Zeros of Sampled Data Systems Represented by FIR Models

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Abstract

In this note, we investigate zero locations of FIR linear systems that are finite length approximations of sampled continuous time systems. For linear systems with rational transfer functions, it is shown that with a relative degree higher than two and a short sampling interval, the resultant FIR sampled data system is always nonminimum phase.

Keywords: zeros, FIR systems, nonminimum phase systems, sampled data systems

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1 Introduction

Consider a stable continuous time system with rational transfer function

$$G(s) = k_1 \frac{(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}, \quad n > m$$

and let

$$H(z) = h_1(h)z^{-1} + h_2(h)z^{-2} + ... = \sum_{i=1}^{\infty} h_i(h)z^{-i}$$

be its sampled data system with a Zero Order Hold and the sampling interval h as shown in the following figure.

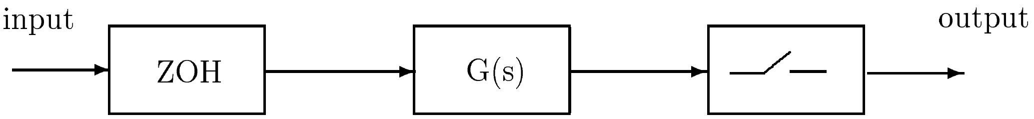

Fig. 1, H(z)

It is well known [1] that $H(z)$ has two sets of zeros as $h \to 0$. The first set of zeros converge to 1 as $h \to 0$. The second set of zeros are due to the relative degree $m - n$. As $h \to 0$, $H(z)$ always has at least one zero converging to -1 when $n - m = 2$ and one zero strictly outside the unit circle when $n - m > 2$. We study the following question in the note: If the discrete time system is modeled by a FIR system, can we say that the relative degree higher than 2 also always leads to a zero of the FIR system strictly outside the unit circle as $h \to 0$? This problem is important. Note that in many signal processing and communication applications, $H(z)$ is approximated by an FIR system with finite delay taps and the inverse is sought to recover the system input signal. In communication, for instance, channel equalization is essentially the inverse of a FIR channel model. If the FIR system is nonminimum phase, the implementation of the inverse can become problematic, possibly leading to unrecoverable error propagation.

Let $T > 0$ be a fixed time of interest and let $l = \lceil \frac{T}{h} \rceil$ be the smallest integer that is larger than or equals to T/h. Then, the $l-$th order FIR approximation of $H(z)$ by keeping only first l terms of $H(z)$ can be obtained

$$\widehat{H}_l(z) = \sum_{l=1}^{l} h_l z^{-l} = h_1 z^{-1} + h_2 z^{-2} + ... + h_l z^{-l}.$$

The main contribution of this note is to show that $\widehat{H}_l(z)$ always has at least one zero strictly outside the unit circle if the relative degree $n - m$ is higher than 2 and h is small.

2 Main results

Define the polynomial

$$B_{n-m-1}(z^{-1}) = \sum_{k=1}^{1}(-1)^{1-k}k^{n-m}\binom{n-m+1}{1-k} + \sum_{k=1}^{2}(-1)^{2-k}k^{n-m}\binom{n-m+1}{2-k}z^{-1}$$
$$+ ... + \sum_{k=1}^{n-m}(-1)^{n-m-k}k^{n-m}\binom{n-m+1}{n-m-k}z^{-(n-m-1)}. \tag{2.1}$$

From [1], $B_{n-m-1}(z^{-1})$ always contains at least one zero strictly outside the unit circle when $n-m-1 \geq 2$. We list $B_{n-m-1}(z^{-1})$ and its unstable zeros for $1 \leq n-m-1 \leq 4$.

$n-m-1$	$B_{n-m-1}(z^{-1})$	unstable zeros of $B_{n-m-1}(z^{-1})$
1	$1+z^{-1}$	-1
2	$1+4z^{-1}+z^{-2}$	-3.732
3	$1+11z^{-1}+11z^{-2}+z^{-3}$	$-1, -9.899$
4	$1+26z^{-1}+66z^{-2}+26z^{-3}+z^{-4}$	$-2.322, -23.20$

The main result is the following theorem.

