EIT/FE Exam
EE Review
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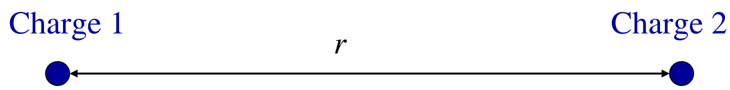
Basic Electricity Outline

- Charge, Force, Electric Field, Work and Energy
- Work, Energy and Voltage
- The Atom
- Current, Resistance and Ohm's Law
- Power and Energy
- Conductors, Resistors and Insulators
- Schematics & models
- DC Circuits

Force between Two Charges



- Consider two charges separated by a distance r
- The force on charge 2 because of charge 1 is:

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon r^2}\mathbf{a}_{r12}$$

- Just like gravity!

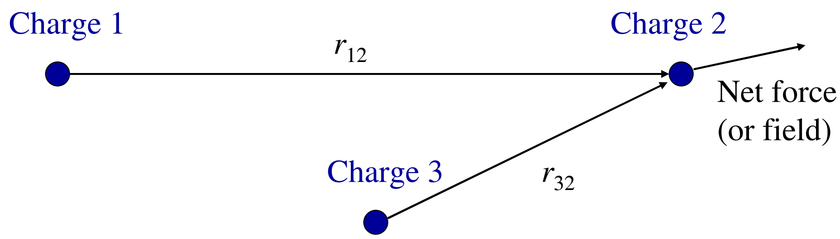
Electric Field

- We sometimes find it convenient to picture the effect one charge has on another by defining an electric field vector
- The electric field at point 2 due to the charge at 1 is

$$\mathbf{E}_1 = \frac{Q_1}{4\pi\varepsilon r^2}\mathbf{a}_{r12}$$

- And the force is then given by $\mathbf{F}_2 = Q_2\mathbf{E}_1 = \dfrac{Q_1 Q_2}{4\pi\varepsilon r^2}\mathbf{a}_{r12}$

Electric Field

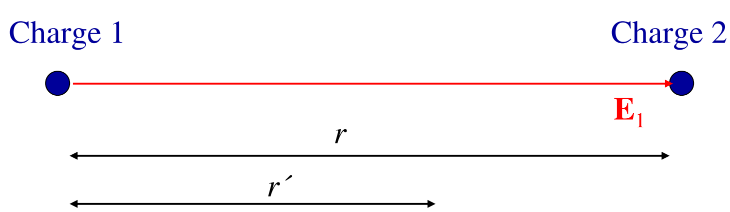


- If there are more charges, the fields and forces add as vectors

$$\mathbf{E}_{Net} = \mathbf{E}_1 + \mathbf{E}_3$$

$$\mathbf{F}_{Net} = \mathbf{F}_1 + \mathbf{F}_3 = Q_2\left(\mathbf{E}_1 + \mathbf{E}_3\right)$$

Electric Field and Voltage



- It takes work to move a charge in a field
- For example, the work required to move Charge 2 to the left is

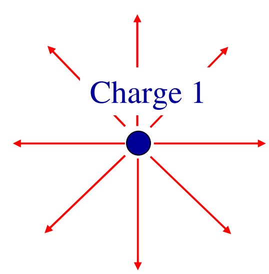
$$W = -Q_2 \int_r^{r'} E_1 dr = -Q_2 \int_r^{r'} \frac{Q_1 dr}{4\pi\varepsilon r^2} = \frac{-Q_1 Q_2}{4\pi\varepsilon} \int_r^{r'} \frac{dr}{r^2}$$

$$= \frac{Q_1 Q_2}{4\pi\varepsilon}\left(\frac{1}{r'} - \frac{1}{r}\right)$$

Electric Field and Voltage

- Because it takes work to move a charge in a field, charge possesses electrical potential energy when in a field
- Voltage is the electrical potential energy per unit charge and is always measured as the difference in potential energy between two points – if a second point is not explicitly stated, one has been assumed (e.g., ground)
  - Note that potential only makes sense for a conservative field!

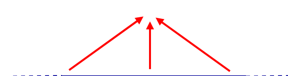
Electric Fields



- We saw that the field from a point charge is given by

$$\mathbf{E}_1 = \frac{Q_1}{4\pi\varepsilon r^2}\mathbf{a}_{r12}$$

- The field from an infinitely long line of charge is



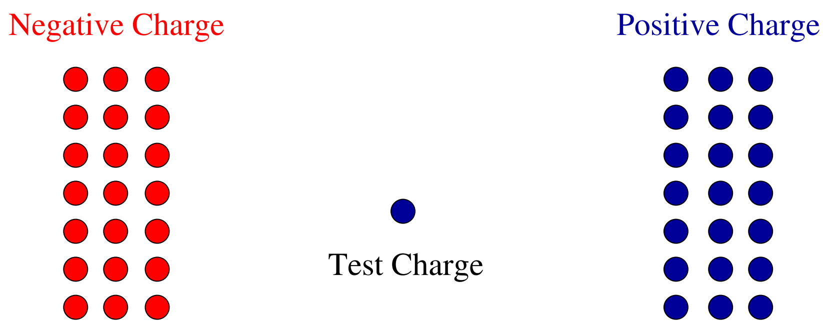
$$\mathbf{E}_L = \frac{\rho_L}{2\pi\varepsilon r}\mathbf{a}_r$$

- The field from an infinite sheet of charge is

$$\mathbf{E}_S = \frac{\rho_S}{2\varepsilon}\mathbf{a}_z$$

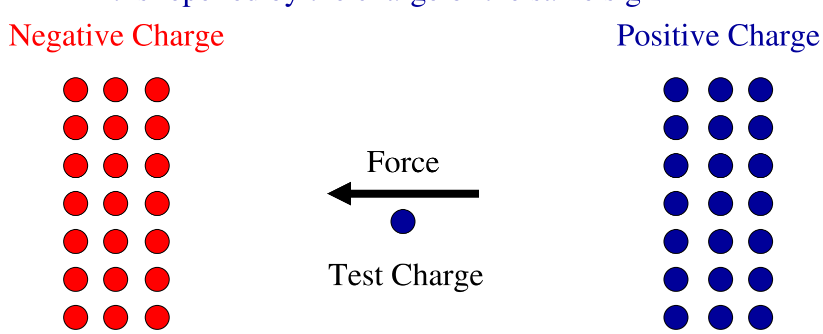
Summary: Consider Test Charge

- Consider a spatial distribution of charges
- Now consider a test charge:



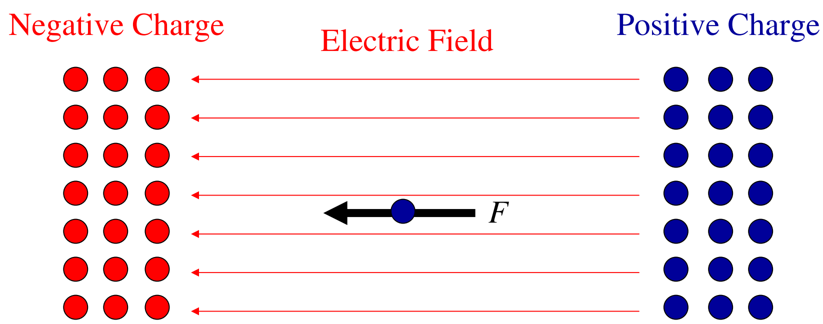
Force

- There is a force on this test charge
  - it is attracted by the opposite charge
  - it is repelled by the charge of the same sign



Electric Field

- We can visualize this force by saying that there is an *Electric Field* in the region, then if the electric field is E, the force on the test charge is $F = qE$



Work & Energy

- Work is equal to force times distance for a constant force (for a variable force we integrate)
- The work done in moving some particle is equal to the change in the particle's energy
- Consider lifting an object off of the floor; the gravitational force is approximately constant (mg), so the work done in lifting to a height h is mgh
- Equating the work done with the gravitational potential energy of the particle, we say the potential energy is $U_m = mgh$. We can think of h as the gravitational potential energy per unit weight (weight = mg)

Electrical Potential Energy

- If a particle has electric charge, then it may also have an electrical potential energy, U_{el}
- We find U_{el} by equating it with the work required to move the charge in a static electric field
  - If we move in opposition to the field, U_{el} increases and the inner product is negative
- Allowing for the possibility of a field that varies with position, we have (in one dimension, assuming the charge moves from 0 to x)

$$U_{el}(x) = -\int_0^x \vec{F}(\lambda) \cdot d\vec{\lambda} = -q\int_0^x \vec{E}(\lambda) \cdot d\vec{\lambda}$$

Voltage

- It is convenient to define the *voltage* between two points in space, V_{21}, as the integral of the electric field, E ($V_{21} > 0 \Rightarrow$ the potential at point 2 is higher)

$$V_{21} = -\int_{x_1}^{x_2} \vec{E}(\lambda) \cdot d\vec{\lambda}$$

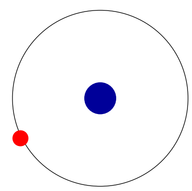
- If we define x_1 to be 0 and assume $V(x_1) = 0$, then we see from our previous result that the voltage is the electrical potential energy *per unit charge*, i.e.,

$$U_{el}(x) = -q\int_0^x \vec{E}(\lambda) \cdot d\vec{\lambda} \Rightarrow V(x) = -\int_0^x \vec{E}(\lambda) \cdot d\vec{\lambda} = \frac{U_{el}(x)}{q}$$

