Name: Solutions

Instructions: This test consists of 4 problems and 13 pages, including the cover sheet. Please make sure that you have all of them. This is an open-book, open-notes test. State any assumptions you make and show complete work to receive credit. The time limit is 80 minutes. The problems are weighted as shown below.

Grading:

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Typo: Problem 4, $V_{DD} = 6V$
1 CMOS Inverter Amplifier

Figure 1 shows a CMOS inverter biased as a linear amplifier. For this problem, use the transistor parameters in Table 1.

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<tr>
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<td>0.1 V$^{-1}$</td>
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Table 1: Problem 1 Transistor Parameters.

**Problem 1.1 (3 points)** Assume $V_i = 1.5$ V. For what range of $V_O$ will the circuit best act as an amplifier? What is the output voltage swing?

Want $M1 + M2$ in saturation:

$V_{BS1} = V_T = 1.5$ V, $V_{OV1} = 0.5$ V $\Rightarrow$ $V_{out} \geq 0.5$ V

$V_{BS2} = V_{DD} - V_T = 1.5$ V, $V_{OV2} = 0.5$ V $\Rightarrow$ $V_{out} \leq 2.5$ V

$0.5 \leq V_{out} \leq 2.5$ V

Output swing = 2 V as amplifier

$= 3$ V ($0 \leq V_O \leq V_{DD}$) for output levels

**Problem 1.2 (2 points)** Find the power dissipation assuming the bias conditions in Problem 1.1.

$I_{D1} = I_{D2} = \frac{k'}{2} \left( \frac{W}{L} \right) (V_T - V_{th})^2 = \frac{300 \mu A/V^2}{2} (2) (1.5 V - 1)^2$

$= 75 \mu$A ($86.2 \mu$A if $\lambda = 0.1$ V$^{-1}$)

$P = I_{D1}V_{DD} = 225 \mu$W
Figure 1: CMOS inverter as linear amplifier.
Problem 1.3 (2 points) Find the small-signal output resistance $R_o$ assuming the bias conditions in Problem 1.1.

Small Signal model:

\[ V_i = V_{gs1} = V_{gs2} \]

\[ r_{o1} = \frac{1}{\lambda I_0} = \frac{1}{(0.1)(75 \mu A)} = 133 \, k\Omega \quad (116 \, k\Omega \, \text{for} \, I_0 = 78 \mu A) \]

\[ r_{o2} = r_{o1} \]

\[ R_{out} = r_{o1} || r_{o2} = \frac{66.5 \, k\Omega}{58.6} \, k\Omega \]

Problem 1.4 (3 points) Find the small-signal gain assuming the bias conditions in Problem 1.1.

\[ A_V = -G_m R_o = -(g_{m1} + g_{m2}) \left( r_{o1} || r_{o2} \right) \]

\[ g_{m1} = k_n \left( \frac{W}{L} \right)_1 (V_{GS1} - V_{TH}) = (300 \, \mu A/V^2) (2)(1.5 \, V - 1 \, V) = 300 \, \mu A/V \]

\[ g_{m2} = k_p \left( \frac{W}{L} \right)_2 (V_{GS2} - V_{TH}) = (100 \, \mu A/V^2)(6)(1.5 \, V - 1 \, V) = 300 \, \mu A/V \]

\[ A_V = 39.9 \approx 40 \]

Problem 1.5 (2 points) Suppose that $\gamma$ is nonzero and that through source-well biasing, $V_{th} = 1.25 \, V$ and $V_{tp} = -1.25 \, V$. How does this affect the small-signal gain (assume the bias conditions in Problem 1.1)?

