

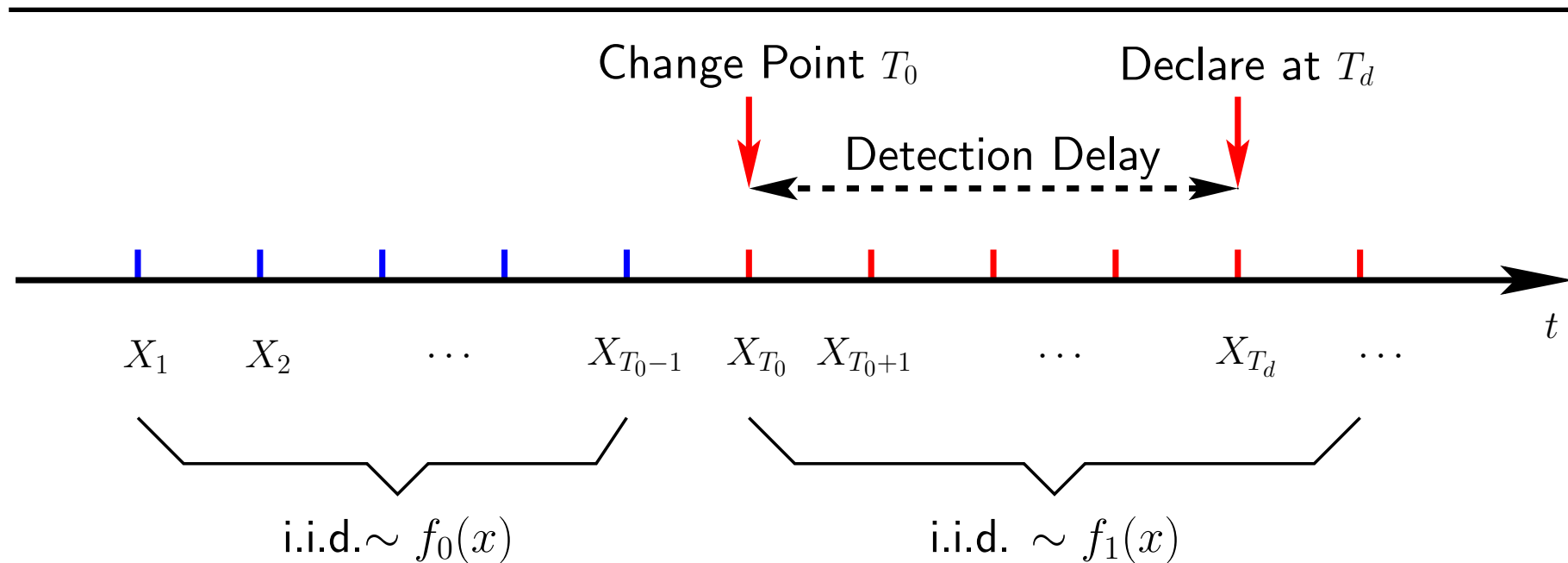
Quickest Change Detection in Multiple On-Off Processes

Qing Zhao, Jia Ye

Department of Electrical and Computer Engineering
University of California, Davis, CA 95616

