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Simple Integral-Action Controllers for SISO and MIMO Systems

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Abstract— **A simple low order controller synthesis is developed for certain classes of linear, time-invariant, multi-input multi-output plants. The order of these controllers depends on the number of right-half plane plant poles rather than the order of the plant to be stabilized. Furthermore, the controller's poles and zeros are all in the stable region with the exception of one pole at the origin for the integral-action design requirement. The integral-action in the controller achieves asymptotic tracking of step input references with zero steady-state error. The freedom available in the design parameters may be used for additional performance objectives although the only goal here is stabilization and tracking of constant references.**

I. Introduction

In this paper we show that it is possible to design very simple low order controllers to stabilize linear time-invariant (LTI), multi-input multi-output (MIMO) plants that have restrictions on their zeros that lie in the region of instability. The pole locations are not restricted as well as the zeros that lie in the stable region. An additional objective is to design these LTI controllers with integral-action so that the closed-loop system achieves asymptotic tracking of constant reference inputs.

Controllers stabilizing a complex plant and achieving a specified performance are usually at least as complex as the plant itself [15]. The issues of computation and implementation of such controllers are dealt with in control system design using reduction approaches such as a) designing the high-order controller and then approximating it with a low-order one within an acceptable loss of performance; b) reducing the order of the plant model with the prospect that a low order model will lead to a low order controller (see e.g., [1], [2], [3], [5], [8], [10], [11], [12]). Model reduction is not the objective of this work; what is developed here is a direct simple controller design that stabilizes the original plant without reducing it.

Asymptotic tracking of constant reference inputs is achieved with poles duplicating the dynamic structure of the exogenous signals that the regulator has to process. Due to this internal model principle, integral-action controllers have poles at the origin of the complex plane [6]. The standard method of designing controllers with integral-action starts by augmenting the plant dynamics with an extra state, which is the integral of the output error. In the MIMO case with m inputs and outputs, the planted augmented by adding the integrators to the input of the plant then has m more states than the n states of the original plant. Using a full-order observer and state feedback to move the $(n+m)$ eigenvalues to the region of stability, the resulting $m \times m$ controller is always strictly-proper, has $(n+m)$ eigenvalues, with m at the origin and the remaining eigenvalues may be anywhere in the complex plane. For the class of plants we consider here, a simpler integral-action controller design can be achieved. The plants have no restrictions as far as the location of the poles are concerned (stable or unstable) and the zeros in the region of stability are also not restricted. However, we assume the zeros in the region of instability are on the positive real axis and have "large" magnitude (including infinity). The case where the unstable zeros are all at infinity is particularly interesting: Plants that have blocking-zeros at infinity of multiplicity r can be stabilized using the proposed design, which gives r-th order controllers. These $m \times m$ controllers have exactly rm eigenvalues, where m are at the origin and the remaining $(r-1)m$ are all in the region of stability (open left-half complex plane). Furthermore, they are bi-proper and they have stable inverse. For single-input single-output (SISO), the comparison is easy to illustrate. An SISO plant $(m = 1)$ with n poles that has zeros at infinity has relative degree $r < n$. The proposed r-th order integral-action controller design has r eigenvalues (with one at $s = 0$ and the rest in the open left-half complex plane), whereas full-order observer and state feedback design based on an augmented plant would result in a strictly-proper controller of order $(n + 1)$, with one eigenvalue at $s = 0$ and some of the n eigenvalues possibly in the right-half plane.

The paper is organized as follows: Section II gives the problem formulation, and defines the class of plants considered. All results apply to square MIMO plants and obviously SISO plants are included as special cases. The main results in Section III are grouped into subsections depending on the restrictions imposed on the zeros in the unstable region. Proposition 1 provides a systematic procedure of constructing first order integral-action controllers when none of the zeros of the plant are in the unstable region. Proposition 2 considers blocking-zeros in the unstable region, particularly at infinity, and Proposition 3 extends to the case of transmission-zeros at infinity. In all cases the plants under consideration may have unrestricted zeros in the stable region. There are no constraints on plant poles anywhere in the complex plane. Illustrative MIMO examples are also given, and a controller order comparison is provided with the standard integral-action design method based on full-order observer and state-feedback applied to an augmented plant model. Section IV contains remarks and possible future extensions. Proofs of all propositions are collected in the Appendix.

