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Simultaneous and strong simultaneous stabilisation of some classes of MIMO systems

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Simultaneous and strong simultaneous stabilisation of some classes of MIMO systems

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Sufficient conditions are derived for simultaneous stabilisability of any finite number of linear, time-invariant, multi-input multi-output (MIMO) systems. Four general MIMO plant classes with arbitrary finite number of plants are shown to be simultaneously stabilisable using simple proportional + derivative controllers. For some of these plant classes it is also possible to achieve simultaneous stabilisation and asymptotic tracking of step-input references with zero steady-state error by using integral-action controllers. Three of these general MIMO plant classes are shown to be strongly simultaneously stabilisable.

Keywords: simultaneous stabilisation; integral action; PID control; robust asymptotic tracking

1. Introduction

Control problems involving simultaneous stabilisation of a finite family of plants arise in many practical applications. For example, linearisation of nonlinear process models at various operating points results in a set of models to be controlled simultaneously. The requirement of maintaining stability under sensor or actuator failures also leads to dynamic models corresponding to failure modes to be all controlled using a common controller for reliable operation (Stoustrup and Blondel 2004). The robust control problem deals with controller design for an infinite number of plant models all within a neighbourhood of a nominal model, which represent perturbations of the nominal plant. Simultaneous control deals with finitely many plants that may or may not be topologically close (Sourlas and Manousiouthakis 1999).

The simultaneous stabilisation problem is recognised as one of the hard open problems in linear systems theory. Conditions for the existence of simultaneously stabilising controllers have been explored extensively, e.g. Vidyasagar (1985) and Blondel (1994). The well established result that the simultaneous stabilisation of n plants is equivalent to strong stabilisation of $(n-1)$ plants leads to explicit conditions for existence of simultaneously stabilising controllers for $n=2$: two plants are simultaneously stabilisable if and only if a related system is strongly stabilisable, i.e. can be stabilised using a stable controller. Strong stabilisability of this single system can in turn be checked via the parity interlacing property (PIP) of the positive real poles and (blocking) zeros (Youla, Bongiorno, and Lu 1974; Vidyasagar 1985; Blondel 1994). Unlike the case of two plants, there are no *necessary and sufficient* conditions available for simultaneous stabilisability of *three or more* plants (Blondel, Campion, and Gevers 1993; Blondel 1994; Blondel, Gevers, Mortini, and Rupp 1994). An algorithm for simultaneous stabilisation of four scalar plants in groups is given in Jia and Ackermann (2001). Closed-loop performance issues in addition to simultaneous stabilisation have also been explored to a lesser extent, e.g. Sourlas and Manousiouthakis (1999).

The problem considered here is the simultaneous stabilisation of a finite class of linear, time-invariant (LTI) multi-input multi-output (MIMO; unstable or stable) plants using linear *time-invariant* output-feedback controllers. Since it is not always possible to stabilise two or more plants with a common LTI controller, alternative strategies such as time-varying or sampled-data controllers have been developed to overcome the limitations (see e.g. Kabamba and Yang (1991) and Miller and Kennedy (2002)). This work deals with the problem using time-invariant controllers. Recognising the fact that explicit existence conditions for the general case of three or more arbitrary plants are not possible to obtain (Blondel 1994), the goal of this work is to identify some important classes of practically relevant plants such that simultaneous stabilisation is achievable. An additional objective is to design the common stabiliser

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as a low-order LTI controller with integral-action, which provides asymptotic tracking of constant reference inputs.

Simultaneous stabilisation is challenging even without the order constraint on the controller. The question we attempt to answer here is the following: can we identify any classes of plants that can be simultaneously stabilised using up to second-order controllers such as proper Proportional + Derivative (PD) controllers or proper Proportional + Integral + Derivative (PID) controllers? In order to avoid confusion regarding the use of the 'derivative' term, we define the form of the controller considered throughout this article as the (realisable) PID form given in (1),

$$C_{\mathrm{PID}} = K_P + \frac{K_I}{s} + \frac{K_D s}{\tau s + 1}, \qquad (1)$$

where (the matrices) K_P, K_I, K_D are called the proportional constant, the integral constant, and the derivative constant, respectively Goodwin, Graebe, and Salgado 2001. A (fast) pole is included in the denominator of the derivative term (with $\tau > 0$) so that the transfer-function C_{PID} in (1) is proper. Any controller designed with poles at $s = 0$, duplicating the dynamic structure of the exogenous signals that the regulator has to process, obeys the well-known internal model principle (Francis and Wonham 1975). So if the integral term is non-zero, the PID controller has integral-action and hence, a system stabilised by using a PID controller achieves asymptotic tracking of step-input references with zero steady-state error. Equivalently, constant output disturbances are rejected asymptotically due to the integral term in the PID controller. Restricting the controller order to be one or two by using PD/PID controllers makes the problem clearly more difficult than simultaneous stabilisation without the order constraint. The use of PD controllers (or proportional controllers when also $K_D = 0$) further implies that the problem becomes *strong* simultaneous stabilisation using a (stable) controller whose order is restricted to be at most one.

Although PID controllers are widely used and preferred due to their simplicity, they can stabilise only certain plants. A basic necessary (but not sufficient) condition for PID stabilisability is strong stabilisability (Gündeş and Özgüler 2007). While several rigorous PID design methods exist mostly for single-input single-output (SISO) systems or using numerical methods (see e.g. Aström and Hagglund (1995), Silva, Datta, and Bhattacharyya (2002) and Lin, Wang, and Lee (2004)), simultaneous stabilisation while achieving asymptotic tracking or PID designs that achieve simultaneous closed-loop stability of MIMO systems have not been explored extensively. A simultaneous PID controller synthesis for stable plants was presented in Gündeş (2008). Since strong stabilisability is necessary for the plant classes we aim to simultaneously PID stabilise, we expect that there will be certain restrictions on the positive real-axis zeros and poles due to the PIP. On the other hand, satisfying the PIP is not even sufficient. For example, the simple SISO plant $G = \frac{1}{(s-p)^3}$ is not PID stabilisable for any non-negative p although G is strongly stabilisable for all p. Although a second-order controller cannot be found, there exist third-order controller (with integral-action) that stabilises G. Since the goal is to identify plants that can be simultaneously stabilised using controllers up to second-order, we are imposing constraints on the transmission-zeros in the extended right-half complex plane (RHP), and we are limiting the number of zeros at infinity. There are no limitations on the open left-half plane (OLHP) zeros and the poles may also be anywhere in the complex plane.

Three classes of plants are shown to be simultaneously PID stabilisable; another class is simultaneously PD stabilisable. The first is the class of plants that have no extended RHP zeros, which is strongly simultaneously stabilisable using PD controllers and also simultaneously stabilisable using PID controllers. The second class has any number of plants all with one zero at infinity (i.e. relative degree one in the SISO case) but no other RHP zeros; this class is strongly simultaneously stabilisable using PD controllers and under a sufficient condition, also simultaneously stabilisable using PID controllers. The third class allows the plants to have two zeros at infinity (i.e. relative degree two in the SISO case) but no other RHP zeros; this class is simultaneously stabilisable using PID controllers under a sufficient condition, but the methods used here cannot determine if such plants are simultaneously PD stabilisable in the general case. The fourth class includes plants with one positive real-axis zero, which may be at the origin. We also consider unions of these classes and determine simultaneous PD/PID stabilisability of a finite set that combines plants from some of these classes. In all cases considered, there are no restrictions on the number or location of the plant poles anywhere in the complex plane; the sets to be simultaneously stabilised may include any finite number of stable and unstable plants. The zeros of these plants in the OLHP are similarly unrestricted. In Propositions 1, 2, 3, 4, we develop systematic methods of simultaneous PD/PID synthesis for each of these plant classes. The synthesis approach does not depend on numerical algorithms and is not restricted to three or four plants; any number of plants within the specified classes are simultaneously PD/PID stabilised. The PD/PID controllers proposed here provide robust simultaneous stabilisation of any finite number of plants in the plant classes under consideration.

Although we discuss continuous-time systems here, all results also apply to discrete-time systems with appropriate modifications.

The following notation is used: $\mathcal{U}$ denotes the extended closed right-half plane, i.e. $\mathcal{U}=\{s\in\mathbb{C}\mid \mathcal{R}e(s)\geq 0\}\cup\{\infty\}$; $\mathbb{R}$, $\mathbb{R}_+$ denote real and positive real numbers; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S}\subset\mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in $\mathbf{S}$; $M\in\mathcal{M}(\mathbf{S})$ is called unimodular if $M^{-1}\in\mathcal{M}(\mathbf{S})$; I_m is the $m\times m$ identity matrix; we use I when the dimension is unambiguous. The H_∞-norm of $M(s)\in\mathcal{M}(\mathbf{S})$ is denoted by $\|M(s)\|$ (i.e. the norm $\|\cdot\|$ is defined as $\|M\|:=\sup_{s\in\partial\mathcal{U}}\bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$). For simplicity, we drop (s) in transfer matrices such as $G(s)$ where this causes no confusion. We use coprime factorisations over $\mathbf{S}$; i.e. for $G\in\mathbf{R}_\mathbf{p}^{m\times m}$, $C\in\mathbf{R}_\mathbf{p}^{m\times m}$, $G=Y^{-1}X$ denotes a left-coprime-factorisation (LCF), $C=ND^{-1}$ denotes a right-coprime-factorisation (RCF), where $X,Y,N,D\in\mathbf{S}^{m\times m}$, $\det Y(\infty)\neq 0$, $\det D(\infty)\neq 0$. Let $\operatorname{rank} G(s)=r\leq m$; then $z\in\mathcal{U}$ is a transmission-zero of G if $\operatorname{rank} X(z)<r$ and it is a blocking-zero of G if $X(z)=0$. We refer to poles and zeros in the region of instability $\mathcal{U}$ as $\mathcal{U}$-poles and $\mathcal{U}$-zeros.

