r_\ell - 1}\phi_\ell(s)}{\beta^{r_\ell - 1}s + \chi_\ell(s)}\right]_{\ell=1}^{m} Y_o(\infty)(I + G_j C_t)^{-1} G_j (X Y_o - Y_j X)\right\|^{-1}.$$

The simultaneously stabilizing controllers in (31) have integral-action if and only if $Q(0) = 0$.

(ii) (Robustness of the simultaneously stabilizing controllers). By standard robustness arguments, the simultaneously stabilizing controllers C_r or C_t in Propositions 1 and 2, achieve robust simultaneous stability under 'sufficiently small' plant uncertainty for the plant classes considered. For the set $\mathcal{G}$, the controller C_r in (8) robustly simultaneously stabilizes the additively perturbed plants $G_j + \Delta_j$ for all $\Delta_j \in \mathbf{S}^{m\times m}$ such that $\|\Delta_j\| < \|C_r(I + G_j C_r)^{-1}\|^{-1}$. For multiplicative perturbations, C_r robustly simultaneously stabilizes the plants $G_j(I + \Delta_j)$ under all pre-multiplicative perturbations $\Delta_j \in \mathbf{S}^{m\times m}$ such that $\|\Delta_j\| < \|C_r G_j (I + C_r G_j)^{-1}\|^{-1}$. Similarly, C robustly simultaneously stabilizes the plants $(I + \Delta_j)G_j$ under all post-multiplicative perturbations $\Delta_j \in \mathbf{S}^{m\times m}$ such that $\|\Delta_j\| < \|G_j C_r (I + G_j C_r)^{-1}\|^{-1}$. Some of the free controller parameter choices in the synthesis may be used to maximize the allowable perturbation magnitudes. Similar robust stability conclusions apply to the plant class $\mathcal{P}$ with the controllers C_1 or C_0. For the plant set $\mathcal{G}^t$, the controller C_t in (26) of Proposition 2 robustly simultaneously stabilizes the perturbed plants where the uncertainties satisfy similar bounds.

Example 2. Consider the linear model of the VZ-4 doak, a vertical take-off and landing aircraft [10]:

$$G_j = \begin{bmatrix} \dfrac{s+z_j}{(s+0.8223)(s^2-0.6401s+0.5326)} & 0 \\ \dfrac{-1.08}{s-p_j} & \dfrac{1}{s-p_j} \end{bmatrix}, z_j > 0.$$

The states of the system are forward velocity, downward velocity, pitch rate and pitch angle. The outputs are pitch angle and altitude rates; the inputs are elevator angle and thrust. The parameters for the nominal plant G_o are $z_o = 0.137$, $p_o = -0.137$. The set of poles for each G_j is $\{p_j, -0.8223, 0.3201 \pm j0.6559\}$. These plants have no $\mathcal{U}$-zeros except at infinity (*i.e.*, $z_j > 0$), and therefore they can be written as $G_j = Y_j^{-1}X$, where

$$Y_j^{-1} = \begin{bmatrix} \dfrac{(s+0.8223)(s^2-0.6401s+0.5326)}{(s+z_j)(s+a)^2} & 0 \\ 0 & \dfrac{-(s-p_j)}{(s+a)} \end{bmatrix}^{-1},$$

$$X = \begin{bmatrix} \dfrac{1}{(s+a)^2} & 0 \\ \dfrac{1.08}{(s+a)} & \dfrac{1}{(s+a)} \end{bmatrix},$$

for any $a > 0$. Following Proposition 2, we design a simultaneously stabilizing controller C_t as in (26), with $Y_o(\infty) = Y_j(\infty) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $F_j = I$. With $r = 1$, we have

$$X = \frac{1}{(s+a)} \begin{bmatrix} \dfrac{1}{(s+a)} & 0 \\ 1.08 & 1 \end{bmatrix} = \frac{1}{(s+a)} X_t$$