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Quickest Change Detection

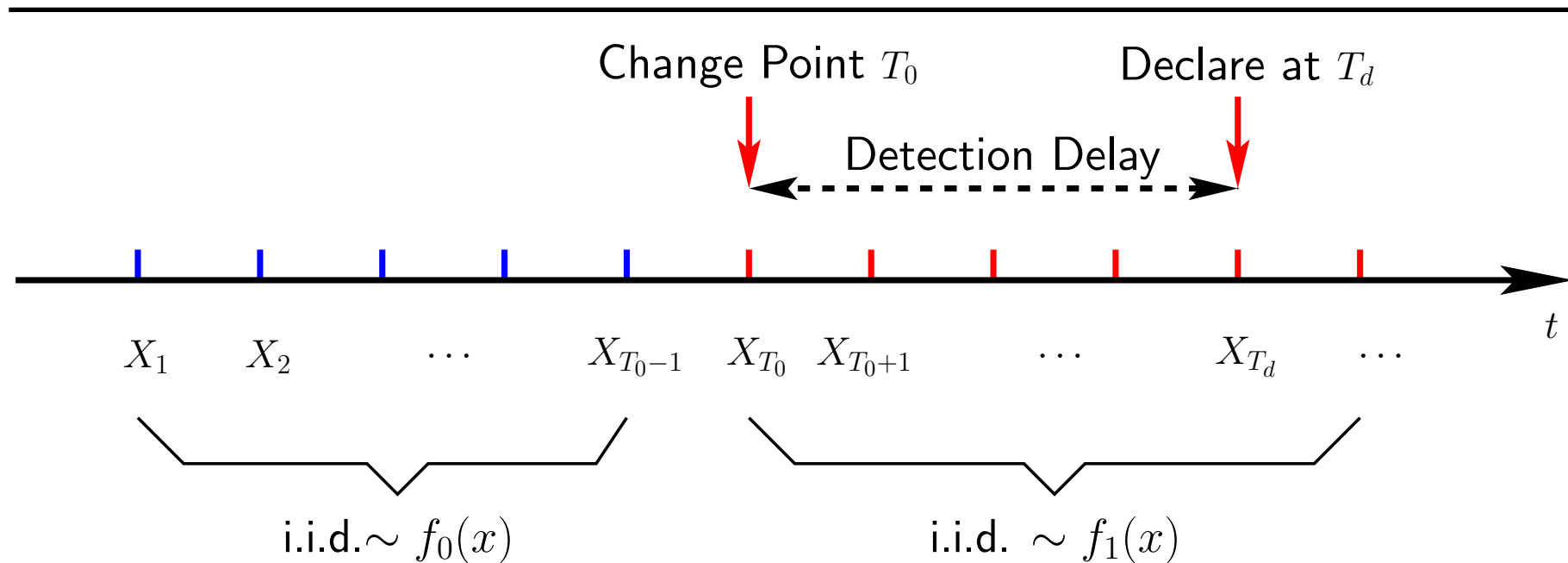


► **Quickest Detection:** \min $\underbrace{\mathbb{E}[(T_d - T_0)^+]}$ subject to $\underbrace{\Pr[T_d < T_0] \leq \zeta}$

Detection Delay Reliability Constraint

► **Tradeoff:** Detection delay vs. detection reliability.

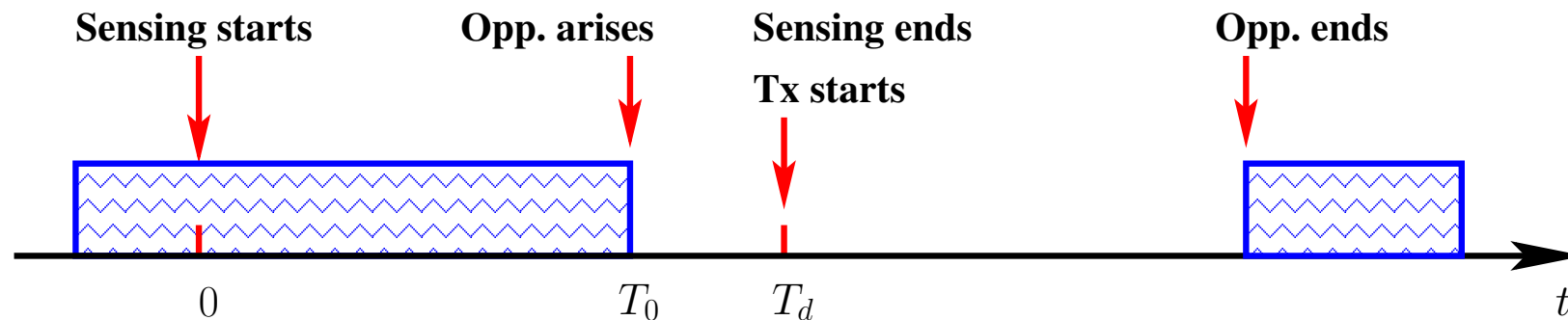
Quickest Change Detection



► **Quickest Detection:** $\min \underbrace{\mathbb{E}[(T_d - T_0)^+]}_{\text{Detection Delay}}$ subject to $\underbrace{\Pr[T_d < T_0] \leq \zeta}_{\text{Reliability Constraint}}$