2. Problem description

Consider the standard LTI, MIMO unity-feedback system Sys(G, C) shown in Figure 1, where $G\in\mathbf{R}_\mathbf{p}^{m\times m}$, and $C\in\mathbf{R}_\mathbf{p}^{m\times m}$ denote the plant's and the controller's transfer-functions, and it is assumed that the feedback system is well-posed, G and C have no unstable hidden-modes, and $\operatorname{rank} G(s)=m$. The objective is to design a single stabilising controller C that achieves asymptotic tracking of step-input references with zero steady-state error for a finite class of plants simultaneously.

Let $G=Y^{-1}X$ be an LCF and $C=ND^{-1}$ be an RCF, where $Y,X,D,N\in\mathbf{S}^{m\times m}$, $\det Y(\infty)\neq 0$, det $D(\infty)\neq 0$. Then C stabilises $G\in\mathcal{M}(\mathbf{S})$ if and only if

$$M := YD + XN \tag{2}$$

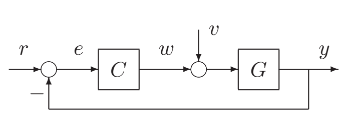

Figure 1. Unity-feedback system Sys(G, C).

is unimodular (Vidyasagar 1985; Gündeş and Desoer 1990). Let the (input-error) transfer-function from r to e be denoted by H_{er} and let the (input–output) transfer-function from r to y be denoted by H_{yr}; then

$$\begin{aligned} H_{er} &= (I+GC)^{-1} = I - GC(I+GC)^{-1} \\ &= I - GH_{wr} = I - H_{yr}. \end{aligned} \tag{3}$$

Definition 1:

(i) The system Sys(G, C) is said to be stable if the closed-loop transfer-function from (r, v) to (y, w) is stable.

(ii) The stable system Sys(G, C) is said to have integral-action if H_{er} has blocking-zeros at $s=0$.

(iii) The controller C is said to be an integral-action controller if C stabilises G and the denominator D of any RCF $C=\mathrm{ND}^{-1}$ has blocking-zeros at $s=0$, i.e. $D(0)=0$.

Suppose that Sys(G, C) is stable and that step input references are applied to the system. Then the steady-state error $e(t)$ due to all step input vectors at $r(t)$ goes to zero as $t\to\infty$ if and only if $H_{er}(0)=0$. Therefore, by Definition 1, the stable system Sys(G, C) achieves asymptotic tracking of constant reference inputs with zero steady-state error if and only if it has integral-action. Write $H_{er}=(I+GC)^{-1}=DM^{-1}Y$. Then by Definition 1, Sys(G, C) has integral-action if $C=ND^{-1}$ is an integral-action controller since $D(0)=0$ implies that $H_{er}(0)=(DM^{-1}Y)(0)=0$.

The simplest integral-action controllers are in PID form (with $K_D=0$ for first-order and $K_D\neq 0$ for second-order controllers). Here we only consider the proper PID controller form given in (1), where a pole is typically included in the derivative term due to implementation issues so that C_{PID} in (1) is proper. The only $\mathcal{U}$-pole of the PID controller in (1) is at zero. The constants K_P, K_D, K_I may be negative; in the scalar case, this would imply that the zeros of C_{PID} may be in the unstable region $\mathcal{U}$. The integral-action in the PID controller is present when $K_I\neq 0$. Subsets of the PID controller in (1) are obtained as C_{PI}, C_{PD}, C_{ID}, C_I, C_D, C_P, by setting one or two of the three constants K_P, K_I, K_D equal to zero.

3. Simultaneous controller synthesis

We now explore the problem of simultaneous PD/PID stabilisation for finite sets of plants. The plant classes we consider have restrictions on their $\mathcal{U}$-zeros. The poles are completely unrestricted, and there are no restrictions on the OLHP zeros. The sets of systems to be simultaneously stabilised include stable and unstable plants. Four specific plant classes are investigated in the next four sections.

3.1 *Simultaneous controller synthesis for plants with no $\mathcal{U}$-zeros*

Consider a finite set $\mathcal{G}^{no}$ of plants that have *no* $\mathcal{U}$-zeros. The plants in this set have no zeros in the extended right-half complex plane $\mathcal{U}$ including infinity; in the SISO case, this implies that the relative degree is zero for these plants. There is an arbitrary number of plants in this finite class and they may have poles anywhere in $\mathbb{C}$. These plants may have (transmission and blocking) zeros anywhere in $\mathbb{C}\setminus\mathcal{U}$. There may be any number of stable as well as unstable plants in the set under consideration.

The plants in $\mathcal{G}^{no}$ are strongly simultaneously stabilisable using PD controllers and they are also simultaneously PID stabilisable as shown in Proposition 1, which presents a systematic simultaneously stabilising PD and PID controller synthesis procedure for this plant class.

Proposition 1 (PD/PID synthesis for simultaneous stabilisation of $G_k \in \mathcal{G}^{no}$)**:** *Let $\mathcal{G}^{no}$ be a finite set of plants with no $\mathcal{U}$-zeros. Simultaneously stabilising PD controllers and PID controllers exist for all plants $G_k \in \mathcal{G}^{no}$, and can be designed as follows*: *Choose any $K_D \in \mathbb{R}^{m\times m}$, $\tau \in \mathbb{R}_+$. Choose any non-singular $\hat{K}_P \in \mathbb{R}^{m\times m}$. Choose any $g \in \mathbb{R}_+$. Define $\Theta_k \in \mathcal{M}(\mathbf{S})$ and $\alpha_n \in \mathbb{R}_+$ as in* (4):

$$\Theta_k := \left(G_k^{-1} + \frac{s}{\tau s+1}K_D\right)\hat{K}_P^{-1}, \qquad \alpha_n := \max_{G_k\in\mathcal{G}^{no}} \|\Theta_k\|. \tag{4}$$

For any $\alpha > \alpha_n$, the PD controller C_{PD} given by (5) and the PID controller C_{PID} given by (6) both simultaneously stabilise all plants $G_k \in \mathcal{G}^{no}$:

$$C_{\text{PD}} = \alpha\hat{K}_P + \frac{s}{\tau s+1}K_D, \tag{5}$$

$$C_{\text{PID}} = C_{\text{PD}} + \frac{1}{s}K_I = \alpha\hat{K}_P + \frac{s}{\tau s+1}K_D + \frac{\alpha g}{s}\hat{K}_P. \tag{6}$$

Remarks: In Proposition 1, the choice of $g \in \mathbb{R}_+$ is completely arbitrary in the integral constant $K_I := \alpha g\hat{K}_P$. This is interesting because (6) implies that any integral term can be added to a stabilising PD controller to make it into a stabilising PID controller as long as the sign of K_I remains the same as that of the proportional constant $K_P := \alpha\hat{K}_P$.

In the PD controller (5) and the PID controller (6) of Proposition 1, the choice of the derivative constant matrix K_D is completely free. For $K_D = 0$, (5) is a P controller and (6) is a PI controller.

A major advantage of the simultaneous PD/PID controller synthesis method in Proposition 1 is that if additional plants join the original set $\mathcal{G}^{no}$, then the design does not need to start over. Instead, only the scalar α has to be adjusted as needed in (5) and (6) by calculating $\|\Theta_k\|$ for the newly added plants. Another implication is that the controllers C_{PD} and C_{PID} stabilise any number of other plants that have no $\mathcal{U}$-zeros and whose $\|\Theta_k\|$ is less than the α chosen to simultaneously stabilise the original plant set. Similarly, plants can be deleted from the set to be simultaneously stabilised without any need to modify the controller; if desired, a smaller α value can be selected if $\alpha_n = \|\Theta_d\|$ of a deleted plant G_d.

Example 1: Consider the set $\mathcal{G}^{no} = \{G_1, G_2, G_3, G_4, G_5\}$ of non-strictly-proper (relative-degree zero) SISO plants given in (7):

$$G_k = \frac{(-1)^k(s+6)^k}{(s-3)^k}, \quad k = 1,2,3,4, \qquad G_5 = \frac{-0.1(s^2+8s+25)}{(s-2)(s-5)}. \tag{7}$$

Following Proposition 1 to design PD/PID controllers for $\mathcal{G}^{no}$, choose $K_D = 5$, $\tau = 0.05$, $\hat{K}_p = 20$ completely arbitrarily. By (4), $\alpha_n = \max_{k=1,\ldots,5}\{\|\Theta_k\|\} = \max\{4, 6, 4, 6, 4.995\} = 6$; we choose $\alpha = 8 > \alpha_n$. Then the PD controller

$$C_{\text{PD}} = \alpha\hat{K}_p + \frac{K_D s}{\tau s+1} = 160 + \frac{5s}{0.05s+1} \tag{8}$$

as in (5) strongly simultaneously stabilises the plants in $\mathcal{G}^{no}$. We add an integral term to this PD controller by choosing any $g \in \mathbb{R}_+$; for example, if $g = 2$, then

$$C_{\text{PID}} = C_{\text{PD}} + \frac{\alpha g}{s}\hat{K}_p = 160 + \frac{5s}{0.05s+1} + \frac{320}{s}. \tag{9}$$

The design offers flexibility in the various free parameter choices. With the current parameter choices, the closed-loop poles of Sys(G_1, C_{PID}) are $\{-1.848, 8.951 \pm j2.543\}$, of Sys$(G_2, C_{\text{PID}})$ are $\{-13.91, -1.81, -4.49 \pm j3.52\}$, of Sys$(G_3, C_{\text{PID}})$ are $\{-1.784, -3.074 \pm j2.718, -12.659 \pm j\ 5.616\}$, of Sys (G_4, C_{PID}) are $\{-16, -1.76, -6.31 \pm j7.44, -2.53 \pm j2.16\}$, of Sys$(G_5, C_{\text{PID}})$ are $\{-8.49, -6.87, -3.26 \pm j0.59\}$, which all have reasonable damping.

Now append the plants G_k for $k = 6, 7, 8$ given in (10) to the set $\mathcal{G}^{no}$, where

$$G_k = \frac{(s+5)(s+z_k)^2}{(s^2+16)(s-10)}, \qquad z_k = 1, \ \text{for } k=6;\ 0.5 \text{ for } k=7;\ 0.4 \text{ for } k=8. \tag{10}$$

We compute $\|\Theta_6\| = 5.05$, $\|\Theta_7\| = 6.40$, $\|\Theta_8\| = 10.0$. Since $\|\Theta_6\|$, $\|\Theta_7\|$ are both less than $\alpha = 8$, the controllers C_{PD} and C_{PID} in (8), (9) stabilise G_6 and G_7 simultaneously with the original set of five plants. In order to simultaneously stabilise the entire set including G_8, we need $\alpha > 10$. For example, choosing $\alpha = 16$, the controllers $2C_{\mathrm{PD}}$, and $2C_{\mathrm{PID}}$ simultaneously stabilise all eight plants.