as in (15); then,

$$X_t^{-1} = \begin{bmatrix} (s+a) & 0 \\ -1.08(s+a) & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \dfrac{1}{(s+a)} & 0 \\ 0 & 1 \end{bmatrix},$$

where $\rho_1 = 1$, $\rho_2 = 0$, $r_1 = 2$, $r_2 = r = 1$. If we choose $\varphi_1(s) = (s+2)$, $\varphi_2(s) = 1$, $g = 3$, then $\phi_1(s) = (s+2)(s+3)$, $\phi_2(s) = (s+3)$, $\chi_1(s) = s$, $\chi_2(s) = 0$. If the plants G_j all have $z_j = z_0 = 0.317$, then $\|\Psi_{tj}\| = \max\{4.9548, |3+p_j|\}$, and $\|\Psi_{to}\| = 4.9548$. Suppose that $\mathcal{G}^t$ contains finitely many plants $G_j \in \mathcal{G}^t$, where $-15 < p_j < 7$, and $0 < z_j < 7$. Then we can choose $\alpha = 12$ satisfying (27), and $\beta = 18$ satisfying (28). By (26), an integral-action controller stabilizing any number of plants $G_j \in \mathcal{G}^t$ is given by

$$C_t = 12 \begin{bmatrix} \dfrac{18(s+2)(s+3)}{s(s+21)} & 0 \\ \dfrac{-1.08 \times 18(s+2)(s+3)}{s(s+21)} & \dfrac{-(s+3)}{s} \end{bmatrix}.$$

The order of each entry in the integral-action controller C_t does not exceed $r_\ell \le 2$. The poles of C_t are all in the open left-half plane $\mathbb{C}\backslash\mathcal{U}$ except for the one at $s = 0$ due to the integral-action requirement.

IV. CONCLUSIONS

This work identified some important classes of any finite number of plants that can be simultaneously stabilized using the same simple controller. The plant classes considered here have restrictions only on their zeros in the region of instability, while the poles are completely unconstrained. These restrictions are due to the difficulties involving simultaneous stabilization of three or more plants with low order tracking controllers. Systematic synthesis procedures are proposed for each plant class, where the controller parameters and the design choices are explicitly defined. The proposed designs allow freedom in the parameters, which should be used to satisfy additional performance criteria that the design may require.

While asymptotic tracking of constant reference inputs is achieved by the integral term in the designed controllers, performance objectives beyond tracking (equivalently, disturbance rejection) were not considered within the scope of this paper. The goal of this study was to establish simultaneous stabilizability using low order controllers, and it was shown that these controllers achieve robust stability under sufficiently small additive and multiplicative plant uncertainty.

V. APPENDIX

5.1 PROOFS

Proof of Lemma 1. If all plants $G_j \in \mathcal{G}$ as in (2) can be simultaneously stabilized by an integral-action controller, then by Definition 1, there exists $C = ND^{-1}$ such that $D(0) = 0$

satisfying (1); *i.e.*, $M_o = XN + Y_oD$ and $M_j = XN + Y_jD$ are unimodular for all $G_j \in \mathcal{G}$. Since $D(0) = 0$, we have $M_o(0) = X(0)N(0) = M_j(0)$, which implies $\det M_j(0)M_o^{-1}(0) = \det I = 1 > 0$. By (2), $X(\infty) = 0$ implies $M_o(\infty) = Y_o(\infty)D(\infty)$, $M_j(\infty) = Y_j(\infty)D(\infty)$. Since $M_jM_o^{-1}$ is unimodular, $\det(M_j(s)M_o^{-1}(s))$ must have the same sign at all $s \in U$ (including $s = 0$ and $s = \infty$). Therefore, $\det(M_j(\infty)M_o(\infty)^{-1}) = \det(Y_j(\infty)Y_o(\infty)^{-1}) = \det F_j > 0$.