Theorem 2.1 *Consider $G(s)$, $H(z)$ and $\widehat{H}_l(z)$ with $l = \lceil \frac{T}{h} \rceil$ for some fixed $T > 0$. As $h \to 0$, $\widehat{H}_l(z)$ has two sets of zeros. The first set of zeros are either inside the unit circle or converge to unit circle as $h \to 0$. The second set of zeros are unstable zeros of $B_{n-m-1}(z^{-1})$ which are strictly outside the unit circle.*

The above result tells us that if $G(s)$ has the relative degree $n-m$ higher than 2, the FIR approximation $\widehat{H}_l(z)$ of $H(z)$ is always non-minimum phase provided that the sampling interval h is small.

Proof: The proof is based on the fact [1] that as $h \to 0$, $n-m-1$ zeros of $H(z)$ converge to the zeros of $B_{n-m-1}(z^{-1})$, and is proceeded as follows: Any fixed z_0 with magnitude larger than 1 is a zero of $\widehat{H}_l(z)$ as $h \to 0$ if and only if it is a zero of $B_{n-m-1}(z^{-1})$. To show that, note from [1] as $h \to 0$

$$H(z) \to h_1(h)z^{-1}\frac{(1-e^{z_1h}z^{-1})(1-e^{z_2h}z^{-1})...(1-e^{z_mh}z^{-1})B_{n-m-1}(z^{-1})}{(1-e^{p_1h}z^{-1})(1-e^{p_2h}z^{-1})...(1-e^{p_nh}z^{-1})}.$$

At any fixed z_0 with $|z_0| = 1+\delta_1$ for some $\delta_1 > 0$, define $z = z_0\lambda$. Then,

$$h_1(h)z^{-1}\frac{(1-e^{z_1h}z^{-1})(1-e^{z_2h}z^{-1})...(1-e^{z_mh}z^{-1})B_{n-m-1}(z^{-1})}{(1-e^{p_1h}z^{-1})(1-e^{p_2h}z^{-1})...(1-e^{p_nh}z^{-1})}$$

$$= h_1(h) z_0^{-1}\lambda^{-1} \frac{(1 - e^{z_1 h} z_0^{-1}\lambda^{-1})...(1 - e^{z_m h} z_0^{-1}\lambda^{-1}) B_{n-m-1}(z_0^{-1}\lambda^{-1})}{(1 - e^{p_1 h} z_0^{-1}\lambda^{-1})(1 - e^{p_2 h} z_0^{-1}\lambda^{-1})...(1 - e^{p_n h} z_0^{-1}\lambda^{-1})}$$

$$= h_1(h) z_0^{-1}\lambda^{-1} \sum_{i=0}^{\infty} \alpha_i(h)\lambda^{-i} = h_1(h) z_0^{-1}\lambda^{-1} \{\sum_{i=0}^{l-1} \alpha_i(h)\lambda^{-i} + \sum_{i=l}^{\infty} \alpha_i(h)\lambda^{-i}\}. \qquad (2.2)$$

Clearly,

$$\widehat{H}_l(z) = \widehat{H}_l(z_0\lambda) = h_1(h) z_0^{-1}\lambda^{-1} \sum_{i=0}^{l-1} \alpha_i(h)\lambda^{-i}.$$

Now, since $1 < |z_0| = 1 + \delta_1$, it follows that

$$|\frac{e^{p_i h}}{z_0}| \le 1 - \delta_2$$

for some $1 > \delta_2 > 0$ uniformly in $h > 0$. Therefore, at $z = z_0$, the coefficients α_i's of equation (2.2) are bounded by some constants $M > 0$, $0 \le \rho < 1$,

$$|\alpha_i(h)| \le M\rho^i, \quad \forall i$$

uniformly in $h > 0$. Then, it follows that at $\lambda = 1$ or $z = z_0\lambda$,