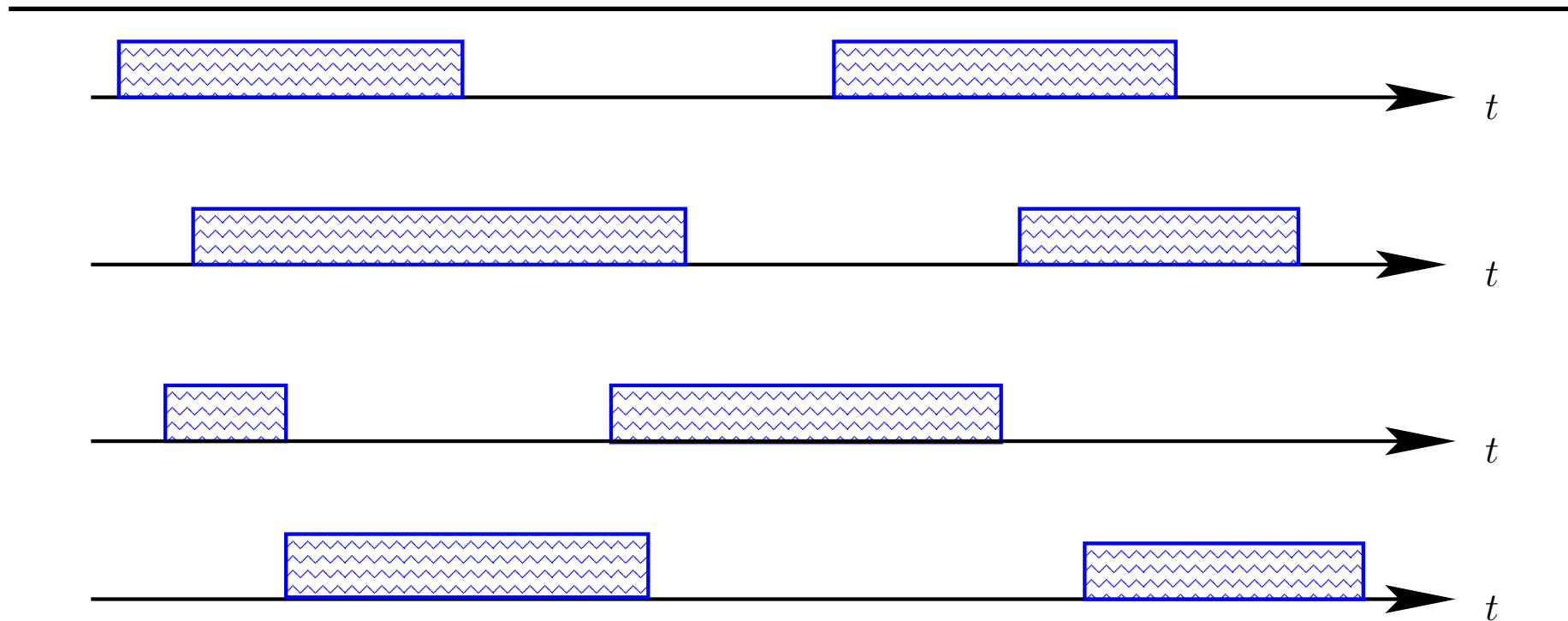
- Bayesian: Shiryaev'61, Borovkov'98, Tartakovsky&Veeravalli'05.
- Minimax: CUSUM (Page'54, Lorden'71).

Application in Cognitive Radio



- ▶ **Measurements:** $\{X_1, X_2, \dots, X_{T_0-1}\}$ are i.i.d with distribution $f_0(x)$;
 $\{X_{T_0}, X_{T_0+1}, \dots\}$ are i.i.d with distribution $f_1(x)$.
- ▶ **Stopping Time:** At time $t = T_d$, the user declares an opportunity.
- ▶ **Quickest Detection:** $\min \underbrace{\mathbb{E}[(T_d - T_0)^+]_{\text{Detection Delay}}}$ subject to $\underbrace{\Pr[T_d < T_0] \leq \zeta}_{\text{Interference Constraint}}$