Proof of Proposition 1. (i) Let C_r be as in (8); then $C_r^{-1} \in M(S)$ since $g > 0$ and $\varphi(s)$ is Hurwitz. Define

$$\chi(s) := [(s+\beta)^{r-1} - \beta^{r-1}],\, d := (s+g)\chi + \beta^{r-1}s. \quad (33)$$

(a) Write C_r in (8) as $C_r = ND^{-1} = \dfrac{\alpha\beta^{r-1}(s+g)\varphi(s)}{d}Y_o(\infty)$, where $N = \alpha\beta^{-1}I$, $D = \alpha\beta^{r-1}C_r^{-1}$. By (1), C_r stabilizes each $G_j \in \mathcal{G}$ as in (2) if and only if

$$\begin{aligned} M_j &= XN + Y_jD = \beta^{r-1}\alpha X + \beta^{r-1}\alpha Y_jC_r^{-1} \\ &= \frac{\beta^{r-1}\alpha}{(s+a)^r}I + \frac{[(s+g)\chi + \beta^{r-1}s]}{(s+g)\varphi(s)}Y_j(s)Y_o(\infty)^{-1} \end{aligned}$$

is unimodular. With $(s+a)^rY_j = G_j^{-1}$, define

$$W_j := \frac{\alpha}{(s+a)}I + \frac{s}{(s+a)(s+g)\varphi(s)}G_j^{-1}(s)Y_o(\infty)^{-1}.$$

Since F_j is positive definite and $\alpha > 0$, the eigenvalues of $-\alpha F_j^{-1}$ are in the open left-half plane; hence, $(sF_j + \alpha I)^{-1} \in \mathcal{M}(\mathbf{S})$ for $\alpha > 0$. By symmetry of F_j, the maximum eigenvalue of $(\alpha I + j\omega F_j)^{-1}(\alpha I - j\omega F_j)^{-1} = (\alpha^2 I + \omega^2 F_j^2)^{-1}$ is $\bar{\sigma}^2 = \dfrac{1}{\alpha^2}$ obtained for $\omega = 0$; hence, $\|(sF_j + \alpha I)^{-1}\| = \dfrac{1}{\alpha}$. Write W_j as

$$\begin{aligned} W_j &= \frac{1}{(s+a)}(F_js + \alpha I)\Big[(sF_j + \alpha I)^{-1}\alpha I \\ &\quad + (sF_j + \alpha I)^{-1}\frac{s}{(s+g)\varphi(s)}(s+a)^rY_j(s)Y_o(\infty)^{-1}\Big] \\ &= \frac{1}{(s+a)}(F_js + \alpha I) \\ &\quad \left(I + (sF_j + \alpha I)^{-1}\left[\frac{s}{(s+g)\varphi(s)}(s+a)^rY_j(s)Y_o(\infty)^{-1} - sF_j\right]\right) \\ &= \frac{1}{(s+a)}(F_js + \alpha I)(I + (sF_j + \alpha I)^{-1}\Psi_j). \end{aligned}$$

For α satisfying (9), $\|(sF_j + \alpha I)^{-1}\|\|\Psi_j\| \le \dfrac{1}{\alpha}\|\Psi_j\| < 1$; therefore, W_j is unimodular. With $\dfrac{(s+a)^{r-1}}{\varphi(s)}W_j^{-1}Y_j(s)Y_o(\infty)^{-1} = \left(I + \dfrac{s}{\alpha(s+g)\varphi(s)}G_j^{-1}(s)Y_o(\infty)^{-1}\right)^{-1}\dfrac{1}{\alpha\varphi(s)}G_j^{-1}(s)Y_o(\infty)^{-1}$, write M_j as

$$\begin{aligned} M_j &= \frac{\beta^{r-1}}{(s+a)^{r-1}}W_j + \frac{\chi}{\varphi(s)}Y_j(s)Y_o(\infty)^{-1} \\ &= \frac{(s+\beta)^{r-1}}{(s+a)^{r-1}}W_j\left(\frac{\beta^{r-1}}{(s+\beta)^{r-1}}I + \frac{\chi}{(s+\beta)^{r-1}}\frac{(s+a)^{r-1}}{\varphi(s)}W_j^{-1}Y_j(s)Y_o(\infty)^{-1}\right) \\ &= \frac{(s+\beta)^{r-1}}{(s+a)^{r-1}}W_j\left(I + \frac{\chi}{s(s+\beta)^{r-1}}\left[s\frac{(s+a)^{r-1}}{\varphi(s)}W_j^{-1}Y_j(s)Y_o(\infty)^{-1} - sI\right]\right) \\ &= \frac{(s+\beta)^{r-1}}{(s+a)^{r-1}}W_j\left[I + \frac{\chi}{s(s+\beta)^{r-1}}\Phi_j\right]. \end{aligned}$$