3.2 *Simultaneous controller synthesis for plants with one zero at infinity*

Consider a finite set $\mathcal{G}^{1\infty}$ of plants that all have *exactly* one blocking-zero at infinity and no other (transmission and blocking) $\mathcal{U}$-zeros. There is an arbitrary number of plants in this set and they may have poles anywhere in $\mathbb{C}$. These plants may have (transmission and blocking) zeros anywhere in $\mathbb{C}\backslash\mathcal{U}$. There may be any number of stable as well as unstable plants in the set under consideration. In the SISO case, the relative degree is one for these plants. More specifically, what is meant by one blocking-zero at infinity for the MIMO case is that the plants $G_i \in \mathcal{G}^{1\infty}$ are expressed as

$$G_i = Y_i^{-1} \frac{1}{s+a} I = \left[\frac{1}{s+a} G_i^{-1}\right]^{-1} \left[\frac{1}{s+a} I\right], \tag{11}$$

where $G_i = Y_i^{-1} X$ is an LCF, with $X = \frac{1}{s+a} I$ for each G_i, and $Y_i = \frac{1}{s+a} G_i^{-1}$ is stable for any $a \in \mathbb{R}_+$.

Each *individual* plant G_i as described in (11) is PD and PID stabilisable (Gündeş and Özgüler 2007). However, existence of a single integral-action controller that simultaneously stabilises all plants $G_i \in \mathcal{G}^{1\infty}$ requires additional conditions. For $G_i \in \mathcal{G}^{1\infty}$, let

$$Y_i(\infty)^{-1} = [(s+a)\, G_i(s)]|_{s\to\infty}. \tag{12}$$

Designate an arbitrary plant $G_o = Y_o^{-1} X \in \mathcal{G}^{1\infty}$ as the nominal plant, with $Y_o(\infty)^{-1} = [(s+a) G_o(s)]|_{s\to\infty}$. It is shown in Proposition 2 that the plants in $\mathcal{G}^{1\infty}$ are strongly simultaneously stabilisable using PD controllers and they are also simultaneously PID stabilisable if all eigenvalues of W_i are real and positive for all $G_i \in \mathcal{G}^{1\infty}$, where $W_i \in \mathbb{R}^{m\times m}$ is given by

$$W_i := Y_i(\infty)\, Y_o(\infty)^{-1} = (G_i^{-1} G_o)(\infty). \tag{13}$$

Furthermore, the finite set $\mathcal{G}^{1\infty} \cup \mathcal{G}^{no}$ consisting of plants with one blocking-zero at infinity as in (11) and plants with no $\mathcal{U}$-zeros is also strongly simultaneously stabilisable using PD controllers and is also simultaneously PID stabilisable under the same assumptions for $G_i \in \mathcal{G}^{1\infty}$ (no additional conditions are required for the plants $G_k \in \mathcal{G}^{no}$). Proposition 2(i) presents a systematic simultaneously stabilising PD and PID controller synthesis procedure for the plant class $\mathcal{G}^{1\infty}$ and Proposition 2(ii) extends the procedure to the combined class $\mathcal{G}^{1\infty} \cup \mathcal{G}^{no}$.

Proposition 2 (PD/PID synthesis for simultaneous stabilisation of $\mathcal{G}^{1\infty}$ and of $\mathcal{G}^{1\infty} \cup \mathcal{G}^{no}$):

(i) *Let $\mathcal{G}^{1\infty}$ be a finite set of plants as in* (11). *Let $G_o \in \mathcal{G}^{1\infty}$ be an arbitrary member designated as the nominal plant. With $Y_i(\infty)$ as in* (12) *and W_i as in* (13), *suppose that all eigenvalues of W_i are real and positive for all $G_i \in \mathcal{G}^{1\infty}$. Under these assumptions, simultaneously stabilising PD controllers and PID controllers exist for all plants $G_i \in \mathcal{G}^{1\infty}$, and can be designed as follows*: *Choose any $K_D \in \mathbb{R}^{m\times m}$, $\tau \in \mathbb{R}_+$. Choose any $g \in \mathbb{R}_+$. Define $\Phi_i \in \mathcal{M}(\mathbf{S})$ and $\beta_\infty \in \mathbb{R}_+$ as in* (14); *define $\Psi_i \in \mathcal{M}(\mathbf{S})$ and $\rho_\infty \in \mathbb{R}_+$ as in* (15):

$$\Phi_i := \left(G_i^{-1} + \frac{s}{\tau s+1} K_D\right) Y_o(\infty)^{-1} - sW_i, \quad \beta_\infty := \max_{G_i \in \mathcal{G}^{1\infty}} \|\Phi_i\|, \tag{14}$$

$$\Psi_i = \frac{s}{s+g}\left(G_i^{-1} + \frac{s}{\tau s+1} K_D\right) Y_o(\infty)^{-1} - sW_i, \quad \rho_\infty := \max_{G_i \in \mathcal{G}^{1\infty}} \|\Psi_i\|. \tag{15}$$

For any $\beta > \beta_\infty$, the PD controller C_{PD} given by (16) *simultaneously stabilises all plants $G_i \in \mathcal{G}^{1\infty}$*:

$$C_{\mathrm{PD}} = \beta\, Y_o(\infty) + \frac{s}{\tau s+1} K_D. \tag{16}$$

For any $\rho > \rho_\infty$, the PID controller C_{PID} given by (17) *simultaneously stabilises all plants $G_i \in \mathcal{G}^{1\infty}$*:

$$C_{\mathrm{PID}} = \rho Y_o(\infty) + \frac{s}{\tau s+1} K_D + \frac{\rho g}{s}\, Y_o(\infty). \tag{17}$$

(ii) *Let $\mathcal{G}^{1\infty}$ be a finite set of plants as in* (11). *Let $\mathcal{G}^{no}$ be a finite set of plants that have no $\mathcal{U}$-zeros as in Proposition* 1. *Under the assumptions and definitions of part* (i) *above, simultaneously stabilising PD controllers and PID controllers exist for all plants G_i, $G_k \in \mathcal{G}^{1\infty} \cup \mathcal{G}^{no}$, and can be designed as follows*: *Let $G_o \in \mathcal{G}^{1\infty}$ and Y_o be as in part* (i) *above. Choose any $K_D \in \mathbb{R}^{m\times m}$, $\tau \in \mathbb{R}_+$. Choose any $g \in \mathbb{R}_+$. Define $\Phi_i \in \mathcal{M}(\mathbf{S})$, $\beta_\infty \in \mathbb{R}_+$, $\Psi_i \in \mathcal{M}(\mathbf{S})$ and $\rho_\infty \in \mathbb{R}_+$, as in* (14), (15). *Let $\hat{K}_P = Y_o(\infty)$ in the definition* (4) *of $\Theta_k \in \mathcal{M}(\mathbf{S})$ and $\alpha_n \in \mathbb{R}_+$, and for $G_k \in \mathcal{G}^{no}$, define $\rho_n \in \mathbb{R}_+$ as in* (18):

$$\frac{s}{s+g}\Theta_k = \frac{s}{s+g}\left(G_k^{-1} + \frac{s}{\tau s+1} K_D\right) Y_o(\infty)^{-1}, \quad \rho_n := \max_{G_k \in \mathcal{G}^{no}} \left\|\frac{s}{s+g}\Theta_k\right\|. \tag{18}$$

For any $\beta > \max\{\beta_\infty, \alpha_n\}$, the PD controller C_{PD} given by (16) simultaneously stabilises all plants

G_i, $G_k \in \mathcal{G}^{1\infty} \cup \mathcal{G}^{no}$. For any $\rho > \max\{\rho_\infty, \rho_n\}$, the PID controller C_{PID} given by (17) simultaneously stabilises all plants G_i, $G_k \in \mathcal{G}^{1\infty} \cup \mathcal{G}^{no}$.

In the PD controller (16) and the PID controller (17) of Proposition 2, the choice of the derivative constant matrix K_D is completely free. For $K_D = 0$, (16) is a P controller and (17) is a PI controller.

In Example 2, we apply the systematic design procedure of Proposition 2(i) to a set of three SISO plants each with one zero at infinity. We then combine this class with the plant set considered in Example 1 and follow Proposition 2(ii) to design PD and PID controllers stabilising all eight plants.

Example 2:

(i) Consider the set $\mathcal{G}^{1\infty} = \{G_1, G_2, G_3\}$ of strictly-proper (relative-degree one) plants:

$$G_1 = \frac{1}{20(s-3)}, \quad G_2 = \frac{0.1(s+3)}{(s-2)(s-5)}, \quad G_3 = \frac{s+10}{25(s^2+6s+18)}. \tag{19}$$

The plant G_3 is stable and the others are unstable. Since $Y_1(\infty) = 20$, $Y_2(\infty) = 10$, $Y_3(\infty) = 25$ are all positive, $W_i = Y_i(\infty) Y_o^{-1}(\infty) > 0$ for any of these plants that might be designated as the nominal plant. Let $G_o = G_1$; then $W_1 = 1$, $W_2 = 0.5$, $W_3 = 1.25$. Since these choices are completely arbitrary, let's choose $K_D = 5$, $\tau = 0.05$. By (14), $\beta_\infty = \max\{\Phi_1, \Phi_2, \Phi_3\} = \max\{3, 3.5261, 2.25\} = 3.5261$; we choose $\beta = 5 > \beta_\infty$. Then the PD controller $C_{\mathrm{PD}} = 100 + \frac{5s}{0.05s+1}$ as in (16) simultaneously stabilises the plants in $\mathcal{G}^{1\infty}$. We design a PID controller for the plants in $\mathcal{G}^{1\infty}$ by choosing any $g \in \mathbb{R}_+$. For example, if $g = 4$, $\rho_\infty = \max\{\Psi_1, \Psi_2, \Psi_3\} = \max\{5.8722, 93, 90\} = 93$ by (15); we choose $\rho = 100 > \rho_\infty$. Then the PID controller $C_{\mathrm{PID}} = 2000 + \frac{5s}{0.05s+1} + \frac{8000}{s}$ as in (17) simultaneously stabilises the plants in $\mathcal{G}^{1\infty}$. The closed-loop poles of $\mathrm{Sys}(G_1, C_{\mathrm{PID}})$ are $\{-99.23, -18.38, -4.39\}$, of $\mathrm{Sys}(G_2, C_{\mathrm{PID}})$ are $\{-196.86, -18.53, -4.96, -2.65\}$, of $\mathrm{Sys}(G_3, C_{\mathrm{PID}})$ are $\{-76.35, -17.83, -11.85, -3.97\}$.