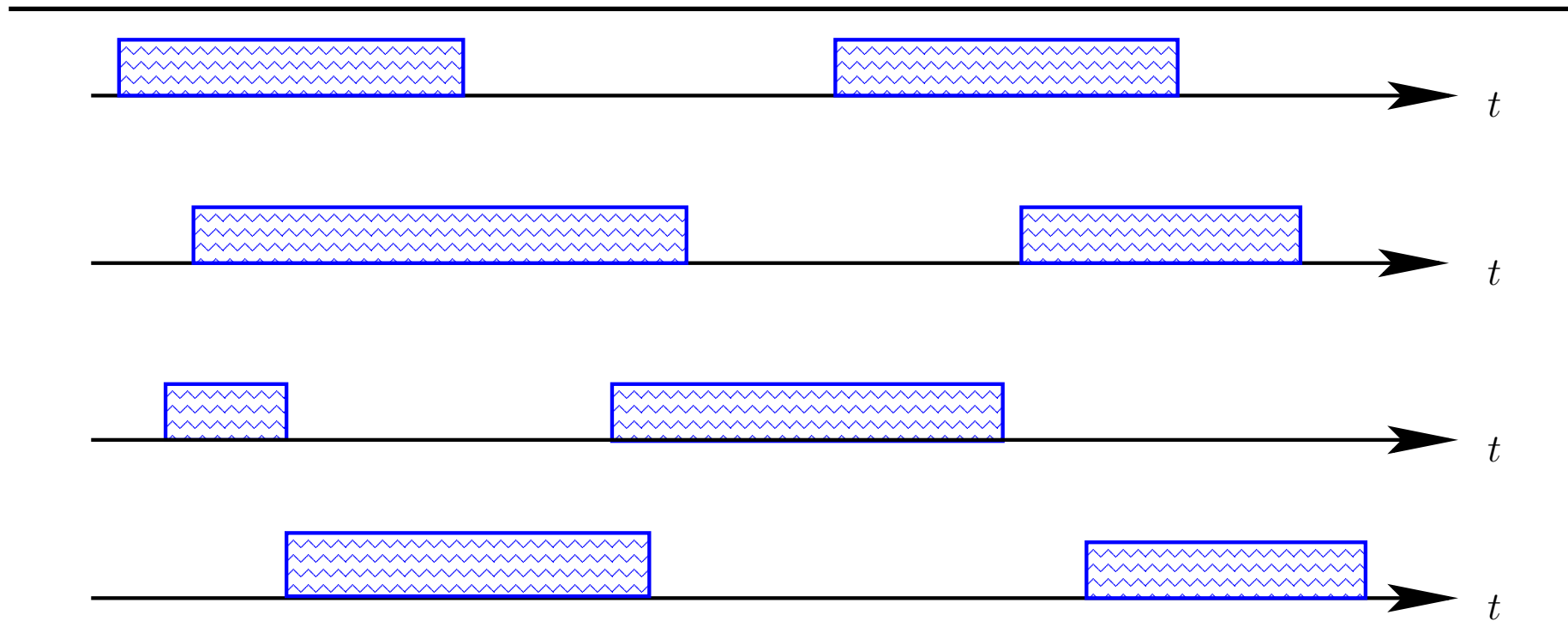
Quickest Detection in Multiple On-Off Processes



► Two Fundamental Differences:

- Channel occupancy is an **on-off process with multiple change points**.
- There are **multiple** channels available.

Quickest Detection in Multiple On-Off Processes



▶ Quickest Detection of Idle Periods in Multiple On-Off Processes:

- Continue, switch, or declare?

▶ Tradeoffs:

- Whether to declare: delay vs. reliability.
- Whether to switch: loss of data vs. avoiding bad realizations.

Outline

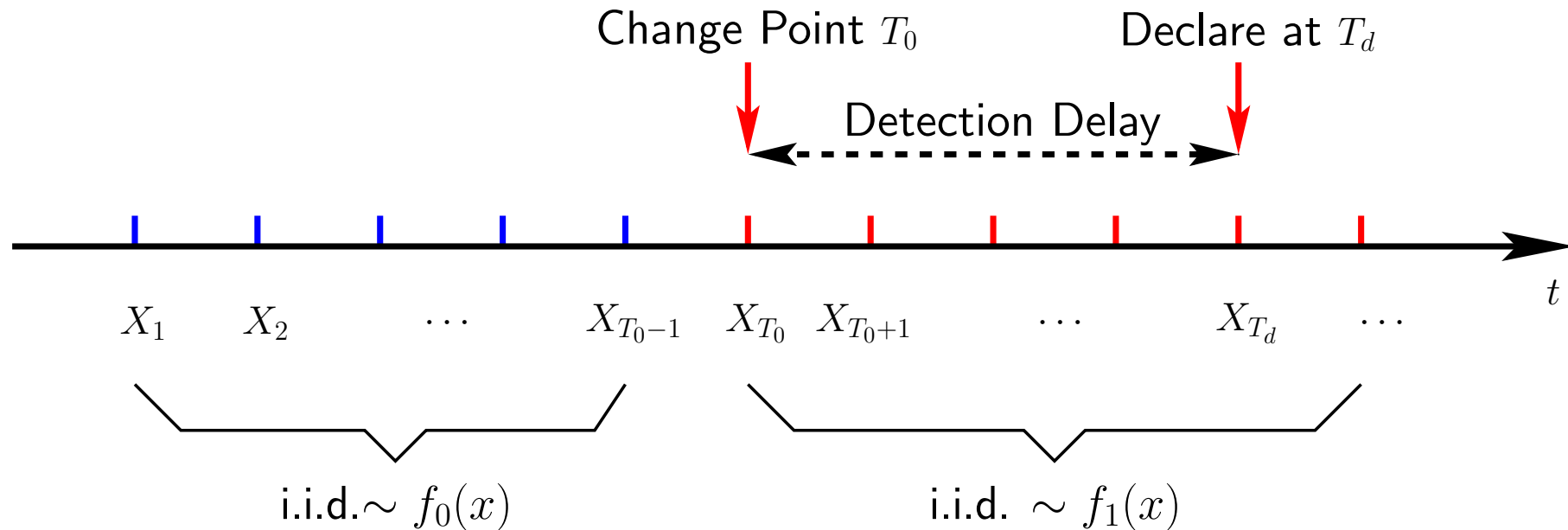
- ▶ Quickest change detection in a single stochastic process
 - Shiryaev's algorithm