Although we discuss continuous-time systems here, all results apply also to discrete-time systems with appropriate

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modifications. The following fairly standard notation is used:

Notation: Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{C}$ denote real, positive real, and complex numbers, respectively. The extended closed right-half plane is $\mathcal{U} = \{ s \in \mathbb{C} \mid \mathcal{R}e(s) \geq 0 \} \cup \{\infty\}$; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in $\mathbf{S}$; I is the identity matrix (of appropriate dimension). A transfer-matrix $M \in \mathcal{M}(\mathbf{S})$ is called unimodular iff $M^{-1} \in \mathcal{M}(\mathbf{S})$. The H_∞-norm of $M \in \mathcal{M}(\mathbf{S})$ is denoted by $\|M\|$ (i.e., the norm $\| \cdot \|$ is the usual operator norm $\|M\| := \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$). For simplicity, we drop (s) in transfer-matrices such as $P(s)$ where this causes no confusion. We use coprime factorizations over $\mathbf{S}$; i.e., for $P \in {\mathbf{R_p}}^{m \times m}$, $P = D^{-1}N$ denotes a left-coprime-factorization (LCF), where $N \in \mathbf{S}^{m \times m}$, $D \in \mathbf{S}^{m \times m}$, $\det D(\infty) \neq 0$. For full-rank P, we say that $z \in \mathcal{U}$ is a $\mathcal{U}$-zero of P if $\text{rank} N(z) < m$; these zeros include both transmission-zeros and blocking-zeros in $\mathcal{U}$. If $z \in \mathcal{U}$ is a blocking-zero of P, then $P(z) = 0$ and equivalently $N(z) = 0$

II. Problem Description

Consider the standard LTI, MIMO unity-feedback system $Sys(P, C)$ shown in Fig. 1, where $P \in {\mathbf{R_p}}^{m \times m}$ and $C \in {\mathbf{R_p}}^{m \times m}$ denote the plant's and the controller's transfer-matrices, respectively. It is assumed that the feedback system is well-posed, P and C have no hidden-modes in the unstable region, and the plant $P \in {\mathbf{R_p}}^{m \times m}$ is full normal rank m. The objective is to design a low order stabilizing controller C with integral-action, so that the closed-loop system achieves asymptotic tracking of step-input references with zero steady-state error.

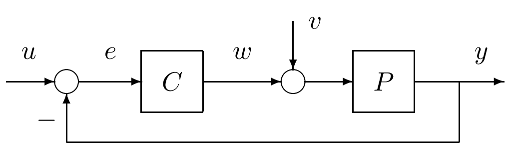

Fig. 1. Unity-Feedback System $Sys(P, C)$.

Let $P = D^{-1}N$ be a left-coprime-factorization (LCF) of the plant and $C = N_c D_c^{-1}$ be a right-coprime-factorization (RCF) of the controller, where N, D, N_c, $D_c \in \mathcal{M}(\mathbf{S})$ have appropriate sizes, $\det D(\infty) \neq 0$, $\det D_c(\infty) \neq 0$. The system $Sys(P, C)$ is said to be stable if the closed-loop transfer-function from (u, v) to (y, w) is stable. The controller C is said to stabilize P if C is proper and the system $Sys(P, C)$ is stable. The controller C stabilizes $P \in \mathcal{M}(\mathbf{R_p})$ if and only if

$$M := DD_c + NN_c \tag{1}$$

is unimodular [13], [7].

Let the (input-error) transfer-function from u to e be denoted by H_{eu} and let the (input-output) transfer-function from u to y be denoted by H_{yu}; then

$$H_{eu} = (I + PC)^{-1} = I - PC(I + PC)^{-1} = I - H_{yu} \ . \tag{2}$$

Definition 1: i) The system $Sys(P, C)$ is stable if the closed-loop transfer-function from (u, v) to (y, w) is stable.
ii) The stable system $Sys(P, C)$ has integral-action if H_{eu} has blocking-zeros at $s = 0$.
iii) The controller C is an integral-action controller if C stabilizes P and the denominator D of any RCF $C = N_c D_c^{-1}$ has blocking-zeros at $s = 0$, i.e., $D_c(0) = 0$. □

Suppose that the system $Sys(P, C)$ is stable and that step input references are applied to the system. Then the steady-state error $e(t)$ due to step inputs at $u(t)$ goes to zero as $t \to \infty$ if and only if $H_{eu}(0) = 0$. Therefore, by Definition 1, the stable system $Sys(P, C)$ achieves asymptotic tracking of constant reference inputs with zero steady-state error if and only if it has integral-action. Write $H_{eu} = (I + PC)^{-1} = D_c \ M^{-1} D$. Then by Definition 1, $Sys(P, C)$ has integral-action if $C = N_c D_c^{-1}$ is an integral-action controller since $D_c(0) = 0$ implies $H_{eu}(0) = (D_c M^{-1} D)(0) = 0$.

Lemma 1 states the basic necessary condition on the plant P for existence of integral-action controllers.

Lemma 1: (*Necessary condition for integral-action*): Let $P \in {\mathbf{R_p}}^{m \times m}$. Let $\text{rank} P(s) = m$. If the system $Sys(P, C)$ has integral-action, then P does not have transmission-zeros at $s = 0$. □

In order to design controllers with integral-action, we assume from now on that the plants under consideration have no zeros at $s = 0$, i.e., $\text{rank} P(0) = m$.

III. Low order controller synthesis

The plants under consideration here for low order stabilizing controller synthesis have no restrictions on their poles; there are no restrictions on the zeros in the open left-half complex plane $\mathbb{C} \setminus \mathcal{U}$, and at infinity. However, the finite $\mathcal{U}$-zeros are restricted. In order to design controllers with integral-action, based on the necessary condition of Lemma 1, we assume everywhere that the plant has no zeros at $s = 0$, i.e., $\text{rank} P(0) = m$.