- V is, therefore, analogous to h

The Atom

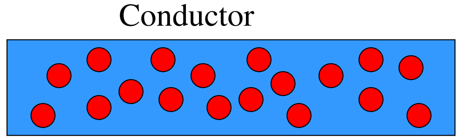
- Start with a model of the atom - the Bohr model
  - protons and neutrons make up the nucleus, protons have positive charge
  - electrons have negative charge and move around the nucleus in different orbits
  - electrons in the outermost orbit can easily be removed if that shell is not full
  - These electrons then become mobile charge (e.g., they are free to move under the influence of an electric field)



Electrical Conductor

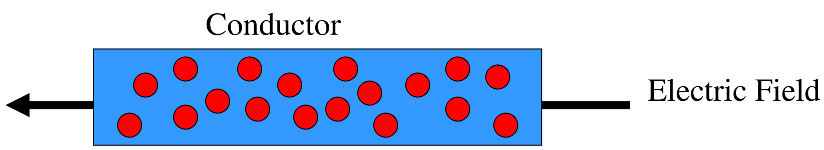
- A conductor is a material that contains a substantial number of mobile charges - for now let's just assume that they are always electrons

Conductor



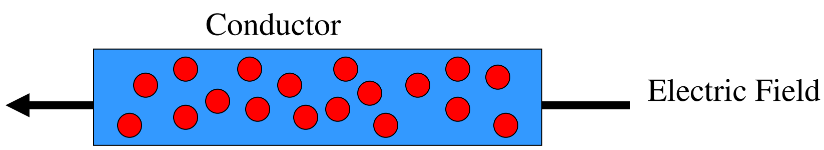
Conductor in a Field

- Now suppose we apply an electric field to the conductor
- According to Newton's second law ($F = ma$), the electrons will continuously accelerate while in the field



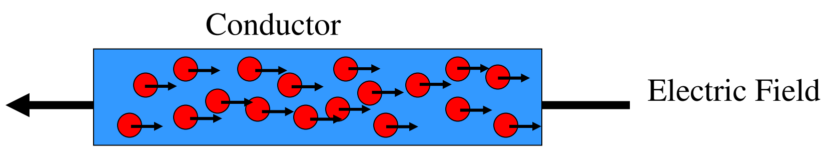
Conductor in a Field

- But, what do we actually observe when the mobile electrons in a conductor are exposed to an electric field?



Conductor in a Field

- We find that the electrons move with a *constant* average velocity! Why?



Electrical Resistance

- We find that the electrons move with a *constant* average velocity! Why?

- Our simplest model of this situation is that the electrons occasionally collide with atoms in the conductor and loose energy in each collision
- These collisions are the cause of *resistance* - which is the electrical analog of friction

Ohm's Law

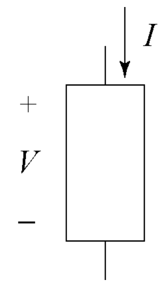
- Experimentally, we find that the current flowing in a conductor is proportional to the field (or voltage) and inversely proportional to the resistance
- Stating this law, called Ohm's law, in terms of voltage, as is customary, we have $I = V/R$
  - V is the voltage *across* the conductor (remember, voltage is always a potential *difference*, therefore it appears across things, it does not flow through them)
  - I is the electric current (charge per unit time) *through* the conductor defined as positive in the direction a *positive* charge would move
  - R is the resistance of the conductor

Units for Ohm's Law

- Voltage is measure in *volts* in honor of Alessandro Volta (1745-1827)
  - One volt is one Joule per Coulomb of charge (charge is measured in Coulombs, in honor of Charles Augustin de Coulomb, 1736-1806)
- Current is measured in *amperes* (amps) in honor of Andre Marie Ampere (1775-1836)
  - One ampere is one Coulomb per second
- Resistance is measured in *ohms*, Ω, in honor of Georg Simon Ohm (1787-1854)
  - One ohm means that a one volt drop will produce a current of one ampere

Power and Energy

- Power is the derivative of energy with respect to time (i.e., it is the rate of exchange of energy)
- Remembering that voltage is electrical potential energy per unit charge, we find the power dissipated in the element is



$$P = \frac{dU_{el}}{dt} = \frac{d(QV)}{dt} = V\frac{dQ}{dt} = VI$$

where we have assumed that the voltage is constant over time and have recognized that current is the time-derivative of charge

- Power is measured in Watts in honor of James Watt (1736 – 1819)

Current in a Conductor

- When a current flows in a conductor, energy is dissipated (i.e., it is lost to the electrical circuit because it has been converted into heat)
- The heat is caused by the energy transferred to the material by the electrons colliding with the atoms (in fact, heat is modeled as vibrations in the atoms comprising the conductor)

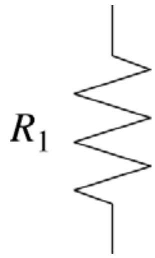
Conductors, Resistors, & Insulators

- Under normal circumstances, all materials have some non-zero, finite electrical resistance
- Nevertheless, we often find it convenient to define
  - Conductors - materials that conduct electric current with very little voltage across them (we usually approximate the voltage across them as zero)
  - Resistors - materials that have significant resistance and, therefore, require a significant voltage across them to produce current through them
  - Insulators - materials that do not allow significant currents to flow (although they will if the voltage gets large enough to break them down)
- Ideal conductors and insulators do not dissipate any energy because either v or i is zero ($P = VI$)

Schematics and Models

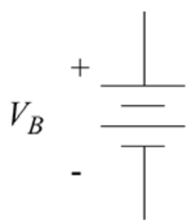
- We use *schematics* to show how conductors, resistors, batteries, and other components are connected together.
- Each of the elements in a schematic is a representation of a real or theoretical element
  - if the element is a real physical device (e.g., a real resistor or a transistor), then we must remember that many different models can be used to represent the element depending on the desired analysis and accuracy
  - in the EIT exam you deal with idealized circuits, so the symbols represent specific idealized models (e.g., a resistor symbol implies that Ohm's law is satisfied)

Schematic Symbols - Resistor



- This symbol is used for a resistor. In the EIT it will always imply that Ohm's law holds (i.e., that $V = IR$)
- But, if the symbol represents a *real* resistor in some circuit, we may need to use a more complex model for that element.
  - For example, all real resistors are nonlinear and they contain other *parasitic* elements like capacitance and inductance

Schematic Symbols - Battery

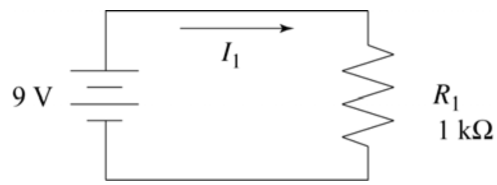


- The symbol shown here represents a battery
- In reality, this symbol is often used for any source of constant voltage (called DC for *direct current*, which simply means the direction does not change, although usually DC also implies that the voltage is constant)
- As with the resistor, we may need a more complex model if this element represents a real battery (e.g., all real batteries have internal resistance)

A Simple DC Circuit

- Here is a very simple example circuit
- This circuit actually uses *three* models:
  - the battery (assumed to be a perfect voltage source)
  - the resistor (assumed to follow Ohm's law perfectly)
  - the wires (assumed to be perfect conductors)
- Using Ohm's law we find

$$I_1 = \frac{V_B}{R_1} = \frac{9\ V}{1\ \text{k}\Omega} = 9\ \text{mA}$$

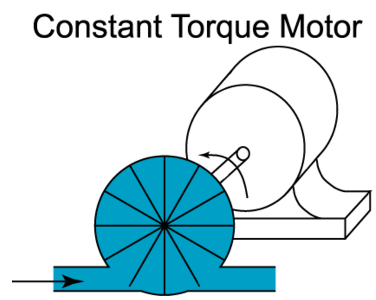


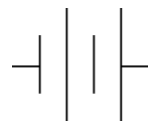
DC Circuits Outline

- Water analogy for electric circuits
- Resistors and sources in series and parallel
- Kirchoff's voltage law
- Kirchoff's current law
- Thevenin and Norton equivalent circuits
- Example circuits
- Inductors
  - Transformers
- DC transient circuit examples
- AC Circuit examples

Water Models – Battery

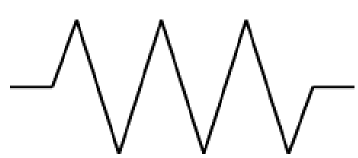
- Our water models will use:
  - Pressure analogous to voltage
  - Electrical current is analogous to water flow
- If a motor with constant torque drives a turbine (i.e., an enclosed paddle wheel), there will be a constant pressure difference between the water entering and the water leaving the turbine independent of the flow
  - This is a "battery" for our water models





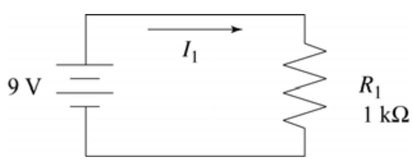
Water Models - Resistor

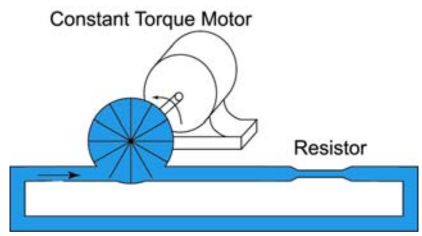
- One possible water model for a resistor is simply a pipe with a narrow region to restrict the flow
- We are assuming with these models that the pipes are completely full of water and that the flow is laminar (so that flow is proportional to pressure drop)
- In real pipes, the flow is rarely laminar – so water in pipes is usually *more* complex than electrical current in wires!