Since $\left\|\dfrac{\chi}{s(s+\beta)^{r-1}}\right\| = \left\|\dfrac{(s+\beta)^{r-1} - \beta^{r-1}}{s(s+\beta)^{r-1}}\right\| = \dfrac{r-1}{\beta}$, it follows that $\left\|\dfrac{\chi}{s(s+\beta)^{r-1}}\right\|\|\Phi_j\| \le \dfrac{(r-1)}{\beta}\|\Phi_j\| < 1$ for β satisfying (10). Therefore, M_j is unimodular for each $G_j \in \mathcal{G}$ and hence, the controller C_r in (8) stabilizes each $G_j \in \mathcal{G}$.

(b) For $r = 1$, Ψ_j in (6) becomes $\Psi_j = s\left[\dfrac{1}{(s+g)}G_j^{-1}(s)Y_o(\infty)^{-1} - F_j\right]$, and the controller C_r in (8) becomes C_1 as in (11). Since $C_1^{-1} \in \mathcal{M}(\mathbf{S})$, C_1 also stabilizes each $P_j \in \mathcal{P}$ if and only if $\tilde{M}_j = I + P_j^{-1}C_1^{-1} = I + \dfrac{s}{\alpha(s+g)}P_j^{-1}Y_o(\infty)^{-1}$ is unimodular. For α satisfying (12), we have $\left\|\dfrac{s}{\alpha(s+g)}P_j^{-1}Y_o(\infty)^{-1}\right\| < 1$ and hence, $\tilde{M}_j$ is unimodular for each $P_j \in \mathcal{P}$.

(ii) The controller C_0 in (13) is the same as C_1 in (11) with $Y_o(\infty)$ replaced by an arbitrary constant nonsingular matrix K; C_0 stabilizes each $P_j \in \mathcal{P}$ if and only if $\tilde{M}_j = I + P_j^{-1}C_0^{-1} = I + \dfrac{s}{\alpha(s+g)}P_j^{-1}K^{-1}$ is unimodular. For α satisfying (14), we have $\left\|\dfrac{s}{\alpha(s+g)}P_j^{-1}K^{-1}\right\| < 1$ and hence, $\tilde{M}_j$ is unimodular for each $P_j \in \mathcal{P}$.

Proof of Proposition 2. Let C_t be as in (26); then an RCF $C_t = ND^{-1}$ is given by

$$N = \alpha X_t^{-1}(s)\mathrm{diag}\left[\frac{\beta^{r_\ell - 1}}{(s+a)^{r_\ell}}\right]_{\ell=1}^{m}(s+a)^r \in \mathcal{M}(\mathbf{S}),$$

$$D = Y_o(\infty)^{-1}\mathrm{diag}\left[\frac{\beta^{r_\ell-1}}{\phi_\ell(s)}\right]_{\ell=1}^{m} s\mathrm{diag}\left[\frac{1}{(s+a)^{r_\ell-(r+\rho_\ell)}}\right]_{\ell=1}^{m}$$
$$+ Y_o(\infty)^{-1}\mathrm{diag}\left[\frac{\chi_\ell(s)}{\varphi_\ell(s)}\right]_{\ell=1}^{m} \in \mathcal{M}(\mathbf{S}),$$