\[ A_V = -(g_{m1} + g_{m2}) \left( r_{o1} || r_{o2} \right) = \frac{-(g_{m1} r_{o1}) r_{o2}}{r_{o1} + r_{o2}} - \frac{r_{o1} (g_{m2} r_{o2})}{r_{o1} + r_{o2}} \]

\[ g_{m1} r_{o1} = \frac{1}{\lambda} \left( \frac{2}{V_{ GS1} - V_{ T1 }} \right) = \left( \frac{1}{0.1} \right) \left( \frac{2}{1.5 - 1} \right) = 40 \quad \text{for} \quad V_{t1} = 1.0 \, \text{V} \]

\[ g_{m1} r_{o1} = 80 \quad \text{for} \quad V_{t1} = 1.25 \]

\[ A_V \, \text{doubles,} \quad A_V \approx 80 \]
2 Differential Amplifier

Figure 2 shows the circuit schematic for a differential amplifier. For this problem, use the transistor parameters in Table 2. Assume the following circuit parameters: $I_{TAIL} = 590\mu A$, $R_{TAIL} = 30\, k\Omega$, $R1 = 3\, k\Omega$, $R2 = 6\, k\Omega$, $(W/L)_1 = (W/L)_2 = 8$.

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Table 2: Problem 2 Transistor Parameters.

Problem 2.1 (3 points) Find $V_I$ such that $V_s = 300 \, mV$.

$I_1 = I_2 \Rightarrow I_1 = \frac{1}{2} \left( I_{TAIL} + \frac{V_s}{R_{TAIL}} \right) = \frac{1}{2} \left( 590\, \mu A + \frac{300 \, mV}{30 \, k\Omega} \right) = 300\, \mu A$

$V_{ov1} = \sqrt{\frac{2I_1}{k' \cdot \left(\frac{W}{L}\right)_1}} = \sqrt{\frac{2 \left( \frac{300\, \mu A}{300\, \mu A} \right)}{\left(\frac{8}{v^2}\right)}} = 0.5 \, V$

$V_I = V_s + V_{o1} + V_{ov1} = 0.3 \, V + 1 \, V + 0.5 \, V = 1.8 \, V$

For $I_1 = \frac{I_{TAIL}}{2}$, $V_I = 1.796 \, V$

Problem 2.2 (3 points) Find $(W/L)_3 = (W/L)_4$ such that $V_O = 1.5 \, V$, assuming the bias conditions you found in Problem 2.1.

$I_3 = I_4 \quad I_3 = I_1 + \frac{V_o}{R_1 + 2 R_2} = \frac{300\, \mu A + \frac{1.5 \, V}{3 \, k\Omega + 2 \left(6 \, k\Omega\right)}}{1.5 \, V} = 400 \, \mu A$

$V_{o3} = \left| \frac{2R_2}{R_1 + 2R_2} \right| - V_{DD} = \left| \frac{12 \, k\Omega}{15 \, k\Omega} \right| \left( V_O - 3 \, V \right) = 1.8 \, V \Rightarrow 400 \, \mu A = \frac{k'}{2} \left( \frac{W}{L} \right)_3 \left( V_{o3} - V_{o4} \right)^2$

$\Rightarrow \left( \frac{W}{L} \right)_3 = \left( \frac{100\, \mu A}{V^2} \right) \left( 1.8 - 1 \, V \right)^2 = 12.5 \quad \left( \frac{W}{L} \right)_4 = 12.5$

For $I_3 = 295\, \mu A$, $(W/L)_3 = 23.6$
Figure 2: Differential amplifier.
Problem 2.3 (4 points) Draw and label the small-signal differential-mode half circuit, and find the differential-mode gain, $\frac{v_{od}}{v_{id}}$, where $v_{id} = v_{i1} - v_{i2}$ and $v_{od} = v_{o1} - v_{o2}$.

$g_{m1} = K_n \left( \frac{W}{L} \right) \frac{V_{ov1}}{V^2} = \frac{300 \mu A}{V} \left( 8 \right) \left( 0.5 \text{V} \right) = 1200 \mu A/V$

$R_o = R_1 = 3 \text{k}\Omega$

$\frac{V_{od}}{V_{id}} = -\left( \frac{1200 \mu A/V}{3 \text{k}\Omega} \right)$

$\frac{V_{od}}{V_{id}} = -3.6$

Problem 2.4 (6 points) Draw and label the small-signal common-mode half circuit, and find the common-mode gain, $\frac{v_{oc}}{v_{ic}}$, where $v_{ic} = 0.5(v_{i1} + v_{i2})$ and $v_{oc} = 0.5(v_{o1} + v_{o2})$.