(ii) Append the set $\mathcal{G}^{no}$ of five plants as in (10) of Example 1 to the set $\mathcal{G}^{1\infty}$ of the three strictly-proper plants in part (i) of this example, and design PD/PID controllers for the set $\mathcal{G}^{1\infty} \cup \mathcal{G}^{no}$ following Proposition 2(ii). Since our choice of $\hat{K}_P = 20$ in Example 1 is in fact the same as $\hat{K}_p = Y_o(\infty)$ here, the value of $\alpha_n = 6$ is the same as computed in Example 1. But the choice of $\beta = 5$ is no longer valid for the set $\mathcal{G}^{1\infty} \cup \mathcal{G}^{no}$ since $\beta > \max\{\beta_\infty, \alpha_n\} = \max\{3.5261, 6\}$; we choose $\beta = 8$. Then the PD controller $C_{\mathrm{PD}} = 160 + \frac{5s}{0.05s+1}$ as in (16) strongly simultaneously stabilises the plants in $\mathcal{G}^{1\infty} \cup \mathcal{G}^{no}$. Continuing on to a PID design with the choice of $g = 4$, we see that the choice of $\rho = 100$ is still valid since $\|\frac{s}{s+g}\Theta_k\| \le \|\frac{s}{s+g}\| \|\Theta_k\| = \|\Theta_k\|$, and hence, $\rho_n \le \alpha_n$. With the previous choice of $\rho = 100 > \max\{\rho_\infty, \rho_n\} = 93$, the same PID controller $C_{\mathrm{PID}} = 2000 + \frac{5s}{0.05s+1} + \frac{8000}{s}$ as in (17) simultaneously stabilises the plants in $\mathcal{G}^{no} \cup \mathcal{G}^{1\infty}$. The closed-loop poles of $\mathrm{Sys}(G_i, C_{\mathrm{PID}})$ for $G_i \in \mathcal{G}^{1\infty}$ in (19) are the same as above since ρ has not changed. The closed-loop poles of $\mathrm{Sys}(G_k, C_{\mathrm{PID}})$ for the five plants $G_k \in \mathcal{G}^{no}$ in (7) are

$\{-18.78, -6.27, -3.92\}$,
$\{-18.84, -3.72, -6.05 \pm j1.48\}$,
$\{-18.73, -9.28, -3.46, -4.84 \pm j2.05\}$,
$\{-18.92, -3.22, -8.18 \pm j3.65, -4 \pm j2\}$,
$\{-18.79, -4.19, -3.98 \pm j2.91\}$, respectively.

In Example 3, we have a class of 2×2 strictly-proper MIMO plants with no transmission-zeros at $s = 0$. These plants represent linearised models of an unstable batch reactor with different sensor settings, each resulting in a different output matrix. This model was considered as an example of an unstable plant also in e.g. Munro (1972), Rosenbrock (1974), Green and Limebeer (1995) and Tabbara, Nešić, and Teel (2007) only for the case of $f_i = 1$, where the goal was not simultaneous stabilisation.

Example 3: A two-input two-output linearised process model of an unstable batch reactor is given by the state-space representation in (20):

$$\dot{x} = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 & f_i & -f_i \\ 0 & 1 & 0 & 0 \end{bmatrix} x. \tag{20}$$

Consider the class of three MIMO strictly-proper plants $\mathcal{G}^{1\infty} = \{G_i : f_i = 1, 2, 3\}$. The transfer-matrix is obtained as a function of f_i:

$$G_i = \frac{1}{d} \begin{bmatrix} g_{11} + f_i h_{11} & g_{12} + f_i h_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

where

$d = s^4 + 11.6680 s^3 + 15.7538 s^2 - 88.2911 s + 5.5406$,
$g_{11} = 0.0008 s^2 + 29.2256 s + 233.6673$,
$g_{12} = -(21.1254 s^2 + 111.0942 s + 26.2766)$,
$g_{21} = 5.6790 s^3 + 42.6665 s^2 - 68.8304 s - 106.8024$,
$g_{22} = 9.4304 s + 15.1503$, $h_{11} = 0.0008 s + 29.7745$,
$h_{12} = -3.1460\, s^3 - 11.5490 s^2 + 21.2688 s - 5.5279$.

The poles of G_i are $\{1.9910, 0.0635, -5.0566, -8.6659\}$. With $Y_i(\infty) = \left[\begin{smallmatrix} 0 & 1/5.679 \\ -1/3.146f_i & 0 \end{smallmatrix}\right]$, the sufficient condition on W_i in (13) for Proposition 2(i) is satisfied since $f_i > 0$ for all $G_i \in \mathcal{G}^{1\infty}$. Choosing $G_o = G_1$ corresponding to $f_1 = 1$ as the nominal plant, we have $Y_o(\infty)^{-1} = \left[\begin{smallmatrix} 0 & -3.146 \\ 5.679 & 0 \end{smallmatrix}\right]$ and $W_i = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1/f_i \end{smallmatrix}\right]$. We choose $K_D = 0$ and compute

$$\beta_\infty = \max_{i=1,2,3} \|\Phi_i\| = \max\{13.9450, 20.2716, 32.1792\}.$$

Choosing $\beta = 46 > \beta_\infty$, the proportional controller $C_p = \beta Y_o(\infty) = \left[\begin{smallmatrix} 0 & 8.10 \\ -14.6217 & 0 \end{smallmatrix}\right]$ as in (16) strongly simultaneously stabilises the plants in $\mathcal{G}^{1\infty}$. For a PI controller, we choose $g = 2$. Then by (15), $\rho_\infty = \max_{i=1,2,3} \|\Psi_i\| = \max\{5.0585, 6.7951, 8.7595\}$. Choosing $\rho = 46 > \rho_\infty$, the PI controller $C_{PI} = (1 + \frac{g}{s})\rho Y_o(\infty) = (1 + \frac{2}{s})C_p$ as in (17) simultaneously stabilises the plants in $\mathcal{G}^{1\infty}$. Other values for the various parameters can be chosen based on additional performance specifications to be satisfied in addition to closed-loop stability and asymptotic tracking of constant reference inputs.

3.3 *Simultaneous controller synthesis for plants with two zeros at infinity*

Consider a finite set $\mathcal{G}^{2\infty}$ of plants that all have *exactly* two blocking-zeros at infinity and no other (transmission and blocking) $\mathcal{U}$-zeros. There is an arbitrary number of plants in this set and they may have poles anywhere in $\mathbb{C}$. These plants may have (transmission and blocking) zeros anywhere in $\mathbb{C}\backslash\mathcal{U}$. There may be any number of stable as well as unstable plants in the set under consideration. In the SISO case, the relative degree is two for these plants. More specifically, what is meant by two blocking-zeros at infinity for the MIMO case is that the plants $G_i \in \mathcal{G}^{2\infty}$ are expressed as

$$G_i = Y_i^{-1} \frac{1}{(s+a)^2} I = \left[\frac{1}{(s+a)^2} G_i^{-1}\right]^{-1} \left[\frac{1}{(s+a)^2} I\right], \tag{21}$$

where $G_i = Y_i^{-1} X$ is an LCF, with $X = \frac{1}{(s+a)^2} I$ for each G_i, and $Y_i = \frac{1}{(s+a)^2} G_i^{-1}$ is stable for any $a \in \mathbb{R}_+$.

Each *individual* plant G_i as described in (21) is PD and PID stabilisable (Gündeş and Özgüler 2007). However, existence of a single integral-action controller that simultaneously stabilises all plants $G_i \in \mathcal{G}^{2\infty}$ requires additional conditions. For $G_i \in \mathcal{G}^{2\infty}$, let

$$Y_i(\infty)^{-1} = [(s+a)^2 G_i(s)]|_{s\to\infty}. \tag{22}$$

Designate an arbitrary plant $G_o = Y_o^{-1} X \in \mathcal{G}^{2\infty}$ as the nominal plant, with $Y_o(\infty)^{-1} = [(s+a)^2 G_o(s)]|_{s\to\infty}$. It is shown in Proposition 3 that the plants in $\mathcal{G}^{2\infty}$ are simultaneously PID stabilisable if all $Y_i(\infty)$ are equal for all $G_i \in \mathcal{G}^{2\infty}$, i.e.

$$W_i = Y_i(\infty)\, Y_o(\infty)^{-1} = (G_i^{-1} G_o)(\infty) = I. \tag{23}$$

Furthermore, the finite set $\mathcal{G}^{2\infty} \cup \mathcal{G}^{no}$ consisting of plants with two blocking-zeros at infinity as in (21) and plants with no $\mathcal{U}$-zeros is also strongly simultaneously stabilisable using PID controllers under the same assumptions for $G_i \in \mathcal{G}^{2\infty}$ (no additional conditions are required for the plants $G_k \in \mathcal{G}^{no}$). Proposition 3(i) presents a systematic simultaneously stabilising PID controller synthesis procedure for the plant class $\mathcal{G}^{2\infty}$ and Proposition 3(ii) extends the procedure to the combined class $\mathcal{G}^{2\infty} \cup \mathcal{G}^{no}$.