In Section III-A, we consider plants with no zeros in the right-half-plane $\mathcal{U}$ including infinity; 1st order integral-action controllers can be designed for these plants. In Section III-B, we consider the case where the $\mathcal{U}$-zeros of the plant P are positive real, and they are blocking zeros. In Section III-C, the $\mathcal{U}$-zeros are only at infinity but instead of appearing in every entry of P with the same multiplicity, they are transmission-zeros.

A. Plants with no right-half-plane zeros including infinity

The plants in this section have no restrictions on their poles, and also no restrictions on the zeros in the open left-half complex plane $\mathbb{C} \setminus \mathcal{U}$. However, there are no $\mathcal{U}$-zeros. Therefore, P^{-1} is stable. Proposition 1 gives a systematic controller synthesis method for such plants.

Proposition 1: (*Controller synthesis for plants with no zeros in $\mathcal{U}$ including infinity*) Let $P \in \mathbf{R_p}^{m\times m}$, where $\text{rank}P(0) = m$. Choose any nonsingular $K \in \mathbf{R_p}^{m\times m}$ and any $g \in \mathbb{R}_+$. Choose $\alpha \in \mathbb{R}_+$ such that

$$\alpha > \| \frac{s}{s+g} P^{-1} K^{-1} \| \ . \tag{3}$$

Then the integral-action controller in (4) stabilizes P:

$$C = \alpha \frac{(s+g)}{s} K \ . \tag{4}$$

□

The 1st order integral-action controller in (4) is a proportional+integral (PI) controller.

B. Plants with large blocking-zeros on the positive real-axis

The plants in this section have no restrictions on their poles, and also no restrictions on the zeros in the open left-half complex plane $\mathbb{C} \setminus \mathcal{U}$. The $\mathcal{U}$-zeros of the plant P are positive real, and they are blocking-zeros (appearing in every entry of P). Therefore, P can be written as

$$P = D^{-1}N = \left(\prod_{i=1}^{r} \frac{(1 - s/z_i)}{(s+a)} P^{-1} \right)^{-1} \prod_{i=1}^{r} \frac{(1 - s/z_i)}{(s+a)} I \ , \tag{5}$$

for any $a \in \mathbb{R}_+$, where $z_i \in \mathbb{R}_+ \cup \{\infty\}$, $i = 1, \ldots, r$, are the $\mathcal{U}$-blocking-zeros of P, and P has no other transmission-zeros in $\mathcal{U}$, i.e.,

$$D = \frac{\prod_{i=1}^{r}(1 - s/z_i)}{(s+a)^r} P^{-1} \in \mathcal{M}(\mathbf{S}) \ .$$

Any number of these r $\mathcal{U}$-blocking-zeros may be at infinity; e.g., if none of the $\mathcal{U}$-zeros is finite, then (5) becomes

$$P = D^{-1}N = \left(\frac{1}{(s+a)^r} P^{-1} \right)^{-1} \frac{1}{(s+a)^r} I \ . \tag{6}$$

Proposition 2 gives a systematic controller synthesis method for plants in the form of (5).

Proposition 2: (*Controller synthesis for plants with blocking-zeros in $\mathcal{U}$*) Let $P \in \mathbf{R_p}^{m\times m}$ be as in (5), with $\text{rank}P(0) = m$. Let $D(\infty)^{-1} = \left(\frac{(s+a)^r}{\prod_{i=1}^{r}(1-s/z_i)} P \right) |_{s\to\infty}$. Choose any monic r-th order strictly-Hurwitz polynomial $\rho(s)$. Define Φ as

$$\Phi := s \left[\frac{\prod_{i=1}^{r}(1 - s/z_i)}{\rho(s)} P^{-1} D(\infty)^{-1} - I \right] \ . \tag{7}$$

If $\sum_{i=1}^{r} \frac{1}{z_i} < \frac{1}{2} \| \Phi \|^{-1}$, then choose $\alpha \in \mathbb{R}_+$ such that $\alpha < z_i$ for $i = 1, \ldots, r$ and

$$\alpha > \frac{r}{\| \Phi \|^{-1} - \sum_{i=1}^{r} \frac{1}{z_i}} \ . \tag{8}$$

Then the r-th order bi-proper integral-action controller in (9) stabilizes P:

$$C = \frac{\alpha^r \rho(s)}{(s+\alpha)^r - \alpha^r \prod_{i=1}^{r}(1 - s/z_i)} D(\infty) \ . \tag{9}$$