$i_1 = g_{m1} \left( V_{ic} - I_1 \cdot 2R_{TIE} \right) \Rightarrow i_1 = \frac{g_{m1} V_{ic}}{1 + 2R_{TIE}g_{m1}}$

$i_2 = \frac{V_{oc}}{R_1 + 2R_2}$

$i_3 = -g_{m3} \left( \frac{2R_2}{2R_2 + R_1} \right) V_{oc}$

$i_1 = i_3 - i_2$

$\frac{V_{oc}}{V_{ic}} = \frac{1.64 \times 10^{-5}}{(0.67 \times 10^{-5} + 8.0 \times 10^{-4})} \Rightarrow \frac{V_{oc}}{V_{ic}} = 0.019$

$g_{m3} = K_p \left( \frac{W}{L} \right)_3 V_{ov3} = \frac{1 \text{mA/V}}{5}$

$i_3 = -\frac{1 \text{mA/V}}{5} \cdot 4V_{oc} = \left( 8 \times 10^{-4} \right) V_{oc}$
3 Current Source

Figure 3 shows a current source circuit. For this problem, use the transistor parameters in Table 3. For the circuit in Figure 3, assume the following circuit parameters: \( R = 10 \, \text{k}\Omega \), \((W/L)_1 = (W/L)_3 = 2\), \((W/L)_2 = 20\).

**Problem 3.1 (2 points)** Find the output current \( I_O \).

\[
V_{gs1} = V_{gs2} + V_R = V_{gs2} + I_0R \quad \Rightarrow \quad V_{ov1} = V_{ov2} + I_0R \quad (V_{t1} = V_{t2} \text{ since } V = 0)
\]
\[
\frac{2(600\,\mu A)}{(300\,\mu A/\sqrt{2})(2)} = 1.414 \quad V = \sqrt{\frac{2I_0}{(300\,\mu A/\sqrt{2})(20)}} + I_0(10 \, \text{k}\Omega)
\]

Solve by iteration or quadratic formula (Widlar current source):

\[
I_0 = 121 \, \mu A
\]
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Table 3: Problem 3 Transistor Parameters.

Problem 3.2 (3 points) Find the minimum output voltage $V_{OUT}(MIN)$.

\[
V_{OUT(\text{min})} = I_o R + V_{OV2} + V_{OV3}
\]

\[
V_{OV3} = \frac{2 I_o}{k'_n (W/L)^3} = \frac{2 (12 \mu A)}{(300 \mu A/V^2)^2} = 0.635 \text{ V}
\]

\[
V_{OV2} = \frac{2 (12 \mu A)}{(300 \mu A/V^2)(20)} = 0.201 \text{ V}
\]

\[
I_o R = (12 \mu A)(10k \Omega) = 1.21 \text{ V} \Rightarrow V_{OUT(\text{min})} = 2.046 \text{ V}
\]

Problem 3.3 (3 points) Find the output resistance $R_o$.

Output branch looks like cascode w/ common-source amp w/ source degeneration:

Cascode: $R_o = R_{O3} \left[ 1 + g_m R_{O2} \right] + R_{O2}$

C.S. Amp w/ degeneration: $R_{O2} = R + R_{O2} (1 + g_m R)$

\[
R_{O2} = 1.09 \ \text{M} \Omega
\]

\[
R_0 = 35.5 \ \text{M} \Omega
\]

Or $R_0 \leq g_{m3} R_{O3} (g_m R_{O2} R) \geq 31.4 \ \text{M} \Omega$
Problem 3.4 (2 points) Resistor $R$ can be used to model the parasitic resistance associated with routing current sources for long distances on an IC. Find the largest value for $R$ such that $I_o$ is within 1% of the current reference value in Figure 3. Assume $(W/L)_1 = (W/L)_3 = (W/L)_2 = 2$.