Proposition 3 (PID synthesis for simultaneous stabilisation of $\mathcal{G}^{2\infty}$ and of $\mathcal{G}^{2\infty} \cup \mathcal{G}^{no}$)**:**

(i) *Let $\mathcal{G}^{2\infty}$ be a finite set of plants as in* (21). *Let $G_o \in \mathcal{G}^{2\infty}$ be an arbitrary member designated as the nominal plant. With $Y_i(\infty)$ as in* (22), *let $W_i = I$ as in* (23) *for all $G_i \in \mathcal{G}^{2\infty}$. Under these assumptions, simultaneously stabilising PID controllers exist for all plants $G_i \in \mathcal{G}^{2\infty}$, and can be designed as follows*:

Choose any $z_1, z_2 \in \mathbb{R}$. Define $\Gamma_i \in \mathcal{M}(\mathbf{S})$ and $\mu_\infty \in \mathbb{R}_+$, as in (24):

$$\begin{aligned} \Gamma_i &:= \frac{s}{(s+z_1)(s+z_2)} G_i^{-1} Y_o(\infty)^{-1} - sI, \\ \mu_\infty &:= 2 \max_{G_i \in \mathcal{G}^{2\infty}} \|\Gamma_i\|. \end{aligned} \tag{24}$$

For any $\mu > \mu_\infty$, the PID controller C_{PID} given by (25) *simultaneously stabilises all plants $G_i \in \mathcal{G}^{2\infty}$*:

$$C_{\mathrm{PID}} = \frac{\mu^2 (s+z_1)(s+z_2)}{s\,(s+2\mu)}\, Y_o(\infty). \tag{25}$$

(ii) *Let $\mathcal{G}^{2\infty}$ be a finite set of plants as in* (21). *Let $\mathcal{G}^{no}$ be a finite set of plants that have no $\mathcal{U}$-zeros as in Proposition* 1. *Under the assumptions and definitions of part* (i) *above, simultaneously stabilising PID controllers exist for all plants $G_i, G_k \in \mathcal{G}^{2\infty} \cup \mathcal{G}^{no}$, and can be designed as follows*: *Let $G_o \in \mathcal{G}^{2\infty}$ and Y_o be as in part* (i) *above. Without loss of generality, suppose that $z_1 = \max\{z_1, z_2\}$. Define $\Gamma_i \in \mathcal{M}(\mathbf{S})$, $\mu_\infty \in \mathbb{R}_+$, as in* (24). *For $G_k \in \mathcal{G}^{no}$, define $\tilde{\Theta}_k \in \mathcal{M}(\mathbf{S})$, $\mu_n \in \mathbb{R}_+$ as in* (26):

$$\tilde{\Theta}_k = \frac{s}{s+z_2} G_k^{-1} Y_o(\infty)^{-1}, \qquad \mu_n := \frac{2}{z_1} \max_{G_k \in \mathcal{G}^{no}} \|\tilde{\Theta}_k\|. \tag{26}$$

For any $\mu > \max\{0.5z_1, \mu_\infty, \mu_n\}$, the PID controller C_{PID} given by (25) *simultaneously stabilises all plants $G_i, G_k \in \mathcal{G}^{2\infty} \cup \mathcal{G}^{no}$.*

Example 4: Consider the class of seven MIMO strictly-proper plants $\mathcal{G}^{2\infty} = \{G_i,\ i = 1, \ldots, 7\}$:

$$G_i = \begin{bmatrix} \frac{-(s-4)}{d_i} & \frac{s(s+5)}{(s-1)d_i} \\ \frac{-10}{d_i} & \frac{-(s+5)}{d_i} \end{bmatrix}, \tag{27}$$
$$d_i = (s - p_i)[(s - a_i)^2 + b_i^2], \quad i = 1, \ldots, 7,$$

where $p_1 = 5,\ a_1 = 1,\ b_1 = 2;\ p_2 = 6,\ a_2 = 2,\ b_2 = 3;\ p_3 = 6,\ a_3 = -2,\ b_3 = 3;\ p_4 = -6,\ a_4 = -2,\ b_4 = 3;\ p_5 = 7,\ a_5 = 3,\ b_5 = 4;\ p_6 = 7,\ a_6 = -3,\ b_6 = 4;\ p_7 = -7,\ a_7 = 3,\ b_7 = 4$. All of these plants are unstable with poles at $\{1, p_i, a_i \pm jb_i\}$, with G_4 having only one $\mathcal{U}$-pole at $s = 1$. Any number of other choices for the poles at $p_i, a_i \pm jb_i$ can be considered for a larger plant set. These plants are in the form of (21), where, for any $a \in \mathbb{R}_+$,

$$G_i = Y_i^{-1} \frac{1}{(s+a)^2} I = \left(\frac{d_i}{(s+1)(s+4)(s+a)^2} \begin{bmatrix} -(s-1) & -s \\ \frac{10(s-1)}{(s+5)} & \frac{-(s-1)(s-4)}{(s+5)} \end{bmatrix} \right)^{-1} \times \frac{1}{(s+a)^2} I. \tag{28}$$

With $Y_i(\infty) = \left[\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right]$ for all $i = 1, \ldots, 7$, the sufficient condition for Proposition 3 is satisfied since all $W_i = I$. Choosing $z_1 = 6$, $z_2 = 9$ completely arbitrarily, from (24) we compute

$$\begin{aligned} \mu_\infty &= \max_{i=1,\ldots,7} \|\Gamma_i\| \\ &= \max\{32.5723, 35.5196, 27.6902, 16.3810, \\ &\qquad 38.4762, 26.720, 24.7885\} \\ &= 38.4762. \end{aligned}$$

Choosing $\mu = 100 > 2\mu_\infty$, we obtain the PID controller as in (25):

$$C_{\text{PID}} = \frac{100(s+6)(s+9)}{s\,(s+200)} \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}. \tag{29}$$

3.4 *Simultaneous controller synthesis for plants with one non-negative zero*

Consider a finite set $\mathcal{G}^z$ of plants that all have *exactly* one blocking-zero at $s = z \in \{\mathbb{R}_+ \cup 0\}$ and no other (transmission and blocking) $\mathcal{U}$-zeros. There is an arbitrary number of plants in this set and they may have poles anywhere in $\mathbb{C}$. These plants may have (transmission and blocking) zeros anywhere in $\mathbb{C} \setminus \mathcal{U}$. There may be any number of stable as well as unstable plants in the set under consideration. In the SISO case, the relative degree is zero for these plants since they have no zeros at infinity. More specifically, what is meant by one blocking-zero at $s = z$ for the MIMO case is that the plants $G_i \in \mathcal{G}^z$ are expressed as

$$G_i = Y_i^{-1} \frac{(s-z)}{a(s-z)+1} I = \left[\frac{(s-z)}{a(s-z)+1} G_i^{-1} \right]^{-1} \left[\frac{(s-z)}{a(s-z)+1} I \right], \tag{30}$$

where $G_i = Y_i^{-1} X$ is an LCF, with $X = \frac{(s-z)}{a(s-z)+1} I$ for each G_i, and $Y_i = \frac{(s-z)}{a(s-z)+1} G_i^{-1}$ is stable for any $a \in \mathbb{R}_+$, $a < z^{-1}$. If $z = 0$, then (30) becomes

$$G_i = Y_i^{-1} \frac{s}{as+1} I = \left[\frac{(s-z)}{as+1} G_i^{-1} \right]^{-1} \left[\frac{s}{as+1} I \right], \tag{31}$$

where $G_i = Y_i^{-1} X$ is an LCF, with $X = \frac{s}{as+1} I$ for each G_i, and $Y_i = \frac{s}{as+1} G_i^{-1}$ is stable for any $a \in \mathbb{R}_+$. Each *individual* plant G_i as described in (30) or (31) is PD stabilisable under certain sufficient conditions on the location of the zero at $s = z \geq 0$. Existence of a single PD controller that simultaneously stabilises all plants $G_i \in \mathcal{G}^z$ requires additional conditions. For $G_i \in \mathcal{G}^z$, let

$$Y_i(z)^{-1} = \left[\frac{1}{(s-z)} G_i(s) \right] \Big|_{s=z}. \tag{32}$$

If $z = 0$, then (32) becomes $Y_i(0)^{-1} = [\frac{1}{s} G_i(s)]|_{s=0}$. Designate an arbitrary plant $G_o = Y_o^{-1} X \in \mathcal{G}^z$ as the nominal plant, with $Y_o(z)^{-1} = [\frac{1}{(s-z)} G_o(s)]|_{s=z}$. Define $Z_i \in \mathbb{R}^{m \times m}$ as

$$Z_i := Y_i(z)\, Y_o(z)^{-1} = (G_i^{-1} G_o)(z). \tag{33}$$

It is shown in Proposition 4 that the plants in $\mathcal{G}^z$ are strongly simultaneously stabilisable using PD controllers if $Z_i = I$ when $z > 0$, and the plants in $\mathcal{G}^z$ are strongly simultaneously stabilisable using PD controllers if all eigenvalues of Z_i are real and positive when $z = 0$, for all $G_i \in \mathcal{G}^z$. Furthermore, the finite set $\mathcal{G}^z \cup \mathcal{G}^{no}$ consisting of plants with one blocking-zero at $s = z$ as in (30) and plants with no $\mathcal{U}$-zeros is also strongly simultaneously stabilisable using PD controllers under the same assumptions for $G_i \in \mathcal{G}^z$. Proposition 4(i) presents a systematic simultaneously stabilising PD controller synthesis procedure for the plant class $\mathcal{G}^z$ and Proposition 4(ii) extends the procedure to the combined class $\mathcal{G}^z \cup \mathcal{G}^{no}$.

Proposition 4 (PD synthesis for simultaneous stabilisation of $\mathcal{G}^z$ and of $\mathcal{G}^z \cup \mathcal{G}^{no}$)**:** *Let $\mathcal{G}^z$ be a finite set of plants as in* (30). *Let $G_o \in \mathcal{G}^z$ be an arbitrary member designated as the nominal plant. Let $Y_i(z)$ be as in* (32) *and Z_i be as in* (33).