□

Remarks: 1) For $r = 1$, the controller in (9) is a proportional+integral (PI) controller, and for $r = 2$, it is a proportional+integral+derivative (PID) controller [9]. **2)** If all of the r zeros of the plant (5) are at infinity as in (6), then Φ in (7) becomes

$$\Phi := s \left[\frac{1}{\rho(s)} P^{-1} D(\infty)^{-1} - I \right] \ , \tag{10}$$

and the condition $\sum_{i=1}^{r} \frac{1}{z_i} = 0 < \| \Phi \|^{-1}$ is obviously satisfied. In this case, $\alpha \in \mathbb{R}_+$ is chosen to satisfy (8) as

$$\alpha > r \| \Phi \| \ , \tag{11}$$

and the r-th order integral-action controller in (9) becomes

$$C = \frac{\alpha^r \rho(s)}{(s+\alpha)^r - \alpha^r} D(\infty) \ . \tag{12}$$

The MIMO controller in (12) is bi-proper. Every entry has the r zeros in $\mathbb{C} \setminus \mathcal{U}$ of the strictly-Hurwitz polynomial ρ, and r poles. The poles are the roots of the polynomial d,

$$d := (s+\alpha)^r - \alpha^r \ , \tag{13}$$

which has one root at $s = 0$ and the remaining $r - 1$ roots in $\mathbb{C} \setminus \mathcal{U}$. □

C. Plants with transmission-zeros at infinity

As in Section III-B, the plants in this section have no restrictions on their poles, and also no restrictions on the zeros in the open left-half complex plane $\mathbb{C} \setminus \mathcal{U}$. The $\mathcal{U}$-zeros of the plant P are at infinity, and every entry in the transfer-matrix of P may have different relative degree and some entries may not even be strictly proper. Hence, the zeros at infinity are not necessarily blocking-zeros. Therefore, the numerator matrix N in any LCF $P = D^{-1}N$ has an improper inverse, which we write as

$$N^{-1} = \left[\frac{n_{ij}}{d_{ij}} \right]_{i,j\in\{1,\ldots,m\}} \ . \tag{14}$$

Since P has no transmission-zeros in $\mathcal{U}$ except at infinity, d_{ij} are strictly-Hurwitz polynomials. Define the integers

$$r_{ij} := \begin{cases} \delta n_{ij} - \delta d_{ij} \ , & \delta n_{ij} > \delta d_{ij} \\ 0 \ , & \delta n_{ij} \le \delta d_{ij} \end{cases} \tag{15}$$

where δ denotes polynomial degree. For $j = 1, \ldots, m$, let

$$r_j := \max_j \{ \ r_{ij} \ \} \ . \tag{16}$$

Let $a \in \mathbb{R}_+$; then

$$\frac{n_{ij}}{d_{ij}(s+a)^{r_j}} \in \mathbf{S} \ , \quad i = 1, \ldots, m \ . \tag{17}$$

Define $\Lambda \in \mathbf{S}^{m\times m}$ as

$$\begin{aligned} \Lambda &:= \text{diag}\,[\lambda_1, \ldots, \lambda_m] \\ &= \text{diag}\left[\tfrac{1}{(s+a)^{r_1}}, \tfrac{1}{(s+a)^{r_2}}, \ldots, \tfrac{1}{(s+a)^{r_m}} \right] \ , \end{aligned} \tag{18}$$

where $\lambda_j = 1$ if $r_j = 0$. Some examples of plants with transmission-zeros at infinity are as follows:

1) Let $P = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s^2-1)} \\ 1 & \frac{1}{s-2} \end{bmatrix}$; note that the transmission-zero at infinity is not a blocking-zero. An LCF of P is $P = D^{-1}N = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s-2}{s+1} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} \\ \frac{s-2}{s+1} & \frac{1}{s+1} \end{bmatrix}$. Then $N^{-1} = \begin{bmatrix} \frac{1}{3}(s+1)^2 & -\frac{1}{3}(s+1) \\ -\frac{1}{3}(s-2)(s+1)^2 & \frac{1}{3}(s+1)^2 \end{bmatrix}$. Here, $r_1 = 3$ and $r_2 = 2$ and Λ as in (18) becomes $\Lambda = \begin{bmatrix} \frac{1}{(s+a)^3} & 0 \\ 0 & \frac{1}{(s+a)^2} \end{bmatrix}$, $a \in \mathbb{R}_+$.

2) In this example, some entries of P are not strictly-proper and $r_1 = 0$. Let $P = \begin{bmatrix} \frac{s+1}{s-1} & \frac{-(s+1)}{s-1} \\ \frac{1}{s-2} & \frac{1}{(s+1)(s-2)} \end{bmatrix}$, with LCF $P = D^{-1}N = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s-2}{s+1} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ \frac{1}{s+1} & \frac{1}{(s+1)^2} \end{bmatrix}$. Then $N^{-1} = \begin{bmatrix} \frac{1}{s+2} & \frac{(s+1)^2}{s+2} \\ \frac{-(s+1)}{s+2} & \frac{(s+1)^2}{s+2} \end{bmatrix}$, and $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{s+a} \end{bmatrix}$, $a \in \mathbb{R}_+$.