Water Model – First Circuit

- The water model equivalent of our first circuit is shown below along with the water equivalent of Ohm's law (in all of our water models we will assume that the narrow regions dominate the resistance – i.e., the "wires" have zero resistance)



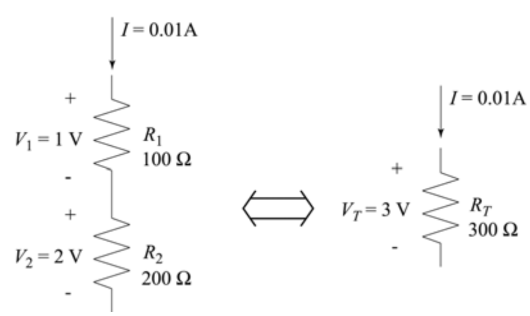


$$I_1 = \frac{V_B}{R_1}$$

$$\text{Flow} = \frac{\text{Pressure}}{\text{Resistance}}$$

Elements in Series

- When the same exact current (not just the same value!) flows through two elements, they are *in series*
- Voltages add for elements in series
- Therefore, resistance adds for resistors in series

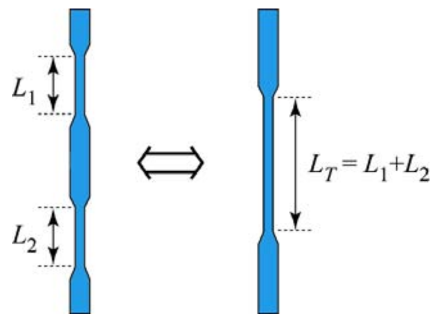


$$V_1 = IR_1 \qquad V_2 = IR_2$$

$$V_T = V_1 + V_2 = I\left(R_1 + R_2\right) = IR_T$$

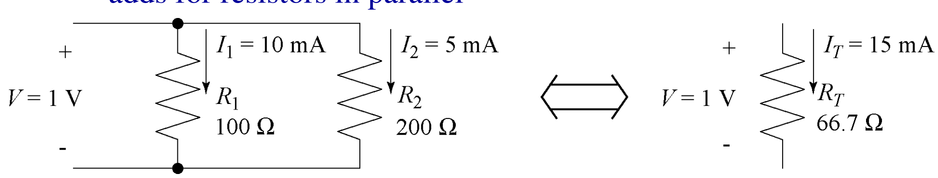
Water Model for Elements in Series

- Two water resistors in series are equivalent to one with the same length of narrow region as the two together
  - This same concept applies to the electrical resistance of a wire with uniform cross section and resistivity, the total resistance is proportional to length



Elements in Parallel

- When the same exact voltage (not just the same value!) appears across two elements, they are *in parallel*
- Currents add for elements in parallel
- Therefore, the reciprocal of resistance (called *conductance*) adds for resistors in parallel



$$I_1 = V/R_1$$

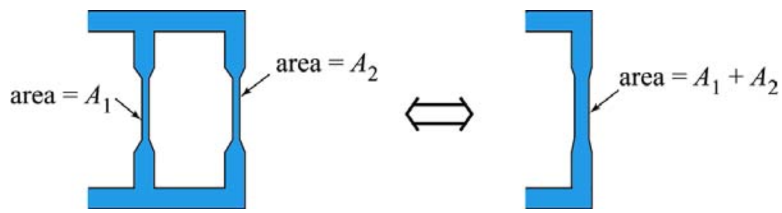
$$I_2 = V/R_2$$

$$I_T = I_1 + I_2 = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V}{R_T}$$

$$\therefore \quad R_T = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2}$$

Water Model for Resistors in Parallel

- Two water resistors in parallel is equivalent to one with the same cross-sectional area as the two combined
  - This same concept applies to electrical resistance – for resistors made with identical materials of equal length, the resistance is inversely proportional to the cross-sectional area

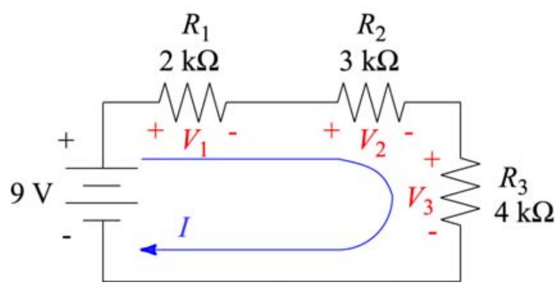


Sources in Series and Parallel

- Voltage sources in series add
  - two equal sources in parallel makes no difference
  - two unequal sources in parallel makes no sense with the simplest model
- Current sources in parallel add
  - two equal sources in series makes no difference
  - two unequal sources in series makes no sense with the simplest model
- Superposition applies for linear circuits; you can analyze the circuit for each source individually and sum the results

Kirchoff's Voltage Law (KVL)

- If you sum all of the voltage rises and drops around any loop, the total must be zero
  - Think about walking around Kemper Hall up and down different stairs, but returning to the same point
  - The signs are arbitrary mathematically, but meaningful physically



$$V_B - V_1 - V_2 - V_3 = 0$$

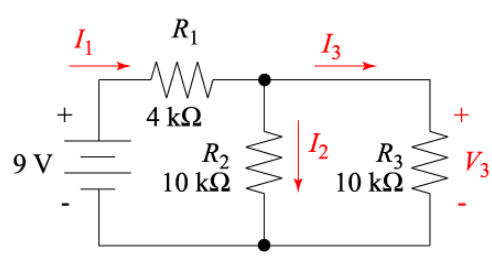
$$V_B = V_1 + V_2 + V_3$$

$$= I\left(R_1 + R_2 + R_3\right)$$

$$I = \frac{V_B}{R_1 + R_2 + R_3} = \frac{9\text{ V}}{9\text{ k}\Omega} = 1\text{ mA}$$

Kirchoff's Current Law (KCL)

- The sum of all currents entering, or leaving, a node must be zero
  - Think of water in pipes
  - Charge can't be created or destroyed



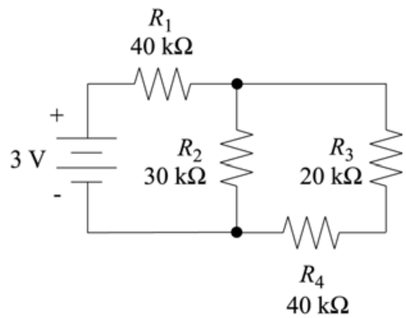
$$I_1 = I_2 + I_3$$

$$R_2 \| R_3 = 5\text{k}\Omega$$

$$I_1 = \frac{V_B}{R_1 + R_2 \| R_3} = \frac{9\text{ V}}{9\text{ k}\Omega} = 1\text{ mA}$$

DC Circuit Analysis - Practice

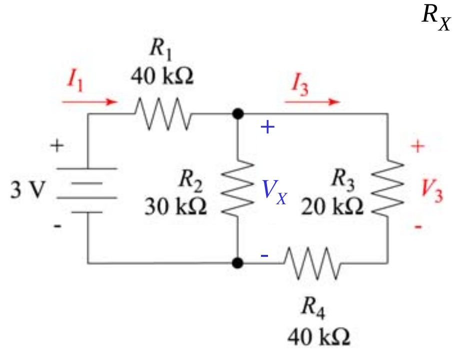
- Consider the circuit shown below



- Are R_3 and R_4 in series or parallel with each other?
- Is R_2 in series or parallel with R_3 and R_4?
- Is R_1 in series or parallel with R_2, R_3 and R_4?

DC Circuit Analysis - Practice

- Now let's find the labeled voltage and currents



$$R_X = R_2 \| (R_3 + R_4) = 30\text{k} \| 60\text{k} = 20\ \text{k}\Omega$$

$$I_1 = \frac{V_B}{R_1 + R_X} = \frac{3\ \text{V}}{60\ \text{k}\Omega} = 50\ \mu\text{A}$$

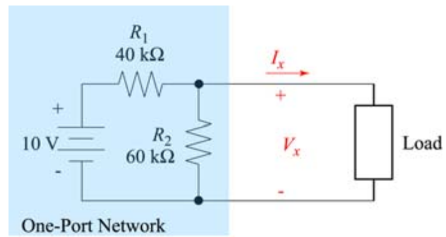
$$I_2 = \frac{V_X}{R_2} = \frac{I_1 R_X}{R_2} = \frac{1\ \text{V}}{30\ \text{k}\Omega} = 33.3\ \mu\text{A}$$

$$I_3 = I_1 - I_2 = 16.7\ \mu\text{A}$$

$$V_3 = I_3 R_3 = 0.33\ \text{V}$$

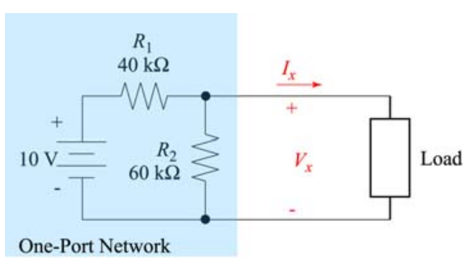
Equivalent Circuits

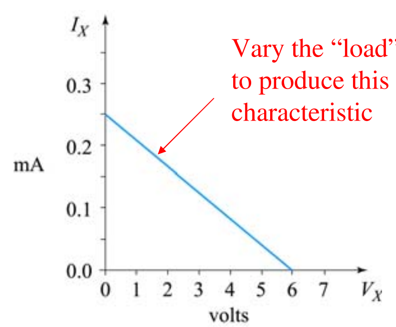
- We can find circuits that, in limited ways, are equivalent to other circuits
- These equivalent circuits may make analysis and/or design much easier
- For example, consider the circuit shown
  - It is a *one-port* network (i.e., it has one pair of terminals to which we can connect another circuit)
  - Can we come up with a circuit that is electrically identical as far as we can determine from the one port available?