$$|\sum_{i=l}^{\infty} \alpha_i(h)\lambda^{-i}| = |\lambda^{-l} \sum_{i=0}^{\infty} \alpha_{i+l}(h)\lambda^{-i}|$$

$$\le M\rho^l \sum_{i=0}^{\infty} \rho^i \to 0$$

as $l \to \infty$ or equivalently $h \to 0$. This shows that at any fixed $|z| = |z_0| > 1$,

$$\widehat{H}_l(z)|_{z=z_0} \to h_1(h) z_0^{-1} \frac{(1 - e^{z_1 h} z_0^{-1})...(1 - e^{z_m h} z_0^{-1}) B_{n-m-1}(z_0^{-1})}{(1 - e^{p_1 h} z_0^{-1})...(1 - e^{p_n h} z_0^{-1})}$$

as $h \to 0$. In other words, if $|z_0| > 1$, then z_0 is a zero of $\widehat{H}_l(z)$ as $h \to 0$ if and only if z_0 is a zero of $B_{n-m-1}(z^{-1})$. This completes the proof.

Finally, we give an example. Consider

$$G(s) = \frac{1}{(s + 2 + j)(s + 2 - j)(s + 1)}$$

with $T = 5$. According to Theorem 2.1, as $h \to 0$, all zeros of $\widehat{H}_l(z)$ either converge to or are inside the unit circle except one which is in the neighborhood of -3.73 because $n - m - 1 = 2$. Figure 2a shows this unstable zero (in "o") with respect to the sampling interval h for $h = .1$, $.05$, $.0333$, $.025$, $.02$, $.0167$, $.0143$, $.0125$, $.0111$ and $.01$. As $h \to 0$, this unstable zero converges to -3.73 as predicted. Figure 2b shows all the zeros of $\widehat{H}_l(z)$ in the complex plan when $h = 0.05$. It is seen that all the zeros except the one near -3.73 converge to the unit circle.

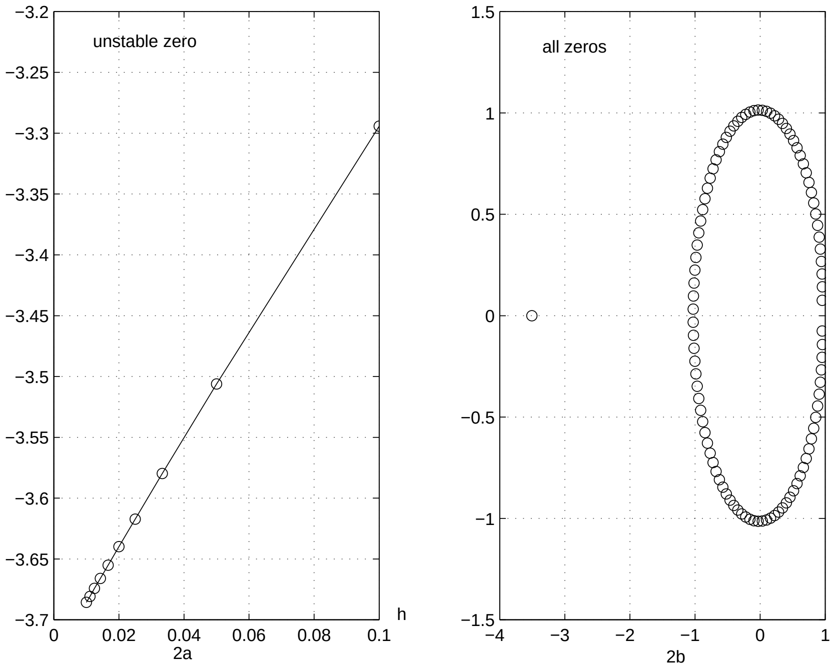

Figure 2: Zero location.

3 Concluding remarks

We have shown that sampled data systems represented by either FIR models or IIR models share similar unstable zeros as the sampling interval $h \to 0$ as long as the continuous time system is a finite order rational transfer function. It would be an interesting problem to study limiting zeros of a sampled data system derived from a continuous time system represented by a non-rational transfer function.

References

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