(i) (a) *If $z > 0$, suppose that $Z_i = I$ for all $G_i \in \mathcal{G}^z$. Under these assumptions, simultaneously stabilising PD controllers exist for all plants $G_i \in \mathcal{G}^z$, and can be* *designed as follows*: *choose any* $K_D \in \mathbb{R}^{m\times m}$, $\tau \in \mathbb{R}_+$. *Define* $\Lambda_i \in \mathcal{M}(\mathbf{S})$ *and* $\varphi_z \in \mathbb{R}_+$ *as in* (34):

$$\Lambda_i := \frac{1}{(s-z)}[\,(s-z)G_i^{-1}Y_o(z)^{-1} - I\,] + \frac{s}{\tau s+1}K_D Y_o(z)^{-1}, \qquad \varphi_z := \min_{G_i \in \mathcal{G}^z} \|\Lambda_i\|^{-1}, \tag{34}$$

If

$$z < \frac{\varphi_z}{2}, \tag{35}$$

then for any $\varphi \in \mathbb{R}_+$, $\varphi < (\varphi_z - z)$, *the PD controller* C_{PD} *given by* (36) *simultaneously stabilises all plants* $G_i \in \mathcal{G}^z$:

$$C_{\mathrm{PD}} = \frac{1}{z+\varphi}\,Y_o(z) + \frac{s}{\tau s+1}K_D. \tag{36}$$

(b) *If* $z=0$, *suppose that all eigenvalues of* Z_i *are real and positive for all* $G_i \in \mathcal{G}^z$. *Under these assumptions, simultaneously stabilising PD controllers exist for all plants* $G_i \in \mathcal{G}^z$, *and can be designed as follows*: *Choose any* $K_D \in \mathbb{R}^{m\times m}$, $\tau \in \mathbb{R}_+$. *Define* $\Lambda_i \in \mathcal{M}(\mathbf{S})$ *and* $\varphi_0 \in \mathbb{R}_+$ *as in* (37):

$$\Lambda_i^0 := \frac{1}{s}[\,sG_i^{-1}Y_o(z)^{-1} - Z_i\,] + \frac{s}{\tau s+1}K_D Y_o(z)^{-1}, \qquad \varphi_0 := \min_{G_i \in \mathcal{G}^z} \|\Lambda_i^0\|^{-1}, \tag{37}$$

For any $\varphi \in \mathbb{R}_+$, $\varphi < \varphi_0$, *the PD controller* C_{PD} *given by* (38) *simultaneously stabilises all plants* $G_i \in \mathcal{G}^z$:

$$C_{\mathrm{PD}} = \frac{1}{\varphi}\,Y_o(0) + \frac{s}{\tau s+1}K_D. \tag{38}$$

(ii) *Let* $\mathcal{G}^z$ *be a finite set of plants as in* (30). *Let* $\mathcal{G}^{no}$ *be a finite set of plants that have no* $\mathcal{U}$-*zeros as in Proposition* 1. *Under the assumptions and definitions of part* (i) *above, simultaneously stabilising PD controllers exist for all plants* G_i, $G_k \in \mathcal{G}^z \cup \mathcal{G}^{no}$, *and can be designed as follows*: *Let* $G_o \in \mathcal{G}^z$ *and* Y_o *be as in part* (i) *above. Choose any* $K_D \in \mathbb{R}^{m\times m}$, $\tau \in \mathbb{R}_+$. *If* $z>0$, *define* $\Lambda_i \in \mathcal{M}(\mathbf{S})$ *and* $\varphi_z \in \mathbb{R}_+$ *as in* (34); *if* $z=0$, *define* $\Lambda_i^0 \in \mathcal{M}(\mathbf{S})\}$ *and* $\varphi_0 \in \mathbb{R}_+$ *as in* (37). *For* $G_k \in \mathcal{G}^{no}$, *let* $\hat{\Theta}$ *and* α_z *be defined as*

$$\hat{\Theta} := \left(G_k^{-1} + \frac{s}{\tau s+1}K_D\right)Y_o(z)^{-1}, \qquad \alpha_z := \min_{G_k \in \mathcal{G}^{no}} \|\,\hat{\Theta}_k\,\|^{-1}. \tag{39}$$

If z *satisfies* (35) *and*

$$z < \alpha_z, \tag{40}$$

then for any $\varphi \in \mathbb{R}_+$, $\varphi < \min\{\varphi_z - z,\ \alpha_z - z\}$, *the PD controller* C_{PD} *given by* (36) *simultaneously stabilises all plants* $G_i, G_k \in \mathcal{G}^z \cup \mathcal{G}^{no}$. *If* $z=0$, *then for any* $\varphi \in \mathbb{R}_+$, $\varphi < \min\{\varphi_0, \alpha_z\}$, *the PD controller* C_{PD} *given by* (38) *simultaneously stabilises all plants* G_i, $G_k \in \mathcal{G}^z \cup \mathcal{G}^{no}$.

In the PD controller (36) for $z>1$, and the PD controller (38) for $z=0$, the choice of the derivative constant matrix K_D is completely free. For $K_D=0$, (36) and (38) are proportional controllers.

Remarks (Robustness of the simultaneously stabilising PD/PID controllers)**:** It follows from standard robustness arguments that the simultaneously stabilising PD and PID controllers in Propositions 1 and 2, the simultaneously stabilising PID controller in Propositions 3, and the PD controller in Propositions 4 all achieve robust simultaneous stability under 'sufficiently small' plant uncertainty for the plant classes considered in those propositions. Let $\Delta \in \mathbf{S}^{m\times m}$ be a stable additive perturbation. For the plant class $\mathcal{G}^{no}$, the PD controller C_{PD} in (5) and the PID controller C_{PID} in (6) of Proposition 1 simultaneously stabilise all $G_k \in \mathcal{G}^{no}$. The PD controller C_{PD} and the PID controller C_{PID} also robustly simultaneously stabilise the additively perturbed plants $G_k + \Delta_k$ for all $\Delta_k \in \mathbf{S}^{m\times m}$ such that

$$\|\Delta_k\| < \|\,C_{\mathrm{PD}}(I+G_kC_{\mathrm{PD}})^{-1}\,\|^{-1}, \qquad \|\Delta_k\| < \|\,C_{\mathrm{PID}}(I+G_kC_{\mathrm{PID}})^{-1}\,\|^{-1}, \tag{41}$$

respectively. For multiplicative perturbations, the PD controller C_{PD} and the PID controller C_{PID} robustly simultaneously stabilise the plants $G_k(I+\Delta_k)$ under all pre-multiplicative perturbations $\Delta_k \in \mathbf{S}^{m\times m}$ such that

$$\|\Delta_k\| < \|C_{\mathrm{PD}}G_k(I+C_{\mathrm{PD}}G_k)^{-1}\|^{-1}, \qquad \|\,\Delta_k\| < \|C_{\mathrm{PID}}G_k(I+C_{\mathrm{PID}}G_k)^{-1}\|^{-1}, \tag{42}$$

respectively. Similarly, the PD controller C_{PD} and the PID controller C_{PID} robustly simultaneously stabilise the plants $(I+\Delta_k)G_k$ under all post-multiplicative perturbations $\Delta_k \in \mathbf{S}^{m\times m}$ such that

$$\|\Delta_k\| < \|G_kC_{\mathrm{PD}}(I+G_kC_{\mathrm{PD}})^{-1}\|^{-1}, \qquad \|\Delta_k\| < \|G_kC_{\mathrm{PID}}(I+G_kC_{\mathrm{PID}})^{-1}\|^{-1}, \tag{43}$$

respectively. Some of the free parameter choices in the proposed controller synthesis method may be used to maximise the allowable perturbation magnitudes in (41), (42) or (43). For example, to maximise $\|\Delta_k\|$ in (41), the choice of the parameters $\hat{K}_p$, K_D should then be formulated into an H_∞ problem to minimise the norms $\|C_{\mathrm{PD}}(I+G_kC_{\mathrm{PD}})^{-1}\|$, $\|C_{\mathrm{PID}}(I+G_kC_{\mathrm{PID}})^{-1}\|$.

Entirely similar robust stability conclusions apply to the plant class $\mathcal{G}^{1\infty}$. The PD controller C_{PD} in (16) and the PID controller C_{PID} in (17) of Proposition 2, which simultaneously stabilise all $G_i \in \mathcal{G}^{1\infty}$, also robustly simultaneously stabilise the additively perturbed plants $G_i + \Delta_i$ for all $\Delta_i \in \mathbf{S}^{m\times m}$ such that $\|\Delta_i\| < \|C_{\mathrm{PD}}(I + G_i C_{\mathrm{PD}})^{-1}\|^{-1}$, $\|\Delta_i\| < \|C_{\mathrm{PID}}(I + G_i C_{\mathrm{PID}})^{-1}\|^{-1}$, respectively. For multiplicative perturbations, the PD controller C_{PD} and the PID controller C_{PID} robustly simultaneously stabilise the plants $G_i(I + \Delta_i)$ under all pre-multiplicative perturbations $\Delta_i \in \mathbf{S}^{m\times m}$ such that $\|\Delta_i\| < \|C_{\mathrm{PD}} G_i(I + C_{\mathrm{PD}} G_i)^{-1}\|^{-1}$, $\|\Delta_i\| < \|C_{\mathrm{PID}} G_i(I + C_{\mathrm{PID}} G_i)^{-1}\|^{-1}$, respectively. Similarly, the PD controller C_{PD} and the PID controller C_{PID} robustly simultaneously stabilise the plants $(I + \Delta_i)G_i$ under all post-multiplicative perturbations $\Delta_i \in \mathbf{S}^{m\times m}$ such that $\|\Delta_i\| < \|G_i C_{\mathrm{PD}}(I + G_i C_{\mathrm{PD}})^{-1}\|^{-1}$, $\|\Delta_i\| < \|G_i C_{\mathrm{PID}}(I + G_i C_{\mathrm{PID}})^{-1}\|^{-1}$, respectively.