Equivalent Circuits

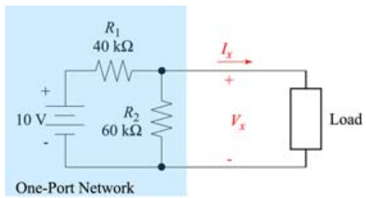
- Consider the *I-V* characteristic shown for the circuit
- Any one-port network that produces the same characteristic is, in this limited sense, equivalent

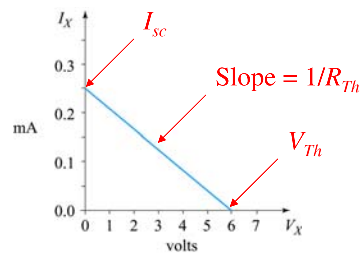


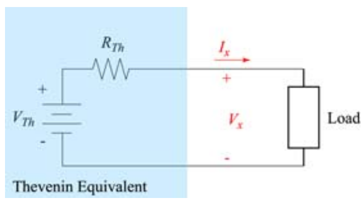


Thevenin Equivalent

- A one-port network comprising a voltage source and a single resistor can be made equivalent to *any* linear one-port network – this is called the Thevenin equivalent





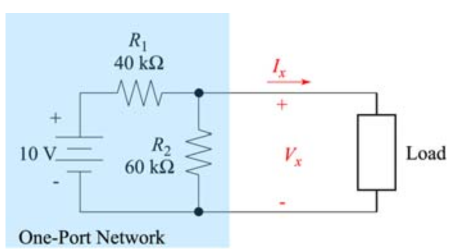


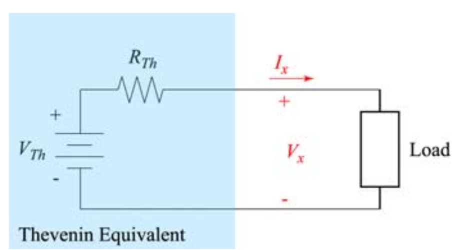
$$V_{Th} = V_X \quad \text{when} \quad I_X = 0$$

$$I_{sc} = I_X \quad \text{when} \quad V_X = 0$$

$$R_{Th} = V_{Th} / I_{sc}$$

Thevenin Equivalent Example





$$V_{Th} = V_X \quad \text{when} \quad I_X = 0$$

$$V_{Th} = \frac{R_2}{R_1 + R_2} V_B = 6 \text{ V}$$

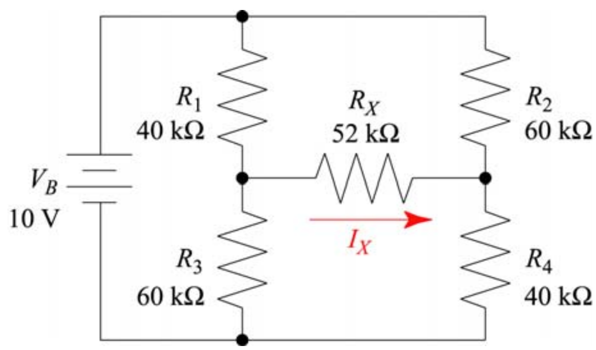
$$I_{sc} = V_B / R_1$$

$$R_{Th} = \frac{V_{Th}}{I_{sc}} = \frac{R_1 R_2}{R_1 + R_2} = R_1 \| R_2$$

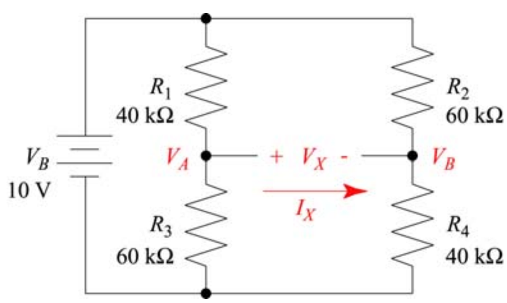
R_{Th} is the resistance seen "looking into" the port with the independent source(s) set to zero

Example Use of Thevenin Equivalent

- Consider finding the current indicated in the circuit below
- It is much easier to do if you find the Thevenin equivalent for the entire circuit driving R_X



Example - Continued



$$V_A = \frac{R_3}{R_1 + R_3} V_B = 6 \text{ V}$$

$$V_B = \frac{R_4}{R_2 + R_4} V_B = 4 \text{ V}$$

$$V_{Th} = V_X \quad \text{when} \quad I_X = 0$$

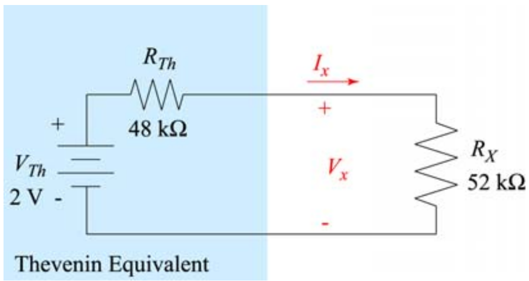
$$V_{Th} = V_A - V_B = 2 \text{ V}$$

To find R_{Th}, set V_B to zero and "look back into" the output port

$$R_{Th} = \left(R_1 \| R_3\right) + \left(R_2 \| R_4\right) = 48 \text{ k}\Omega$$

Example - Continued

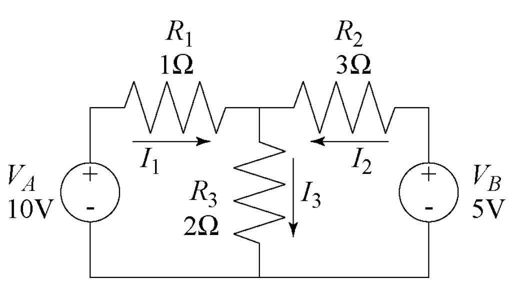
- Using the values just determined for the Thevenin equivalent circuit, we arrive at the much simpler circuit below



$$I_X = \frac{V_{Th}}{R_{Th} + R_X} = \frac{2\text{ V}}{100\text{ k}\Omega} = 20\ \mu\text{A}$$

Another Example

- Consider the circuit shown
- You can analyze this using:
  - Straightforward application of KVL & KCL
  - Superpositon with KVL & KCL
  - Thevenin (or Norton) equivalent
  - Mesh currents



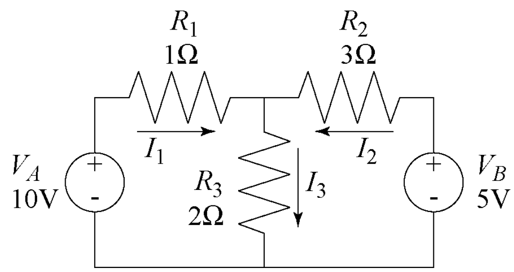
Another Example

- Straightforward application of KVL & KCL
- KVL: $V_A - I_1R_1 - I_3R_3 = 0$
- KVL: $V_B - I_2R_2 - I_3R_3 = 0$
- KCL: $I_3 = I_1 + I_2$
- Combine: $V_A = I_1(R_1 + R_3) + I_2R_3$ & $V_B = I_1R_1 + I_2(R_2 + R_3)$
- Solving:

$$I_1 = \frac{120}{33} = 3.64A$$

$$I_2 = \frac{15}{11} = -0.455A$$

$$I_3 = \frac{105}{33} = 3.18A$$



Another Example

- Straightforward application of KVL & KCL
- KVL: $V_A - I_1R_1 - I_3R_3 = 0$
- KVL: $V_B - I_2R_2 - I_3R_3 = 0$
- KCL: $I_3 = I_1 + I_2$
- Combine: $V_A = I_1(R_1 + R_3) + I_2R_3$ & $V_B = I_1R_1 + I_2(R_2 + R_3)$
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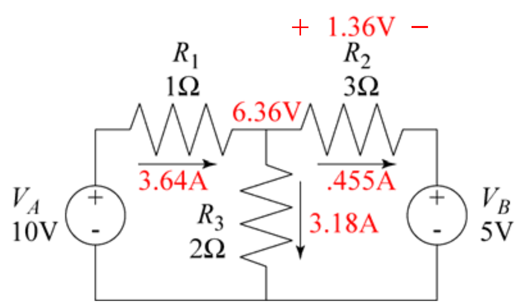
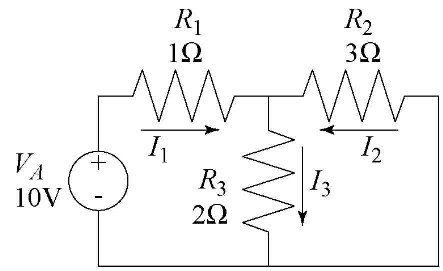