3) In some cases, there may be a blocking-zero at infinity, in addition to transmission-zeros at infinity that do not appear in every entry of P; e.g., $P = \begin{bmatrix} \frac{1}{s-1} & \frac{-1}{s-1} \\ \frac{1}{(s+1)(s-2)} & \frac{1}{(s+1)^2(s-2)} \end{bmatrix}$, with LCF $P = D^{-1}N = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s-2}{s+1} \end{bmatrix}^{-1} \frac{1}{(s+1)} \begin{bmatrix} 1 & -1 \\ \frac{1}{s+1} & \frac{1}{(s+1)^2} \end{bmatrix}$. Then $N^{-1} = (s+1) \begin{bmatrix} \frac{1}{s+2} & \frac{(s+1)^2}{s+2} \\ \frac{-(s+1)}{s+2} & \frac{(s+1)^2}{s+2} \end{bmatrix}$, and with $r_1 = 1$, $r_2 = 2$, Λ as in (18) becomes $\Lambda = \begin{bmatrix} \frac{1}{s+a} & 0 \\ 0 & \frac{1}{(s+a)^2} \end{bmatrix}$, $a \in \mathbb{R}_+$.

Proposition 3 gives a systematic controller synthesis method for MIMO plants with transmission-zeros at infinity.

Proposition 3: (*Controller synthesis for plants with transmission-zeros at infinity*) Let $P \in \mathbf{R_p}^{m\times m}$ have no finite transmission-zeros in $\mathcal{U}$. Let $P = D^{-1}N$ be any LCF of P. Define Λ as in (18). For $j = 1, \ldots, m$, choose any monic r_j-th order strictly-Hurwitz polynomial $\rho_j(s)$. Define Ψ as

$$\Psi := s\left[DD(\infty)^{-1}\mathrm{diag}\left[\tfrac{(s+a)^{r_1}}{\rho_1(s)}, \ldots, \tfrac{(s+a)^{r_m}}{\rho_m(s)}\right] - I\right]. \quad (19)$$

Choose $\alpha \in \mathbb{R}_+$ such that

$$\alpha > \max_j r_j \,\|\, \Psi \,\|\,. \quad (20)$$

Then the integral-action controller in (21) stabilizes P:

$$C = N^{-1}\Lambda\,\mathrm{diag}\left[\tfrac{\alpha^{r_1}\rho_1(s)}{(s+\alpha)^{r_1}-\alpha^{r_1}}, \ldots, \tfrac{\alpha^{r_m}\rho_m(s)}{(s+\alpha)^{r_m}-\alpha^{r_m}}\right] D(\infty)\,. \quad (21)$$

□

Remarks: The poles of the MIMO integral-action controller in (21) are the poles of the stable matrix $N^{-1}\Lambda$ and are the roots of the Hurwitz polynomial d_j defined as

$$d_j := (s+\alpha)^{r_j} - \alpha^{r_j}\,, \quad (22)$$

which has one root at $s = 0$ and the remaining $r_j - 1$ roots in $\mathbb{C} \setminus \mathcal{U}$. □

D. A comparison with integral-action controller design based on augmented plant model

The standard integral-action controller design method is based on augmentation, where the integral of error is included in the state-space representation of the plant. Let $[A_p, B_p, C_p, D_p]$ be a minimal state-space representation of the plant $P \in \mathbb{R}_p^{m\times m}$, where $A_p \in \mathbb{R}^{n\times n}$. A full n-th order observer is designed by choosing $L \in \mathbb{R}^{n\times m}$ such that the eigenvalues of $(A_p - LC_p)$ are in $\mathbb{C}\backslash\mathcal{U}$. The estimator state is used for state-feedback $K_a = [K_n \ \ K_m] \in \mathbb{R}^{m\times(n+m)}$ such that the eigenvalues of $(A_a - B_aK_a)$ are in $\mathbb{C} \setminus \mathcal{U}$, where an $(n+m)$-th dimensional state-space representation of the augmented plant $\frac{1}{s}P \in \mathbb{R}_p^{m\times m}$ gives (A_a, B_a) as

$$A_a = \begin{bmatrix} A_p & 0 \\ -C_p & 0 \end{bmatrix},\ B_a = \begin{bmatrix} B_p \\ -D_p \end{bmatrix}\,, \quad (23)$$

which has $(n+m)$ states. Let C_a denote the integral-action controller designed by this augmentation. Then a state-space representation (A_c, B_c, C_c, D_c) for the controller C_a is

$$\begin{aligned} A_c &= [A_a - B_aK_a + L_a(\ [C \quad 0] - DK_a\ )] \\ &= \begin{bmatrix} A_p - (B_p - LD_p)K_n - LC_p & -(B_p - LD_p)K_m \\ 0 & 0 \end{bmatrix}, \\ B_c &= L_a = \begin{bmatrix} -L \\ I \end{bmatrix},\ C_c = -K_a\,,\ D_c = 0\,. \end{aligned} \quad (24)$$

The transfer function of this controller is given by

$$\begin{aligned} C_a &= C_c(sI_{(n+m)} - A_c)^{-1}B_c + D_c \\ &= -K_a\left[sI - A_a + B_aK_a + L_a(\ [C \quad 0] - DK_a)\right]^{-1} L_a\,. \end{aligned}$$

The controller C_a in (24) has m of its $(n+m)$ eigenvalues at $s = 0$. The other n eigenvalues of C_a may be anywhere in the complex plane. Integral-action controllers designed using full-order observer and state feedback based on an augmented plant model always give a strictly proper controller transfer function.