where $\chi_\ell(s) = 0$ whenever $r_\ell = 1$. In N, the entries of $\mathrm{diag}\left[\frac{1}{(s+a)^{r_\ell}}\right]_{\ell=1}^{m}(s+a)^{r}$ corresponding to $(r+\rho_\ell) > 0$ are $\frac{1}{(s+a)^{\rho_\ell}}$, *i.e.*, the same as the corresponding entries in Λ^{-1}. If $r \neq 0$, $\mathrm{diag}\left[\frac{1}{(s+a)^{r_\ell}}\right]_{\ell=1}^{m}(s+a)^{r} = \Lambda^{-1}$. In D, the entries of $\mathrm{diag}\left[\frac{1}{(s+a)^{r_\ell-(r+\rho_\ell)}}\right]_{\ell=1}^{m}$ are all equal to 1 when $(r+\rho_\ell) > 0$, and they are equal to $\frac{1}{(s+a)}$ only for $(r+\rho_\ell) = 0$ since $r_\ell = 1$ in that case. By (1), C_t stabilizes each $G_j \in \mathcal{G}^t$ if and only if

$$M_j = XN + Y_jD = \alpha XX_t^{-1}\mathrm{diag}\left[\frac{\beta^{r_\ell-1}}{(s+a)^{r_\ell}}\right]_{\ell=1}^{m}(s+a)^{r}$$
$$+ Y_jY_o(\infty)^{-1}\mathrm{diag}\left[\frac{\beta^{r_\ell-1}}{\phi_\ell(s)}\right]_{\ell=1}^{m} s\mathrm{diag}\left[\frac{1}{(s+a)^{r_\ell-(r+\rho_\ell)}}\right]_{\ell=1}^{m}$$
$$+ Y_jY_o(\infty)^{-1}\mathrm{diag}\left[\frac{\chi_\ell(s)}{\phi_\ell(s)}\right]_{\ell=1}^{m}$$
$$= \left[\frac{\alpha}{(s+a)}I + Y_jY_o(\infty)^{-1}\mathrm{diag}\left[\frac{1}{\phi_\ell(s)}\right]_{\ell=1}^{m} s\mathrm{diag}\left[\frac{(s+a)^{r_\ell-1}}{(s+a)^{r_\ell-(r+\rho_\ell)}}\right]_{\ell=1}^{m}\right]\mathrm{diag}\left[\frac{\beta^{r_\ell-1}}{(s+a)^{r_\ell-1}}\right]_{\ell=1}^{m}$$
$$+ Y_jY_o(\infty)^{-1}\mathrm{diag}\left[\frac{\chi_\ell(s)}{\varphi_\ell(s)}\right]_{\ell=1}^{m}$$

is unimodular. Define

$$W_j := \left[\frac{\alpha}{(s+a)}I + Y_jY_o(\infty)^{-1}\mathrm{diag}\left[\frac{1}{\phi_\ell(s)}\right]_{\ell=1}^{m} s\mathrm{diag}\left[\frac{(s+a)^{r_\ell-1}}{(s+a)^{r_\ell-(r+\rho_\ell)}}\right]_{\ell=1}^{m}\right] = \left[\alpha I(F_js+\alpha I)^{-1}\right.$$
$$+ Y_jY_o(\infty)^{-1}\mathrm{diag}\left[\frac{1}{\phi_\ell(s)}\right]_{\ell=1}^{m} \left. s\mathrm{diag}\left[\frac{(s+a)^{r_\ell-1}}{(s+a)^{r_\ell-(r+\rho_\ell)}}\right]_{\ell=1}^{m}(s+a)(F_js+\alpha I)^{-1}\right]\frac{1}{(s+a)}(F_js+\alpha I)$$
$$= \left[I + s(Y_jY_o(\infty)^{-1}\mathrm{diag}\left[\frac{1}{\phi_\ell(s)}\right]_{\ell=1}^{m} \mathrm{diag}\left[\frac{(s+a)^{r_\ell}}{(s+a)^{r_\ell-(r+\rho_\ell)}}\right]_{\ell=1}^{m} - F_j)(F_js+\alpha I)^{-1}\right]\frac{1}{(s+a)}(F_js+\alpha I)$$
$$= (I + \Psi_{tj}(s)(F_js+\alpha I)^{-1})\frac{1}{(s+a)}(F_js+\alpha I).$$