Another Example

- Superpositon with KVL & KCL
- Consider just V_A: $R_2 \| R_3 = \frac{6}{5} = 1.2\Omega$

$$R_1 + R_2 \| R_3 = 2.2\Omega \Rightarrow I_{1A} = 10/2.2 = 4.55A$$

$$V_{3A} = 4.55(1.2) = 5.45V \Rightarrow \begin{array}{l} I_{2A} = -5.45/3 = -1.82A \\ I_{3A} = 5.45/2 = 2.73A \end{array}$$



Another Example

- Superpositon with KVL & KCL
- Consider just V_B: $R_1 \| R_3 = \frac{2}{3} = 0.67\Omega$

$$R_2 + R_1 \| R_3 = 3.67\Omega \Rightarrow I_{2B} = 5/3.67 = 1.36A$$

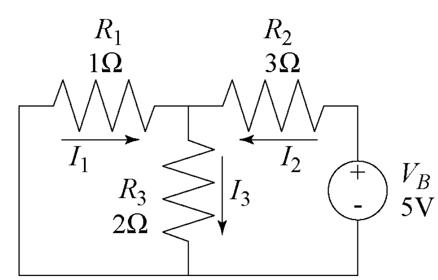
$$V_{3B} = 1.36(0.67) = 0.913V \Rightarrow \begin{array}{l} I_{1B} = -0.913A \\ I_{3B} = 0.913/2 = 0.456A \end{array}$$

- Totals:

$$I_1 = I_{1A} + I_{1B} = 3.64A$$

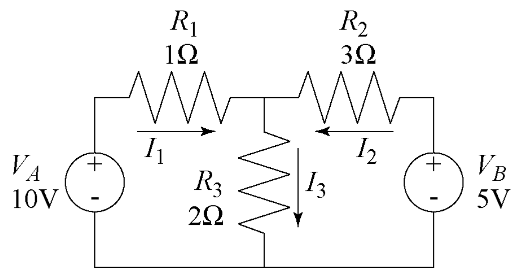
$$I_2 = I_{2A} + I_{2B} = -0.455A$$

$$I_3 = I_{3A} + I_{3B} = 3.18A$$



Another Example

- Thevenin (or Norton) equivalent – remove R_2 and V_B and find Thevenin equivalent looking to the left

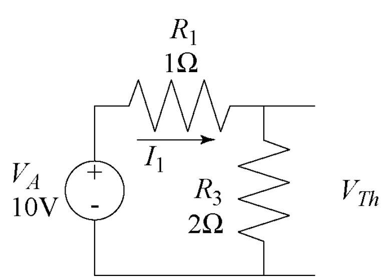


Another Example

- Thevenin (or Norton) equivalent – remove R_2 and V_B and find Thevenin equivalent looking to the left; by inspection,

$$V_{Th} = \frac{R_3}{R_1 + R_3} 10 = \frac{20}{3} = 6.67V$$

$$R_{Th} = R_1 \| R_3 = 0.67\Omega$$

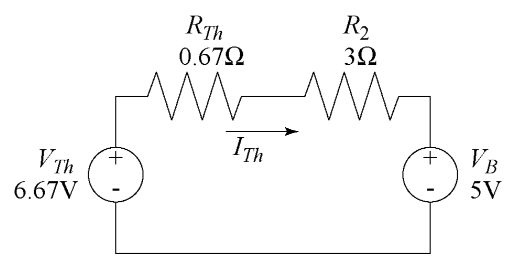


Another Example

- Thevenin (or Norton) equivalent
- Now use Ohm's law to find

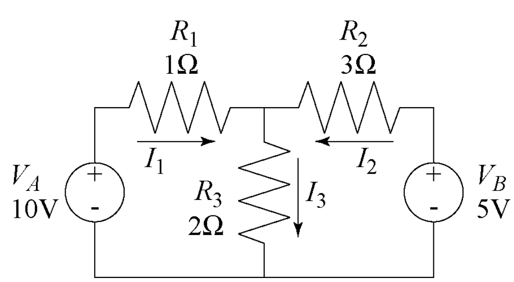
$$I_{Th} = \frac{6.67 - 5}{3.67} = 0.455A = -I_2$$

- Then you can find V_2, V_3 and the other currents



Another Example

- Mesh currents – re-label the currents as loops:



Another Example

- Mesh currents – re-label the currents as loops:
- Now write KVL around each loop:

$$V_A - I_{m1}R_1 - \left(I_{m1} + I_{m2}\right)R_3 = 0$$

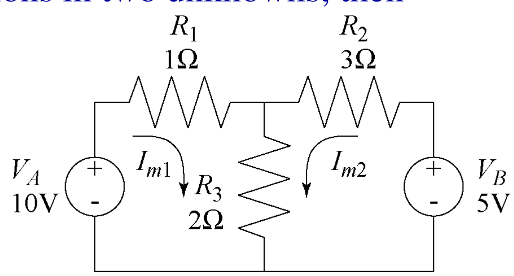
$$V_B - I_{m2}R_2 - \left(I_{m1} + I_{m2}\right)R_3 = 0$$

- Solve the two equations in two unknowns; then

$$I_1 = I_{m1}$$

$$I_2 = I_{m2}$$

$$I_3 = I_{m1} + I_{m2}$$



More DC Examples

- Do more examples on Board

AC Circuits Outline

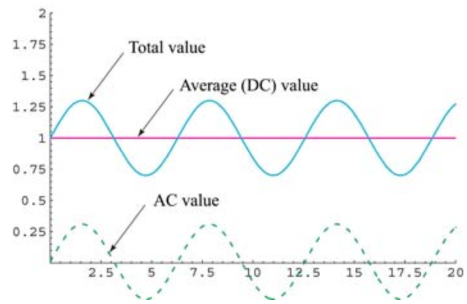
- Uses of DC and AC Circuits
- Notation
- AC Sources
- Time and Frequency Domains
- Capacitors

Uses of DC and AC Circuits

- DC circuits are used to:
  - *Bias* nonlinear elements to put them in a useful operating range, this is necessary, for example, to make amplifiers (DC power is used in virtually all consumer electronics devices)
  - Turn on lights, run small motors and other devices
- AC circuits are used to:
  - Transfer power
  - Transmit information (e.g., cell phone)
  - Store information (e.g., disc drives, CD's)
  - Manipulate information (e.g., computers)
- Note that many systems today use *digital* signals to transmit, store, and manipulate information.

Notation Used

- The standard IEEE notation for time-varying quantities is:
  - v_{AB} is the total instantaneous voltage from A to B
  - V_{AB} is the average (DC) voltage from A to B
  - v_{ab} is the time-varying (AC) voltage from A to B
  - Therefore, we have: $v_{AB} = V_{AB} + v_{ab}$

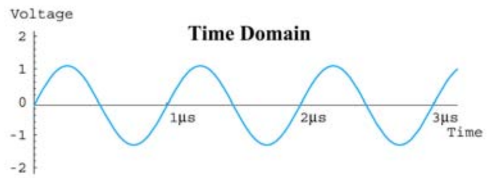


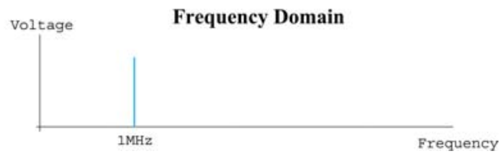
AC Sources

- AC voltages or currents are generated by:
  - microphones
  - musical instruments
  - sensors (e.g., pressure, temperature)
  - AC power generators
  - Other sources
- Most AC voltages and currents are not sinusoidal
- But, all practical periodic functions of time can be represented by summations of scaled and shifted sinusoids (Fourier series)
  - When the function is not periodic, we can still handle it

Time and Frequency Domains

- We can represent a single sine wave in either the *time domain*, or the *frequency domain*
- The frequency domain representation of a sine wave is a vertical line

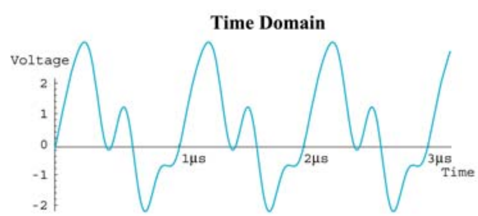


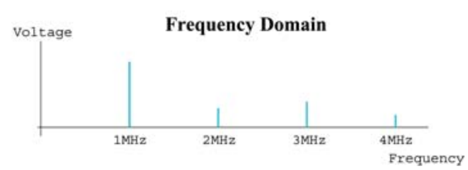


$$v(t) = \sin[2\pi\underbrace{(1\text{ MHz})}_{\text{frequency (cycles/sec = Hertz)}}t]$$

Complex Signals

- More complex signals can be decomposed into a sum of scaled and shifted sine waves (only the amplitude is shown here - there is a phase plot in the frequency domain too)

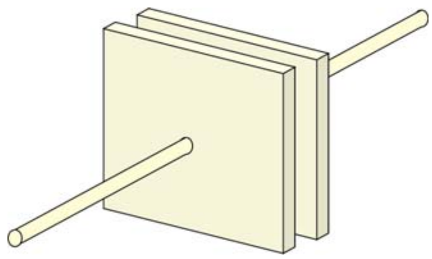




$$v(t) = \sin[2\pi(1\text{ MHz})t] + 0.3\sin[2\pi(2\text{ MHz})t - 0.3]$$
$$-0.4\sin[2\pi(3\text{ MHz})t + 0.5] + 0.2\sin[2\pi(4\text{ MHz})t - 0.1]$$