The order of the bi-proper controllers designed for the plant classes in Section III do not depend on the number of eigenvalues of the plant $P \in \mathbf{R_p}^{m\times m}$. For the class of plants with no $\mathcal{U}$-zeros discussed in Proposition 1, the first order (PI) controller $C \in \mathbf{R_p}^{m\times m}$ in (4) can be realized with m states, with all m eigenvalues at zero. For the class of plants with r blocking-zeros (large, possibly at infinity) discussed in Proposition 2, the controller $C \in \mathbf{R_p}^{m\times m}$ in (9) (and the special case of (12) when all blocking zeros of P are at infinity) can be realized with rm states, with m of the eigenvalues at zero. For the controllers in (12), the remaining $(r-1)m$ eigenvalues are in the open-left-half plane.

We now compare the design for an augmentation based controller C_a to the design in (12) through an illustrative example.

Example 1: In this example we consider a chemical reactor plant obtained by linearizing the model given in [4], where the concentration of the inlet reactant and the rate of heat input are manipulated to regulate the outlet reactant concentration and the reactor temperature. The linearization around one of the operating points gives the unstable plant transfer-matrix in (25), where P has poles at $s = 0.0614 \in \mathcal{U}$ and $s = -0.0167$, and a blocking-zero at infinity:

$$P = \frac{1}{100y} \begin{bmatrix} 1.67s - 0.1232 & -0.00189 \\ 4.143 & 4.184s + 0.1218 \end{bmatrix} ,$$
$$y = (s - 0.0614)(s + 0.167) \ . \quad (25)$$

With $r = 1$, the plant in (25) can be written as in (6), where $N = \frac{1}{(s+a)} I_2$ and

$$D = \frac{100(s + 0.167)}{6.9873(s + a)} \begin{bmatrix} \frac{4.184s+0.1218}{(s+0.0167)} & \frac{0.00189}{(s+0.0167)} \\ \frac{-4.143}{(s+0.0167)} & \frac{1.67s-0.1232}{(s+0.0167)} \end{bmatrix} ,$$

where $a \in \mathbb{R}_+$. Following Proposition 2, we take a simple first order $\rho(s) = (s + 1)$. Then the norm of Φ in (10) is $\| \Phi \| = 1.5$. If we choose $\alpha = 3 > r\| \Phi \|$ satisfying (11), then the first order controller as in (12) becomes

$$C = \frac{(s + 1)}{s} \begin{bmatrix} 179.64 & 0 \\ 0 & 71.7 \end{bmatrix} . \quad (26)$$

For different choices of $\rho(s)$ and α, we would obtain different first order controllers. A minimal state-space realization of the controller in (26) has 2 states, with two eigenvalues both at $s = 0$. The resulting closed-loop poles for the system $Sys(P, C)$ are $\{-1.5452 \pm j0.7759, -1.5826 \pm j0.7025, -0.0168\}$. The closed-loop step responses are shown in Fig. 2.

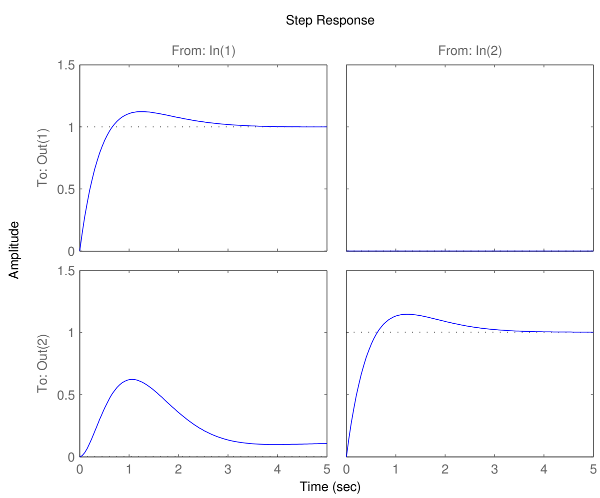

Fig. 2. Step responses for Example 1

We now design an integral-action controller as in (24) based on an augmented plant $\frac{1}{s}P$. A minimal state-space representation of the 2×2 plant in (25) has $n = 4$ states, and the augmented description has $n + m = 6$ states. We choose $L \in \mathbb{R}^{4\times2}$ to place the observer poles (eigenvalues of $(A_p - LC_p)$) at $\{-50, -50, -40, -40\}$. Then we choose $K_a \in \mathbb{R}^{2\times6}$ to place the eigenvalues of $(A_a - B_aK_a)$ at $\{-1.5444 \pm j0.7764, -1.5835 \pm j0.7018, -2, -2\}$. The controller C_a as in (24) then has 6 eigenvalues, at $\{-120575, 120482, -48.498 \pm j10385, 0, 0\}$. Note that C_a has one eigenvalue in $\mathcal{U}$ in addition to the two at $s = 0$. The transfer-function of C_a is strictly-proper, with 5-th order denominator terms. □

This example illustrates that the controller order for the proposed integral-action synthesis method is lower than the augmented plant based full order observer approach since it depends on the relative degree (number of blocking-zeros at infinity) of the plant class under consideration.