For the plant class $\mathcal{G}^{2\infty}$, Proposition 3 proposes simultaneously stabilising PID controllers C_{PID} in (25), which robustly simultaneously stabilise the additively perturbed plants $G_i + \Delta_i$ for all $\Delta_i \in \mathbf{S}^{m\times m}$ such that $\|\Delta_i\| < \|C_{\mathrm{PID}}(I + G_i C_{\mathrm{PID}})^{-1}\|^{-1}$. They robustly simultaneously stabilise the pre-multiplicatively perturbed plants $G_i(I + \Delta_i)$ under all $\Delta_i \in \mathbf{S}^{m\times m}$ such that $\|\Delta_i\| < \|C_{\mathrm{PID}} G_i(I + C_{\mathrm{PID}} G_i)^{-1}\|^{-1}$, and the post-multiplicatively perturbed plants $(I + \Delta_i)G_i$ under all $\Delta_i \in \mathbf{S}^{m\times m}$ such that $\|\Delta_i\| < \|G_i C_{\mathrm{PID}}(I + G_i C_{\mathrm{PID}})^{-1}\|^{-1}$. The PD controllers in Proposition 4 also achieve similar robust simultaneous stabilisation. □

4. Conclusions

This article identified some important plant classes such that any finite number of plants can be simultaneously stabilised using a common PD controller or PID controller. The plant classes considered here have restrictions on the zeros in the region of instability. These restrictions are due to the difficulties involving simultaneous stabilisation of three or more plants with order-restricted controllers whose only unstable pole may be at the origin (PID controller).

The plant classes with at most one blocking-zero at infinity can be simultaneously stabilised using both PD and PID controllers. The existence of PD controllers implies that these plants are strongly simultaneously stabilisable. The existence of simultaneously stabilising PID controllers implies that asymptotic tracking of constant reference inputs is achieved with zero steady-state error. The plants with two blocking-zeros at infinity can be simultaneously PID stabilised. The synthesis method proposed here does not determine existence of PD controllers for this plant class. The class of plants with one (small) non-negative real-axis zero can be simultaneously stabilised using PD controllers. If the zero is at the origin, then these plants would not allow integral-action controllers. Systematic synthesis procedures are proposed for each plant class, where the PD/PID parameters and the design choices are explicitly defined. The proposed designs allow freedom in the parameters, which should be used to satisfy additional performance criteria that the design may require. In each of the illustrative examples, we selected a set of parameters out of infinitely many satisfying the conditions of Propositions 1, 2 and 3. These parameter choices resulted in closed-loop poles in the left-half plane sufficiently far from the origin. While asymptotic tracking of constant reference inputs is achieved with the PID controllers due to the integral term, performance objectives beyond tracking (and equivalently disturbance rejection) were not considered within the scope of this work. The goal of this study was to establish simultaneous stabilisability using PD/PID controllers, and it was shown that these controllers achieve robust stability under sufficiently small additive and multiplicative plant uncertainty.

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Appendix (Proofs)

Proof of Proposition 1: Let $C_{\rm PD}$ be as in (5). For $G_k \in \mathcal{G}^{no}$ we can write $G_k = Y_k^{-1}X = (G_k^{-1})^{-1}I$. So with $X_k = I$, $C_{\rm PD} = {\rm ND}^{-1} = C_{\rm PD}I^{-1}$, (2) becomes

$$M_k = Y_k + XC_{\rm PD} = G_k^{-1} + \alpha\hat{K}_P + \frac{s}{\tau s+1}K_D$$
$$= \left[I + \frac{1}{\alpha}\left(G_k^{-1} + \frac{s}{\tau s+1}K_D\right)\hat{K}_P^{-1}\right]\alpha\hat{K}_P = \left[I + \frac{1}{\alpha}\Theta_k\right]\alpha\hat{K}_P.$$

For $\alpha > \alpha_n > 0$, $\| \frac{1}{\alpha}\Theta_k \| < 1$ implies that M_k is unimodular (since $\hat{K}_P$ is non-singular); hence, $C_{\rm PD}$ stabilises all G_k. Let $C_{\rm PID}$ be as in (6) and write $C_{\rm PID} = {\rm ND}^{-1}$ as

$$C_{\rm PID} = ND^{-1} = \left[\frac{s}{s+e}\left(K_P + \frac{s}{\tau s+1}K_D\right) + \frac{1}{s+e}K_I\right]\left[\frac{sI}{s+e}\right]^{-1}$$
$$= \left[\frac{s}{s+e}C_{\rm PID}\right]\left[\frac{s}{s+e}I\right]^{-1}. \tag{44}$$

Then (2) becomes

$$M_k = \frac{s}{s+e}Y_k + \frac{s}{s+e}XC_{\rm PID}$$
$$= \frac{s}{s+e}\left[\left(1+\frac{g}{s}\right)\alpha\hat{K}_P + \frac{s}{\tau s+1}K_D + G_k^{-1}\right]$$
$$= \frac{(s+g)}{(s+e)}\alpha\hat{K}_P + \frac{s}{s+e}\left(G_k^{-1} + \frac{s}{\tau s+1}K_D\right)$$
$$= \left[I + \frac{1}{\alpha}\frac{s}{(s+g)}\left(G_k^{-1} + \frac{s}{\tau s+1}K_D\right)\hat{K}_P^{-1}\right]\alpha\frac{(s+g)}{(s+e)}\hat{K}_P$$
$$= \left[I + \frac{1}{\alpha}\frac{s}{(s+g)}\Theta_k\right]\alpha\frac{(s+g)}{(s+e)}\hat{K}_P.$$

For $\alpha > \alpha_n$, $\| \frac{1}{\alpha}\frac{s}{(s+g)}\Theta_k \| \le \frac{1}{\alpha}\| \frac{s}{(s+g)} \| \|\Theta_k\| \le \frac{1}{\alpha}\|\Theta_k\| < 1$ implies that M_k is unimodular; hence, $C_{\rm PID} = C_{\rm PD} + \frac{\alpha g}{s}\hat{K}_P$ stabilises all G_k.

Proof of Proposition 2: (i) Let $C_{\rm PD}$ be as in (16). For $G_i \in \mathcal{G}^{1\infty}$ as in (11), with $X = \frac{1}{s+a}I$, $C_{\rm PD} = ND^{-1} = C_{\rm PD}I^{-1}$, (2) becomes

$$M_i = Y_i + XC_{\rm PD} = Y_i + X\beta Y_o(\infty) + X\frac{s}{\tau s+1}K_D$$
$$= \frac{(W_i s + \beta I)}{(s+a)}[(W_i s+\beta I)^{-1}\beta I + (W_i s + \beta I)^{-1}((s+a)Y_i$$
$$+ \frac{s}{\tau s+1}K_D)Y_o(\infty)^{-1}]Y_o(\infty)$$
$$= \frac{(W_i s+\beta I)}{(s+a)}[I + (W_i s + \beta I)^{-1}\Phi_i]Y_o(\infty).$$

By assumption, W_i has real positive eigenvalues implying that $(W_i s + \beta I)^{-1} \in \mathcal{M}(\mathbf{S})$; then $\|(W_i s+\beta I)^{-1}\| = 1/\beta$. If $\beta > \beta_\infty$, then

$$\|(W_i s + \beta I)^{-1}\Phi_i\| \le \|(W_i s+\beta I)^{-1}\|\|\Phi_i\| = \frac{1}{\beta}\|\Phi_i\| < 1$$

implies that M_i is unimodular; hence, $C_{\rm PD}$ stabilises all G_i. Let $C_{\rm PID}$ be as in (17) and write $C_{\rm PID} = [\frac{s}{s+e}C_{\rm PID}][\frac{s}{s+e}I]^{-1}$ as in (44). Then (2) becomes

$$M_i = \frac{s}{s+e}Y_i + \frac{s}{s+e}XC_{\rm PID}$$
$$= \frac{(s+g)}{(s+a)}\frac{s}{(s+e)}\left[\frac{1}{(s+g)}\left(K_P + \frac{g}{s}K_P + \frac{s}{\tau s+1}K_D + (s+a)Y_i\right)\right]$$
$$= \frac{(s+g)}{(s+a)}\left[\frac{\rho}{s+e}Y_o(\infty) + \frac{1}{(s+g)}\frac{s}{(s+e)}(G_i^{-1} + \frac{s}{\tau s+1}K_D)\right]$$
$$= \frac{(s+g)}{(s+a)(s+e)}(W_i s + \rho I)$$
$$\times\left[(W_i s + \rho I)^{-1}\rho I + (W_i s+\rho I)^{-1}\left(G_i^{-1} + \frac{s}{\tau s+1}K_D\right)\right.$$
$$\left.\times\frac{s}{(s+g)}Y_o(\infty)^{-1}\right]Y_o(\infty)$$
$$= \frac{(s+g)}{(s+a)(s+e)}(W_i s+\rho I)[I + (W_i s + \rho I)^{-1}\Psi_i]Y_o(\infty).$$

Following similar steps as for $C_{\rm PD}$, $(W_i s + \rho I)^{-1} \in \mathcal{M}(\mathbf{S})$; then $\|(W_i s+\rho I)^{-1}\| = 1/\rho$. If $\rho > \rho_\infty$, then $\|(W_i s + \rho I)^{-1}\Psi_i\| \le \|(W_i s+\rho I)^{-1}\|\|\Psi_i\| = \frac{1}{\rho}\|\Psi_i\| < 1$ implies that M_i is unimodular; hence, $C_{\rm PID}$ stabilises all G_i. (ii) By (i), the controllers $C_{\rm PD}$ and $C_{\rm PID}$ in (16) and (17) stabilise all $G_i \in \mathcal{G}^{1\infty}$. It remains to show that they also stabilise all

$G_k \in \mathcal{G}^{no}$. For C_{PD} in (16), (2) becomes

$$\begin{aligned} M_k &= Y_k + XC_{\mathrm{PD}} = G_k^{-1} + \beta Y_o(\infty) + \frac{s}{\tau s+1} K_D \\ &= \left[I + \frac{1}{\beta}\left(G_k^{-1} + \frac{s}{\tau s+1} K_D \right) Y_o(\infty)^{-1} \right] \beta Y_o(\infty) \\ &= \left[I + \frac{1}{\beta} \Theta_k \right] \beta Y_o(\infty). \end{aligned}$$