For α satisfying (27), $\|\Psi_{tj}(s)\|\|(F_js+\alpha I)^{-1}\| \le \frac{1}{\alpha}\|\Psi_{tj}(s)\| < 1$; therefore, W_j is unimodular. Note that $W_j(\infty) = F_j$. With $W_j^{-1} = (s+a)(F_js+\alpha I+\Psi_{tj})^{-1}$, write M_j as

$$M_j = W_j\mathrm{diag}\left[\frac{\beta^{r_\ell-1}}{(s+a)^{r_\ell-1}}\right]_{\ell=1}^{m} + Y_jY_o(\infty)^{-1}\mathrm{diag}\left[\frac{\chi_\ell(s)}{\varphi_\ell(s)}\right]_{\ell=1}^{m}$$
$$= W_j\left(\mathrm{diag}\left[\frac{\beta^{r_\ell-1}}{(s+\beta)^{r_\ell-1}}\right]_{\ell=1}^{m} + W_j^{-1}Y_jY_o(\infty)^{-1}\mathrm{diag}\left[\frac{\chi_\ell(s)}{\varphi_\ell(s)}\right]_{\ell=1}^{m}\mathrm{diag}\left[\frac{(s+a)^{r_\ell-1}}{(s+\beta)^{r_\ell-1}}\right]_{\ell=1}^{m}\right)\mathrm{diag}\left[\frac{(s+\beta)^{r_\ell-1}}{(s+a)^{r_\ell-1}}\right]_{\ell=1}^{m}$$
$$= W_j\left(I + s\left[W_j^{-1}Y_jY_o(\infty)^{-1}\mathrm{diag}\left[\frac{(s+a)^{r_\ell-1}}{\varphi_\ell(s)}\right]_{\ell=1}^{m} - I\right]\mathrm{diag}\left[\frac{\chi_\ell(s)}{s(s+\beta)^{r_\ell-1}}\right]_{\ell=1}^{m}\right)\mathrm{diag}\left[\frac{(s+\beta)^{r_\ell-1}}{(s+a)^{r_\ell-1}}\right]_{\ell=1}^{m}$$
$$= W_j\left(I + \Phi_{tj}\mathrm{diag}\left[\frac{\chi_\ell(s)}{s(s+\beta)^{r_\ell-1}}\right]_{\ell=1}^{m}\right)\mathrm{diag}\left[\frac{(s+\beta)^{r_\ell-1}}{(s+a)^{r_\ell-1}}\right]_{\ell=1}^{m}.$$

If $r_\ell = 1$, then $\frac{\chi_\ell}{s(s+\beta)^{r_\ell-1}} = 0$. If $r_\ell > 1$, then $\left\|\frac{\chi_\ell}{s(s+\beta)^{r_\ell-1}}\right\| = \left\|\frac{(s+\beta)^{r_\ell-1}-\beta^{r_\ell-1}}{s(s+\beta)^{r_\ell-1}}\right\| = \frac{r_\ell-1}{\beta}$. Therefore $\left\|\mathrm{diag}\left[\frac{\chi_\ell(s)}{s(s+\beta)^{r_\ell-1}}\right]_{\ell=1}^{m}\right\| = \max_{r_\ell}\frac{(r_\ell-1)}{\beta}$, and it follows that $\left\|\Phi_{tj}\mathrm{diag}\left[\frac{\chi_\ell(s)}{s(s+\beta)^{r_\ell-1}}\right]_{\ell=1}^{m}\right\| \le \max_{r_\ell}\frac{(r_\ell-1)}{\beta}\|\Phi_{tj}\| < 1$ for β satisfying (28). Therefore, M_j is unimodular for each $G_j \in \mathcal{G}^t$ and hence, the controller C_t in (26) stabilizes each $G_j \in \mathcal{G}^t$.

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