IV. Conclusions

For plants whose zeros in the unstable region are 'large' and particularly at infinity, we developed a systematic synthesis methodology that results in a simple integral-action controller whose poles other than the one integrator providing the integral-action all are in the stable region. We investigated both blocking-zeros and transmission-zeros at infinity. The plant classes under consideration do not put any constraints on where the poles are and also the zeros in the stable region are unrestricted. Since the controller has only one integrator but is otherwise stable, the plants here are in fact strongly stabilizable [14]. Low order controllers with integral-action for plants that are not strongly stabilizable would have some poles in the right-half complex-plane.

The proposed controllers for each plant class we considered here have flexibility in the choice of the design parameters (e.g., the numerator polynomial for the controller is chosen arbitrarily). The effect of the parameter choices on the system performance can be explored in future extensions, although the scope of this current work is limited to the challenging goal of low-order stabilization while achieving asymptotic tracking of step-input references with zero steady-state error.

APPENDIX

Proof of Proposition 1:
Let $N_c = I$ and $D_c = \frac{s}{\alpha(s+g)} K^{-1} = C^{-1}$. By (1), $C = N_c D_c^{-1}$ stabilizes $P = (P^{-1})^{-1} I$ if and only if $M = NN_c + DD_c$ is unimodular, where

$$M = I + \frac{1}{\alpha} \frac{s}{(s + g)} P^{-1} K^{-1} \ . \quad (27)$$

A sufficient condition for M to be unimodular is that $\| \frac{1}{\alpha} \frac{s}{(s+g)} P^{-1} K^{-1} \| < 1$, which holds for α satisfying (3); hence, C in (4) stabilizes P, which has no zeros in $\mathcal{U}$. □

Proof of Proposition 2:
Let $d := (s + \alpha)^r - \alpha^r \prod_{i=1}^{r} (1 - s/z_i)$, which becomes (13) when all zeros are at infinity. Let $N_c = \alpha^r I$ and $D_c = \alpha^r C^{-1} = \frac{d}{\rho} D(\infty)^{-1}$; note that C^{-1} is stable by choice of $\rho(s)$. By (1), $C = N_c D_c^{-1}$ stabilizes $P = D^{-1}N$ given by (5) if and only if $M = NN_c + DD_c$ is unimodular, where

$$M = \frac{\alpha^r \prod_{i=1}^{r} (1 - s/z_i)}{(s + a)^r} I + DD(\infty)^{-1} \frac{d}{\rho} \ . \quad (28)$$

Since $a, \alpha \in \mathbb{R}_+$, M is unimodular if and only if $\hat{M} := M\frac{(s+a)^r}{(s+\alpha)^r}$ is unimodular, where $\hat{M}$ can be written as

$$\hat{M} = \frac{\alpha^r \prod_{i=1}^r (1 - s/z_i)}{(s+\alpha)^r} I + DD(\infty)^{-1} \frac{(s+a)^r}{\rho} \frac{d}{(s+\alpha)^r}$$
$$= I + [DD(\infty)^{-1} \frac{(s+a)^r}{\rho} - I] \frac{d}{(s+\alpha)^r}$$
$$= I + s\,[\frac{\prod_{i=1}^r (1 - s/z_i)}{\rho} P^{-1} D(\infty)^{-1} - I] \frac{d}{s\,(s+\alpha)^r}$$
$$= I + \Phi \frac{d}{s\,(s+\alpha)^r} \ . \quad (29)$$

Since $(DD(\infty)^{-1} \frac{(s+a)^r}{\rho} - I)(\infty) = 0$, Φ is proper. A sufficient condition for $\hat{M}$ to be unimodular is that $\| \Phi \frac{d}{s\,(s+\alpha)^r} \| < 1$. For $z_i > \alpha$, the norm $\| \frac{d}{s\,(s+\alpha)^r} \|$ is:

$$\| \frac{d}{s\,(s+\alpha)^r} \| = \| \frac{((s+\alpha)^r - \alpha^r \prod_{i=1}^r (1 - s/z_i))}{s\,(s+\alpha)^r} \|$$
$$= \| \frac{((s+\alpha)^r - \alpha^r}{s\,(s+\alpha)^r} + \frac{\alpha^r - \alpha^r \prod_{i=1}^r (1 - s/z_i))}{s\,(s+\alpha)^r} \|$$
$$\leq \| \frac{((s+\alpha)^r - \alpha^r}{s\,(s+\alpha)^r} \| + \| \frac{\alpha^r - \alpha^r \prod_{i=1}^r (1 - s/z_i))}{s\,(s+\alpha)^r} \|$$
$$\leq \frac{r}{\alpha} + \sum_{i=1}^r \frac{1}{z_i} \ .$$