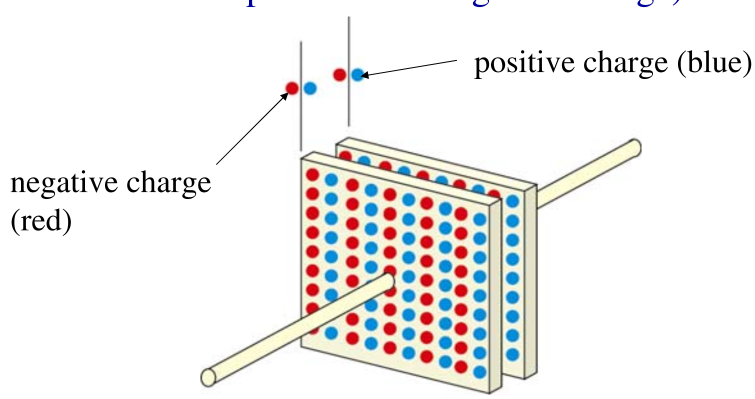
Capacitors

- A capacitor is a component that stores electric charge, just like a reservoir stores water
- A simple capacitor can be made by putting any two conductive plates close to, but not touching, each other



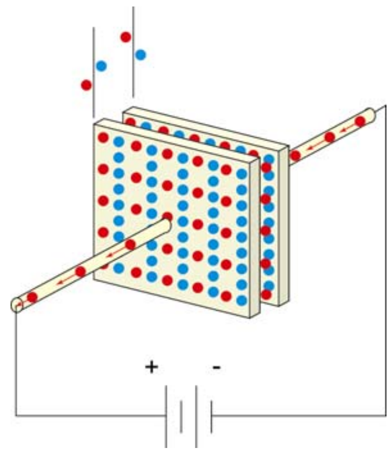
Uncharged Capacitor

- When a capacitor is uncharged, the plates are globally electrically neutral (i.e., they have equal amounts of positive and negative charge)



Charged Capacitor

- When an electrical potential is applied, mobile electrons are pulled from one plate and added to the other so that there is net positive charge on one plate and net negative charge on the other
- This global charge separation sets up an electric field between the plates
- Because the charges on the plates attract each other, the charge separation will persist if the capacitor is removed from the circuit



Capacitor Current

- Current cannot flow in one direction through a capacitor indefinitely - the cap will eventually charge up to the potential driving it and the current will stop
  - charges never really flow "through" the capacitor, we speak of the *displacement* current (i.e., charge moving on one side displaces charge on the other due to the field)
- The capacitance is defined by $C = Q/V$ and is measured in Farads (in honor of Faraday)
- The capacitor current is

$$i(t) = \frac{dq(t)}{dt} = \frac{dCv(t)}{dt} = C\frac{dv(t)}{dt}$$

- And the voltage is

$$v(t) = v(0) + \frac{1}{C}\int_0^t i(t)dt$$

Energy Storage in a Capacitor

- The energy stored in a capacitor can be found by integrating the power delivered up to that time

$$U_c(t) = \int_{-\infty}^{t} p(\alpha) d\alpha$$
$$= \int_{-\infty}^{t} v(\alpha)\left(C\frac{dv(\alpha)}{d\alpha} \right) d\alpha$$
$$= C\int_{-\infty}^{t} v(\alpha) dv(\alpha) = \frac{1}{2} C v^2(t)$$

we have assumed that $v(-\infty) = 0$

and turns out to only be a function of the voltage on the capacitor at the time

DC & Transients in a Capacitor

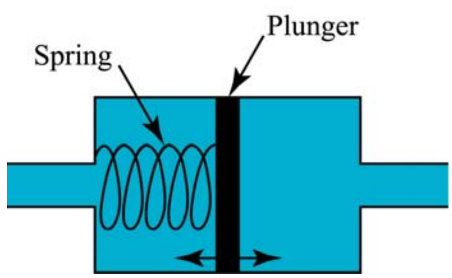
- Because the current is proportional to the time derivative of the voltage, if the voltage is constant (i.e., DC), the current is zero and a capacitor looks like an open circuit

$$i(t) = C\frac{dv(t)}{dt}$$

- Physically, this simply means that the charge on the capacitor is not changing
- Notice that because the charge must change in order to change the voltage, you cannot change the voltage on a capacitor instantly unless you supply an impulse of current, so it looks like a short circuit to step changes in current

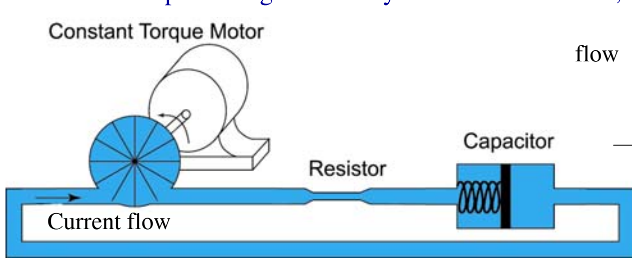
Water Model of a Capacitor

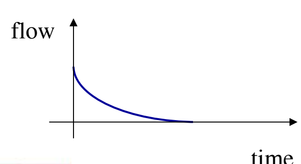
- One water model for a capacitor allows a plunger to move left or right depending on the pressure difference across it
- Energy is stored in the spring, and current only flows while the plunger is moving, i.e., $i = C(dv/dt)$



RC Circuit Example

- Consider the water circuit shown below
  - Turn on the motor at $t = 0$
  - The current flow is largest at the start when the capacitor spring is relaxed
  - The flow decreases as the spring is stretched and applies more back pressure (so the pressure across the resistor is lower)
  - Finally, when the capacitor spring applies a back pressure equal to the pressure generated by the motor & turbine, the flow stops

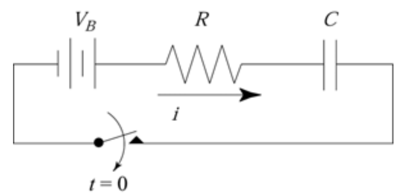




RC Circuit Example

- The equivalent electrical circuit is shown below
  - assume the switch closes at $t = 0$
  - assume the capacitor is initially uncharged
- We have:

$$i(t) = C\frac{dv_C(t)}{dt}$$



using KVL and Ohm's law leads to;

$$i(t) = C\frac{dv_C(t)}{dt} = \frac{V_B - v_C(t)}{R}$$

Solving the Differential Equation

- We had $$i(t) = C\frac{dv_C(t)}{dt} = \frac{V_B - v_C(t)}{R}$$
- Which leads to $$RC\frac{dv_C(t)}{dt} = V_B - v_C(t)$$

  or, $$V_B - v_C(t) - RC\frac{dv_C(t)}{dt} = 0$$
- The solution is $$v_C(t) = V_B\left(1 - e^{-t/RC}\right)$$
- Confirm

$$\frac{dv_C(t)}{dt} = \frac{V_B}{RC}e^{-t/RC}$$ so, $$V_B - V_B\left(1 - e^{-t/RC}\right) - RC\frac{V_B}{RC}e^{-t/RC} = 0$$

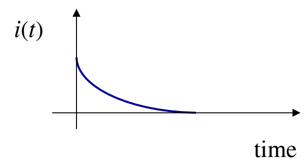
One More Solution

- We found $$i(t) = C\frac{dv_C(t)}{dt} = \frac{V_B - v_C(t)}{R}$$

  and $$v_C(t) = V_B\left(1 - e^{-t/RC}\right)$$

- Therefore

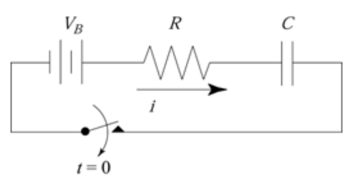
$$i(t) = \frac{V_B - V_B\left(1 - e^{-t/RC}\right)}{R}$$
$$= \frac{V_B}{R}e^{-t/RC}$$



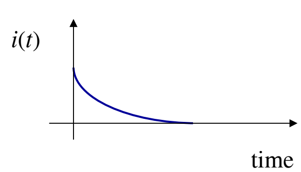
RC is called the *time constant* of the circuit

Develop Your Intuition

- At $t = 0$, $v_C = 0$, so $v_R = V_B$ and $i(0) = V_B/R$, which is the maximum value possible
- With current flowing into the cap, it charges up and $v_C > 0$, so the current decreases
- As the current decreases, rate of change of v_C decreases, and so does the rate of change of the current
- Eventually, $i(\infty) = 0$

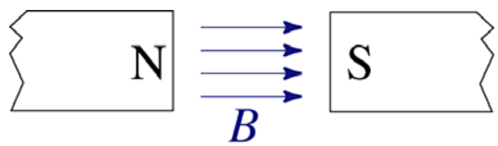


$$i(t) = \frac{V_B}{R}e^{-t/RC} \quad \text{for } t \geq 0$$



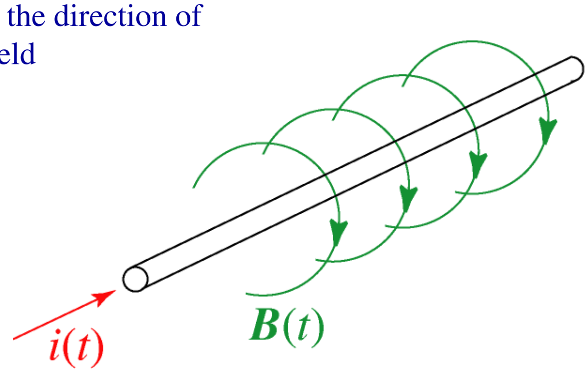
Magnetic Fields

- Magnets also exert force on objects at a distance
- We again represent this force-at-a-distance relationship by visualizing a magnetic field
- As with the electric field we define a flux density, **B**, and picture the flux with lines (you can "see" magnetic flux lines using iron filings)
- Unlike the electric field, there are no magnetic monopoles
- Also unlike **E** fields, the flux lines are NOT lines of force!