For $\beta > \alpha_n$, $\|\frac{1}{\beta}\Theta_k\| < 1$ implies M_k, is unimodular; hence, C_{PD} stabilises all G_k. For C_{PID} in (17),

$$\begin{aligned} M_k &= Y_k \frac{s}{s+e} + XC_{\mathrm{PID}} \frac{s}{s+e} \\ &= \frac{s}{s+e}\left[\left(1 + \frac{g}{s}\right) \rho Y_o(\infty) + \frac{s}{\tau s+1} K_D + G_k^{-1} \right] \\ &= \frac{(s+g)}{(s+e)} \rho Y_o(\infty) + \frac{s}{s+e}\left(G_k^{-1} + \frac{s}{\tau s+1} K_D \right) \\ &= \left[I + \frac{1}{\rho}\frac{s}{(s+g)}\left(G_k^{-1} + \frac{s}{\tau s+1} K_D \right) Y_o(\infty)^{-1} \right] \rho \frac{(s+g)}{(s+e)} Y_o(\infty) \\ &= \left[I + \frac{1}{\rho}\frac{s}{(s+g)} \Theta_k \right] \rho \frac{(s+g)}{(s+e)} Y_o(\infty). \end{aligned}$$

If $\rho > \rho_n$, then $\|\frac{1}{\rho}\frac{s}{(s+g)} \Theta_k\| < 1$ implies that M_k is unimodular; hence, C_{PID} stabilises all G_k. $\square$

Proof of Proposition 3: (i) Let C_{PID} be as in (25) and write $C_{\mathrm{PID}} = ND^{-1} = [\mu^2 Y_o(\infty)][\frac{s(s+2\mu)}{(s+z_1)(s+z_2)} I]^{-1}$. Then (2) becomes

$$\begin{aligned} M_i &= \frac{s(s+2\mu)}{(s+z_1)(s+z_2)} Y_i + \mu^2 X Y_o(\infty) \\ &= \frac{s(s+2\mu)}{(s+z_1)(s+z_2)} \frac{1}{(s+a)^2} G_i^{-1} + \mu^2 \frac{1}{(s+a)^2} Y_o(\infty) \\ &= \frac{(s+\mu)^2}{(s+a)^2}\left[\frac{\mu^2}{(s+\mu)^2} I + \frac{s(s+2\mu)}{(s+\mu)^2 (s+z_1)(s+z_2)} G_i^{-1} Y_o(\infty)^{-1} \right] \\ &\quad \times Y_o(\infty) \\ &= \frac{(s+\mu)^2}{(s+a)^2}\left[I + \frac{(s+2\mu)}{(s+\mu)^2}\left(\frac{s}{(s+z_1)(s+z_2)} G_i^{-1} Y_o(\infty)^{-1} - sI \right) \right] \\ &\quad \times Y_o(\infty) \\ &= \frac{(s+\mu)^2}{(s+a)^2}\left[I + \frac{(s+2\mu)}{(s+\mu)^2} \Gamma_i \right] Y_o(\infty). \end{aligned}$$

If $\mu > \mu_\infty$, then $\|\frac{(s+2\mu)}{(s+\mu)^2} \Gamma_i\| \le \|\frac{(s+2\mu)}{(s+\mu)^2}\| \|\Gamma_i\| = \frac{2}{\mu}\|\Gamma_i\| < 1$ implies M_i is unimodular; hence, C_{PID} stabilises all G_i. (ii) By (i), the controller C_{PID} in (25) stabilises all $G_i \in \mathcal{G}^{2\infty}$ for $\mu > \mu_\infty$. It remains to show that they also stabilise all $G_k \in \mathcal{G}^{no}$ for $\mu > \max\{0.5z_1, \mu_\infty, \mu_n\}$. Writing C_{PID} as in proof of (i) above, for $G_k \in \mathcal{G}^{no}$, M_k becomes

$$\begin{aligned} M_k &= \frac{s(s+2\mu)}{(s+z_1)(s+z_2)} Y_k + X\mu^2 Y_o(\infty) \\ &= \frac{s(s+2\mu)}{(s+z_1)(s+z_2)} G_k^{-1} + \mu^2 Y_o(\infty) \\ &= [I + \frac{(s+2\mu)}{\mu^2(s+z_1)} \frac{s}{(s+z_2)} G_k^{-1} Y_o^{-1}(\infty)] \mu^2 Y_o(\infty) \\ &= [I + \frac{(s+2\mu)}{\mu^2(s+z_1)} \tilde{\Theta}_k] \mu^2 Y_o(\infty). \end{aligned}$$

If $\mu > 0.5z_1$, then $\|\frac{(s+2\mu)}{\mu^2(s+z_1)}\| = 2(\mu z_1)^{-1}$. In addition, $\mu > \mu_n$, then $\|\frac{(s+2\mu)}{\mu^2(s+z_1)} \tilde{\Theta}_k\| \le \frac{2}{\mu z_1}\|\tilde{\Theta}_k\| < 1$ implies that M_k is unimodular; hence, C_{PID} stabilises all G_k. $\square$

Proof of Proposition 4:

(a) If $z > 0$, let C_{PD} be as in (36). For $G_i \in \mathcal{G}^z$ as in (30), with $X = \frac{(s-z)}{a(s-z)+1} I$, $C_{\mathrm{PD}} = \mathrm{ND}^{-1} = C_{\mathrm{PD}} I^{-1}$, (2) becomes

$$\begin{aligned} M_i &= Y_i + XC_{\mathrm{PD}} = Y_i + X\frac{1}{z+\varphi} Y_o(z) + X\frac{s}{\tau s+1} K_D \\ &= \frac{(s+\varphi)}{(a(s-z)+1)}\Big(\frac{(s-z)}{(s+\varphi)} I + \frac{(z+\varphi)}{(s+\varphi)}[(a(s-z)+1) Y_i \\ &\quad + \frac{(s-z)s}{\tau s+1} K_D] Y_o(z)^{-1} \Big) \frac{1}{(z+\varphi)} Y_o(z) \\ &= \frac{(s+\varphi)}{(a(s-z)+1)}\Big(I + \frac{(z+\varphi)(s-z)}{(s+\varphi)}\Big[\frac{(a(s-z)+1) Y_i Y_o(z)^{-1} - I}{(s-z)} \\ &\quad + \frac{s}{\tau s+1} K_D Y_o(z)^{-1} \Big] \Big) \frac{1}{(z+\varphi)} Y_o(z) \\ &= \frac{(s+\varphi)}{(a(s-z)+1)}\Big(I + \frac{(z+\varphi)(s-z)}{(s+\varphi)} \Lambda_i \Big) \frac{1}{(z+\varphi)} Y_o(z). \end{aligned}$$

For $\varphi > z$, $\|\frac{(z+\varphi)(s-z)}{(s+\varphi)}\| = (z+\varphi)$. If z satisfies assumption (35), then for $z < \varphi < \|\Lambda\|^{-1} - z$, we have $\|\frac{(z+\varphi)(s-z)}{(s+\varphi)} \Lambda_i\| \le \|\frac{(z+\varphi)(s-z)}{(s+\varphi)}\| \|\Lambda_i\| = (z+\varphi)\|\Lambda_i\| < 1$ which implies that M_i is unimodular; hence, C_{PD} stabilises all G_i.

(b) If $z = 0$, for $G_i \in \mathcal{G}^z$ as in (31), with $X = \frac{s}{as+1} I$, $C_{\mathrm{PD}} = ND^{-1} = C_{\mathrm{PD}} I^{-1}$, (2) becomes

$$\begin{aligned} M_i &= Y_i + XC_{\mathrm{PD}} = Y_i + X\frac{1}{\varphi} Y_o(0) + X\frac{s}{\tau s+1} K_D \\ &= \frac{(sI+\varphi Z_i)}{(as+1)}\big((sI+\varphi Z_i)^{-1} sI + (sI+\varphi Z_i)^{-1} \varphi[(as+1) Y_i \\ &\quad + \frac{s^2}{\tau s+1} K_D] Y_o(0)^{-1} \big) \frac{1}{\varphi} Y_o(0) \\ &= \frac{(sI+\varphi Z_i)}{(as+1)}\left(I + (sI+\varphi Z_i)^{-1} \varphi s\left[\frac{(as+1) Y_i Y_o(0)^{-1} - I}{s} \right.\right. \\ &\quad \left.\left. + \frac{s}{\tau s+1} K_D Y_o(0)^{-1} \right] \right) \frac{1}{\varphi} Y_o(0) \\ &= \frac{(sI+\varphi Z_i)}{(as+1)}(I + \varphi (sI+\varphi Z_i)^{-1} s \Lambda_i^0) Y_o(0). \end{aligned}$$

By assumption, Z_i has real positive eigenvalues implying $(sI+\varphi Z_i)^{-1} \in \mathcal{M}(\mathbf{S})$; then $\|\varphi(sI+\varphi Z_i)^{-1} s\| = \varphi$. If $\varphi > \varphi_0$, then

$$\|\varphi(sI+\varphi Z_i)^{-1} s \Lambda_i^0\| \le \|\varphi(sI+\varphi Z_i)^{-1} s\| \|\Lambda_i^0\| = \varphi \|\Lambda_i^0\| < 1$$

implies that M_i is unimodular; hence, C_{PD} stabilises all G_i. (ii) By (i), the controllers C_{PD} in (36) and (38) stabilise all $G_i \in \mathcal{G}^z$ for $z > 0$ and $z = 0$, respectively. It remains to show that they also stabilise all $G_k \in \mathcal{G}^{no}$. For C_{PD} in (36), (2) becomes

$$\begin{aligned} M_k &= Y_k + XC_{\mathrm{PD}} = G_k^{-1} + \frac{1}{z+\varphi} Y_o(z) + \frac{s}{\tau s+1} K_D \\ &= [I + (z+\varphi)(G_k^{-1} + \frac{s}{\tau s+1} K_D) Y_o(z)^{-1}] \frac{1}{(z+\varphi)} Y_o(z) \\ &= [I + (z+\varphi)\hat{\Theta}_k] \frac{1}{(z+\varphi)} Y_o(z). \end{aligned}$$

For $\varphi < \alpha_z - z$, $\|(z+\varphi)\hat{\Theta}_k\| < 1$ implies that M_k is unimodular; hence, C_{PD} stabilises all G_k. $\square$