Therefore, if $\sum_{i=1}^r \frac{1}{z_i} < \frac{r}{\alpha}$ and $2 \sum_{i=1}^r \frac{1}{z_i} < \| \Phi \|^{-1}$, then for α satisfying (8),

$$\|\Phi \frac{d}{s(s+\alpha)^r}\| \leq \|\Phi\| \| \frac{d}{s\,(s+\alpha)^r} \| \leq \|\Phi\| [\frac{r}{\alpha} + \sum_{i=1}^r \frac{1}{z_i}] < 1,$$

and hence, $\hat{M}$ is unimodular; equivalently, the controller C in (9) stabilizes P. By Definition 1-(iii), C is an integral-action controller since $d(0) = 0$ implies $D_c(0) = \frac{d}{\rho} D(\infty)^{-1}|_{s=0} = 0$. Since ρ and d are both r-th order polynomials, $C = \frac{\alpha^r \rho}{d} D(\infty)^{-1}$ and C^{-1} are both proper. □

Proof of Proposition 3:
Let d_j be defined as in (22). Let $N_c = N^{-1} \Lambda \,\mathrm{diag}\left[\alpha^{r_1}, \ldots, \alpha^{r_1}\right]$ and $D_c = \alpha^r C^{-1} = D(\infty)^{-1} \mathrm{diag}\left[\frac{d_1}{\rho_1}, \ldots, \frac{d_m}{\rho_m}\right]$; note that C^{-1} is stable by choice of $\rho(s)$. By (1), $C = N_c D_c^{-1}$ stabilizes $P = D^{-1} N$ if and only if $M = N N_c + D D_c$ is unimodular, where

$$M = N N^{-1} \Lambda \,\mathrm{diag}\left[\alpha^{r_1}, \ldots, \alpha^{r_1}\right] + DD(\infty)^{-1} \mathrm{diag}\left[\frac{d_1}{\rho_1}, \ldots, \frac{d_m}{\rho_m}\right] \ .$$

Now since $a, \alpha \in \mathbb{R}_+$, M is unimodular if and only if $\hat{M} := M \mathrm{diag}\left[\frac{(s+a)^{r_1}}{(s+\alpha)^{r_1}}, \ldots, \frac{(s+a)^{r_m}}{(s+\alpha)^{r_m}}\right]$ is unimodular, where $\hat{M}$ can be written as

$$\hat{M} = I + s\,[\,DD(\infty)^{-1} \mathrm{diag}\left[\frac{(s+a)^{r_1}}{\rho_1}, \ldots, \frac{(s+a)^{r_m}}{\rho_m}\right] - I\,]\,\mathrm{diag}\left[\frac{d_1}{s(s+\alpha)^{r_1}}, \ldots, \frac{d_m}{s(s+\alpha)^{r_m}}\right]$$
$$= I + \Psi \,\mathrm{diag}\left[\frac{d_1}{s(s+\alpha)^{r_1}}, \ldots, \frac{d_m}{s(s+\alpha)^{r_m}}\right] \ . \quad (30)$$

Since $(DD(\infty)^{-1} \mathrm{diag}\left[\frac{(s+a)^{r_1}}{\rho_1}, \ldots, \frac{(s+a)^{r_m}}{\rho_m}\right] - I)(\infty) = 0$, Ψ is proper. A sufficient condition for $\hat{M}$ to be unimodular is that $\| \Psi \,\mathrm{diag}\left[\frac{d_1}{s(s+\alpha)^{r_1}}, \ldots, \frac{d_m}{s(s+\alpha)^{r_m}}\right] \| < 1$. The norm

$$\| \left[\frac{d_1}{s(s+\alpha)^{r_1}}, \ldots, \frac{d_m}{s(s+\alpha)^{r_m}}\right] \| = \max_j \| \frac{((s+\alpha)^{r_j} - \alpha^{r_j}}{s\,(s+\alpha)^{r_j}} \| \leq \max_j \frac{r_j}{\alpha} \ .$$

Therefore, for α satisfying (20),

$$\| \Psi \left[\frac{d_1}{s(s+\alpha)^{r_1}}, \ldots, \frac{d_m}{s(s+\alpha)^{r_m}}\right] \| \leq \|\Psi\| \| \left[\frac{d_1}{s(s+\alpha)^{r_1}}, \ldots, \frac{d_m}{s(s+\alpha)^{r_m}}\right] \| \leq \|\Psi\| \max_j \frac{r_j}{\alpha} < 1,$$

and hence, $\hat{M}$ is unimodular; equivalently, the controller C in (21) stabilizes P. By Definition 1-(iii), C is an integral-action controller since $d_j(0) = 0$ implies $D_c(0) = D(\infty)^{-1} \mathrm{diag}\left[\frac{d_1(0)}{\rho_1(0)}, \ldots, \frac{d_m(0)}{\rho_m(0)}\right] = 0$. □

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