$I_o(\text{min}) = (0.99)(600 \, \mu\text{A}) = 594 \, \mu\text{A}$

$I_o(\text{min}) R = V_{ov1} - V_{ov2} \bigg|_{I_o(\text{min})} \Rightarrow R (594 \, \mu\text{A}) = \sqrt{\frac{2(600 \, \mu\text{A})}{(300 \, \mu\text{A})/\sqrt{2}}(2)} - \sqrt{\frac{2(594 \, \mu\text{A})}{(300 \, \mu\text{A})/\sqrt{2}}(2)}$

$R = 119 \, \Omega$
4 Current Source Reference

Figure 4 shows a current reference circuit. For this problem, use the transistor parameters in Table 4. For the circuit in Figure 4, assume the following circuit parameters: \( R = 10 \, \text{k}\Omega \), \((W/L)_1 = (W/L)_2\), \((W/L)_3 = 2\), and \((W/L)_4 = (W/L)_5 = (W/L)_6 = 6\).

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Table 4: Problem 4 Transistor Parameters.

**Problem 4.1 (6 points)** Find \((W/L)_1 = (W/L)_3\) and \((W/L)_7\) such that \( I_o \) is dependent on process parameters only (i.e., independent of bias point to first order).

\[
\begin{align*}
\text{KVL (bottom): } & V_{G53} + V_R = V_{G57} + V_{G52} + (-V_{G7}) \\
& V_R = V_{nV} + V_{o1} + V_{n+} + V_{o2} + |V_{tp}| + |V_{ov7}| - V_{ov3} - V_{tn} \\
& = \underbrace{V_{ov1} + V_{ov2} + |V_{ov7}| - V_{ov3}}_{\text{bias dependent}} + \underbrace{V_{n+} + |V_{tp}|}_{\text{process dependent}} \\
\text{Current mirror (top): } & (W/L)_4 = (W/L)_5 = (W/L)_6 \implies I_{\text{REF}} = I_2 = I_o \\
& V_R = \sqrt{\frac{2I_o}{k'_n (W/L)_1}} + \sqrt{\frac{2I_o}{k'_n (W/L)_2}} + \sqrt{\frac{2I_o}{k'_n (W/L)_7}} - \sqrt{\frac{2I_o}{k'_n (2)}} + 2V \\
\text{Choose } & (W/L)_1 = (W/L)_2 = 9(W/L)_3 \\
\text{Choose } & (W/L)_7 = \frac{k'_n}{k'_p} \cdot 9(W/L)_3 = 27(W/L)_3 \\
\implies & V_R = \frac{1}{3} \sqrt{\frac{I_o}{k'_n}} + \frac{1}{3} \sqrt{\frac{I_o}{k'_n}} + \frac{1}{3} \sqrt{\frac{I_o}{k'_n}} - \sqrt{\frac{I_o}{k'_n}} + 2V = 2V \implies I_o = \frac{V_R}{R} = \frac{2V}{10 \, \text{k}\Omega} \\
& I_o = 200 \, \mu\text{A}
\end{align*}
\]
Figure 4: Current source reference.
Problem 4.2 (6 points) One can model the thermal noise of resistor $R$ as a small-signal current source $i_n$ in parallel with $R$. Quantify the impact of the noise current source on the total output current $I_o + i_o$, assuming the bias point you found in Problem 4.1 and $|i_n| = 1\mu A$.

Small-signal model:

$$i_n \quad \frac{i_c}{i_n} = \frac{R}{R + R_{i3}} \quad \text{(current divider)}$$

$$i_o = 10 \text{ k}\Omega \frac{10 \text{ k}\Omega}{10 \text{ k}\Omega + 2.04 \text{ k}\Omega} = 0.83 \Rightarrow |i_o| = 0.83 \mu A$$

Current gain for common-gate:

$$\frac{i_o}{i_{c}} = \frac{1}{g_{m3}} \quad (\lambda = 0)$$

$$g_{m3} = \sqrt{\frac{2I_0 k_n \lambda}{L}}$$

$$= \sqrt{\frac{2(200 \mu A)(300 \mu A)}{2}}$$

$$= 4.90 \times 10^{-4} \text{ S}$$

$$\frac{|i_o|}{|i_n|} = 0.415 \%$$