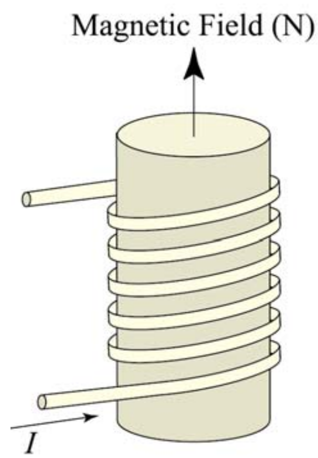
Magnetic Field Produced by Current

- An electric current in a wire produces a magnetic field around the wire that is proportional to the magnitude of the current
  - The direction of the field follows the "right hand rule" put your thumb in the direction of the current, and your fingers wrap in the direction of the magnetic field



Electromagnet

- If we have a coil of wire with a current through the wire, each turn will contribute to the magnetic field and we can produce an electromagnet
- The magneto-motive force (mmf) is the product of the current, *I*, and the number of turns, *N*; mmf = *NI*

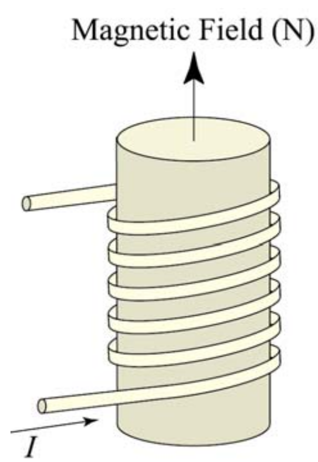


Flux in an Electromagnet

- The flux produced in the electromagnet depends on the mmf = *NI*
- If the coil is a linear coil, we have

$$\phi = \frac{mmf}{\Re} = \frac{NI}{\Re}$$

where the flux, ϕ, is measured in Webers, the mmf is in ampere-turns, and the reluctance, $\Re$, is in ampere-turns/Wb and is the magnetic equivalent of electrical resistance



Note: The flux density is B = ϕ/area in Wb/m^2

Magnetic Fields and Moving Charge

- Magnetic fields are caused by *moving* charges
- If a charge moves in a magnetic field it experiences a force given by $\mathbf{F} = q \times \mathbf{B}$
- There is energy stored in a magnetic field
- Since a DC current produces a constant magnetic field, the power transferred to the field is zero (once it is established); therefore, the voltage across a length of wire is zero (ignoring electrical resistance)
- When the current in a wire is changing, energy must be put into or taken out of the field; therefore, the voltage cannot be zero

Inductor Summary

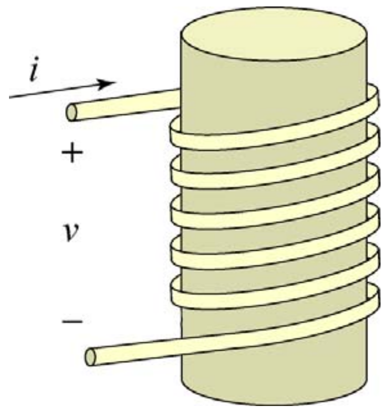
- Inductors are analogous to capacitors
  - For a capacitor, $Q = CV$
  - For an inductor, the magnetic field is proportional to the current, $\phi = NI/\Re$
  - A capacitor stores energy in the electric field
  - An inductor stores energy in the magnetic field
  - Current must flow into or out of a capacitor to change the voltage, so $$i(t) = C\frac{dv(t)}{dt}$$
  - Voltage must appear across an inductor to change the current, so $$v(t) = L\frac{di(t)}{dt}$$

Inductor

- Any wire has the property of inductance
- But, to make use of this property, we make coils so that the magnetic fields reinforce one another to increase the energy stored
- We experimentally find that

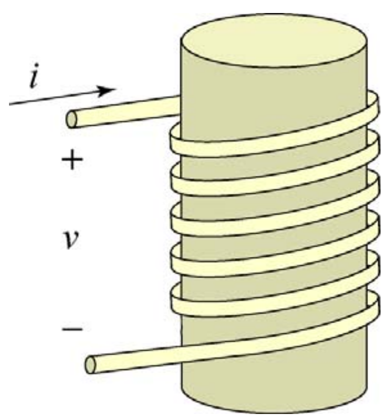
$$v = L\frac{di}{dt}$$

- The inductance, *L*, is measured in Henrys



Inductor

- This relationship makes sense!
  - when di/dt is large you are changing the energy in the field quickly, which requires greater power, so v must be larger
  - when di/dt is negative, the field is supplying energy to the rest of the circuit, so v must be negative



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Inductor Energy

- To increase the current in an inductor we must put energy into its field.
- Energy is the integral of power

$$E(t_2) = \int_{t_1}^{t_2} P(\alpha)\,d\alpha + E(t_1)$$

- Therefore, we must supply power to increase current, which means that the voltage across the inductor must be nonzero

$$P(t) = v(t)i(t)$$

Energy Storage in an Inductor

- The energy stored in an inductor can be found by integrating the power delivered up to that time

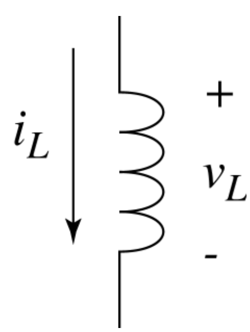
$$
\begin{aligned}
U_l(t) &= \int_{-\infty}^{t} p(\alpha)d\alpha \\
&= \int_{-\infty}^{t} i(\alpha)\left(L\frac{di(\alpha)}{d\alpha}\right)d\alpha \\
&= L\int_{-\infty}^{t} i(\alpha)di(\alpha) = \frac{1}{2}Li^2(t)
\end{aligned}
$$

we have assumed that $i(-\infty) = 0$

and turns out to only be a function of the current through the inductor at the time

Inductor Voltage and Current

- Constant current:
  $v_L = 0$
  (no energy transfer)



- Increasing current:
  $v_L > 0$
  (energy *to* inductor)

- Decreasing current:
  $v_L < 0$
  (energy *from* inductor)

$$v_L(t) = L\frac{di(t)}{dt}$$

Inductors

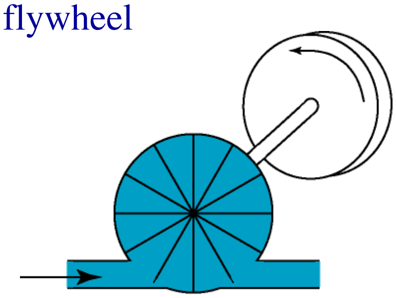
- Store energy in the magnetic field
- Can act as a current source (i.e., it takes work to change the current)
- Initially are an open circuit to step changes
- Steady state are a short circuit to DC
- v leads i

Capacitors

- Store energy in the electric field
- Can act as a voltage source (i.e., it takes work to change the voltage)
- Initially are a short circuit to step changes
- Steady state are an open circuit to DC
- i leads v

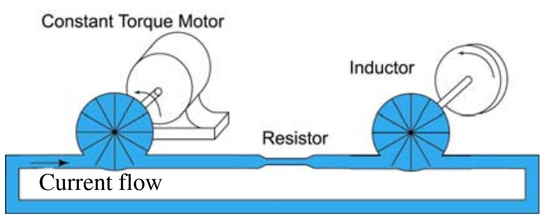
Water Model of an Inductor

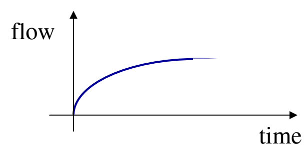
- One water model for an inductor uses a turbine connected to a flywheel
- A pressure difference must be applied across the turbine to change the current flow
- Energy is stored in the flywheel
- The device resists change in the flow (since the flywheel has rotational inertia and wants to keep a constant speed)



RL Circuit Example

- Consider the water circuit shown below
  - Turn on the motor at $t = 0$ (the motor has been off a long time)
  - The current flow is zero at first since it takes time to put energy into the flywheel on the inductor
  - As the flow builds up, there is an increasing pressure drop across the resistor, so the rate at which the flow increases slows down
  - Eventually, the flow reaches a steady-state value and stops changing. At that time, all of the pressure is dropped across the resistor



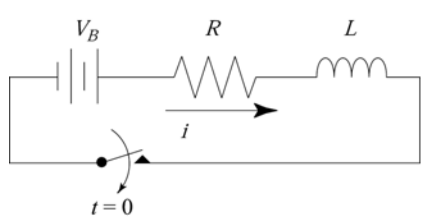


RL Circuit Example

- The equivalent electrical circuit is shown below
  - assume the switch closes at $t = 0$
  - assume the initial inductor current is zero
- We have:

$$v_L(t) = L\frac{di(t)}{dt}$$

using KVL and Ohm's law leads to;



$$i(t) = \frac{V_B - v_L(t)}{R} = \frac{V_B - L\,di(t)/dt}{R}$$

Solving the Differential Equation

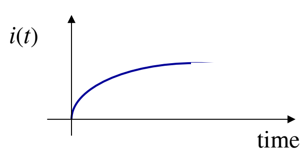
- We had $$i(t) = \frac{V_B - v_L(t)}{R} = \frac{V_B - L\,di(t)/dt}{R}$$
- Which leads to $$Ri(t) = V_B - L\frac{di(t)}{dt}$$

  or, $$V_B - L\frac{di(t)}{dt} - Ri(t) = 0$$
- The solution is $$i(t) = \frac{V_B}{R}\left(1 - e^{-tR/L}\right)$$
- Confirm

$$\frac{di(t)}{dt} = \frac{V_B}{L}e^{-tR/L}$$ so, $$V_B - L\frac{V_B}{L}e^{-tR/L} - V_B\left(1 - e^{-tR/L}\right) = 0$$

Examine the Solution

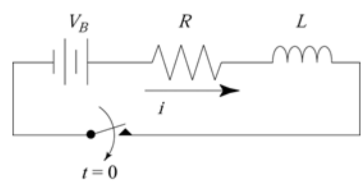
- We found $$i(t) = \frac{V_B}{R}\left(1 - e^{-tR/L}\right)$$



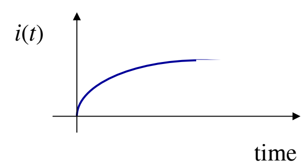
L/*R* is called the *time constant* of the circuit

Develop Your Intuition

- At $t = 0$, $i = 0$, so $v_R = 0$ and $v_L = V_B$, which is the maximum value possible
- With voltage across the inductor, the current increases so $i > 0$ and $v_R > 0$
- With $v_R > 0$, v_L is smaller and the rate of change of the current decreases
- Eventually, $v_L(\infty) = 0$



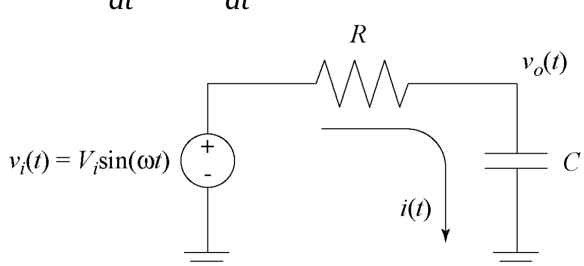
$$i(t) = \frac{V_B}{R}\left(1 - e^{-tR/L}\right) \quad \text{for } t \geq 0$$



AC Circuit Example

- With a sinusoidal source, the steady-state solution to the circuit equation yields sinusoids of the same frequency for all voltages and currents
- Say that $v_C(t) = V_c \sin(\omega t + \phi)$
  where ϕ is some as yet unknown phase angle
- Then

$$i(t) = C\frac{dv_C(t)}{dt} = C\frac{d}{dt}\left[V_c \sin(\omega t + \phi)\right] = \omega C V_c \cos(\omega t + \phi)$$



Impedance of a Capacitor

- We can write the current as

$$i(t) = \omega C V_c \cos\left(\omega t + \phi\right) = \omega C V_c \sin\left(\omega t + \phi + \pi/2\right)$$

- We had $v_C(t) = V_c \sin\left(\omega t + \phi\right)$
- We can then define an *impedance*, which is much like a resistance, but for AC sources, and the complex impedance of a capacitor is

$$Z_c = \frac{v_C(t)}{i_C(t)} = \frac{1}{j\omega C}$$

where the 1/*j* accounts for the 90º phase shift

- The magnitude of the impedance is measured in Ohms, just like resistance

Impedance of an Inductor

- Remember that an inductor has $v_L(t) = L\frac{di}{dt}$
- If the current is $i(t) = I_l \sin\left(\omega t + \phi\right)$
- The voltage becomes

$$v(t) = L\frac{di(t)}{dt} = L\frac{d}{dt}\left[I_l \sin\left(\omega t + \phi\right)\right] = \omega L I_l \cos\left(\omega t + \phi\right)$$

- We can rewrite this as

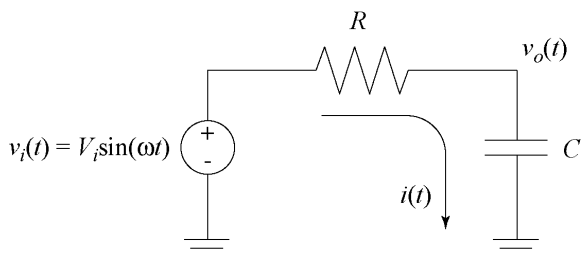
$$v(t) = \omega L I_l \cos\left(\omega t + \phi\right) = \omega L I_l \sin\left(\omega t + \phi + \pi/2\right)$$

- Therefore, the complex impedance is

$$Z_l = \frac{v(t)}{i(t)} = j\omega L$$

Using the Impedance

- Note that Ohm's law applies and $|Z_c|$ drops with increasing frequency ($\omega = 2\pi f$, f in Hertz and ω in rad/sec)
- Therefore, the resistor and capacitor form a voltage divider (similar to two resistors in series), but the impedance of the capacitor drops with increasing frequency



Combining Impedances

- Impedances add in series and parallel just like resistances do, but the quantities are complex
- For two inductors in series, for example, the impedance is (i.e., $L_s = L_1 + L_2$)

$$Z_s = Z_1 + Z_2 = j\omega L_1 + j\omega L_2 = j\omega\left(L_1 + L_2\right)$$

- For two capacitors in parallel the impedance is (i.e., $C_p = C_1 + C_2$)

$$Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{\left(\frac{1}{j\omega C_1}\right)\left(\frac{1}{j\omega C_2}\right)}{\frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}} = \frac{1}{j\omega\left(C_1 + C_2\right)}$$

Combining Impedances

- For two inductors in parallel the impedance is (i.e., $L_p = L_1L_2/(L_1 + L_2)$)

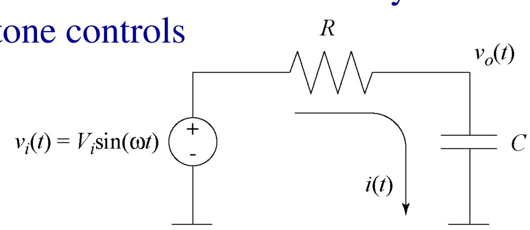
$$Z_p = \frac{Z_1Z_2}{Z_1 + Z_2} = \frac{(j\omega L_1)(j\omega L_2)}{j\omega L_1 + j\omega L_2} = j\omega\frac{L_1L_2}{L_1 + L_2}$$

- For two capacitors in series the impedance is (i.e., $C_s = C_1C_2/(C_1 + C_2)$)

$$Z_s = Z_1 + Z_2 = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2} = \frac{1}{j\omega}\frac{C_1 + C_2}{C_1C_2}$$

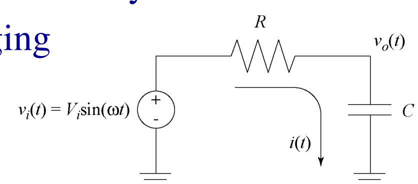
Low-Pass Filter

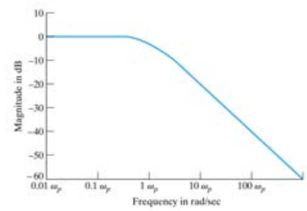
- Because the impedance of the capacitor drops with increasing frequency, the voltage divider from $v_i(t)$ to $v_o(t)$ decreases too
- In other words, this circuit passes low frequencies well, i.e., $v_o(t)$ nearly equal to $v_i(t)$
- But, at high frequencies, $v_o(t)$ is much less than $v_i(t)$ (we say the signal is *attenuated* by the circuit)
- A filter like this would make music sound heavy on the bass (in fact, the tone controls on your stereo are just variable filters)



Low-Pass Filter Intuition

- If you think about the capacitor as a large reservoir of water, saying that it's impedance drops at high frequencies is like saying you can't change the level of the reservoir very fast
- In other words, since changing the voltage on a capacitor requires changing the charge on it, there is a limit to how fast you can do it

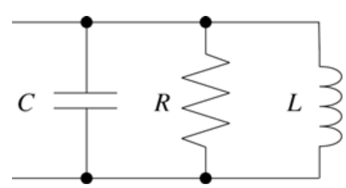




Example: parallel resonant circuit

- The transfer function is

$$Z_p(j\omega) = \frac{V(j\omega)}{I(j\omega)}$$



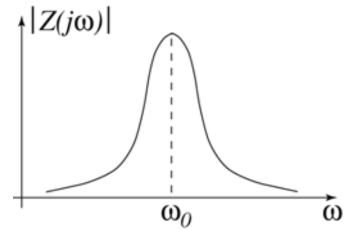
- Which is the parallel combination

$$Z_p(j\omega) = \frac{1}{\dfrac{1}{R} + j\omega C + \dfrac{1}{j\omega L}} = \frac{R}{1 + jR\left(\omega C - \dfrac{1}{\omega L}\right)}$$

Transfer Function

- The resulting transfer function (i.e., v_{out}/i_{in}) is a bandpass filter (BPF).

Note: This response is what you get if you sweep the frequency of a sinusoidal input current, measure the resulting voltage, and take the ratio.



- Where

$$\omega_0 = \frac{1}{\sqrt{LC}}$$