- ▶ Quickest detection in multiple on-off processes
 - A decision-theoretic formulation
 - The optimal detection rule: a threshold policy

- ▶ Simulation examples

- ▶ Conclusion and work in progress

Quickest Change Detection: Classic Bayesian Formulation



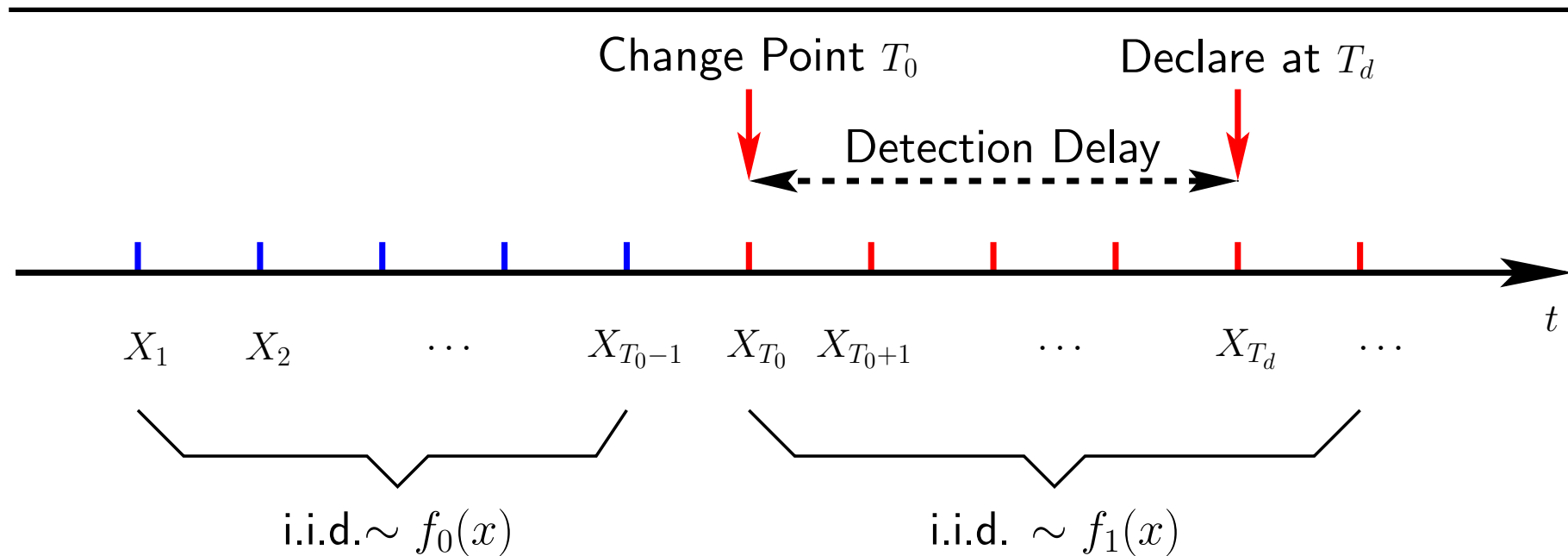
Bayesian Formulation:

- Priori distribution of change point T_0 : geometric

$$\Pr[T_0 = 0] = \lambda_0$$

$$\Pr[T_0 = k] = (1 - \lambda_0)p(1 - p)^{k-1}, \quad \forall k > 0,$$

Shiryaev's Algorithm



- ▶ A sufficient statistic: **a posterior probability** that change has occurred

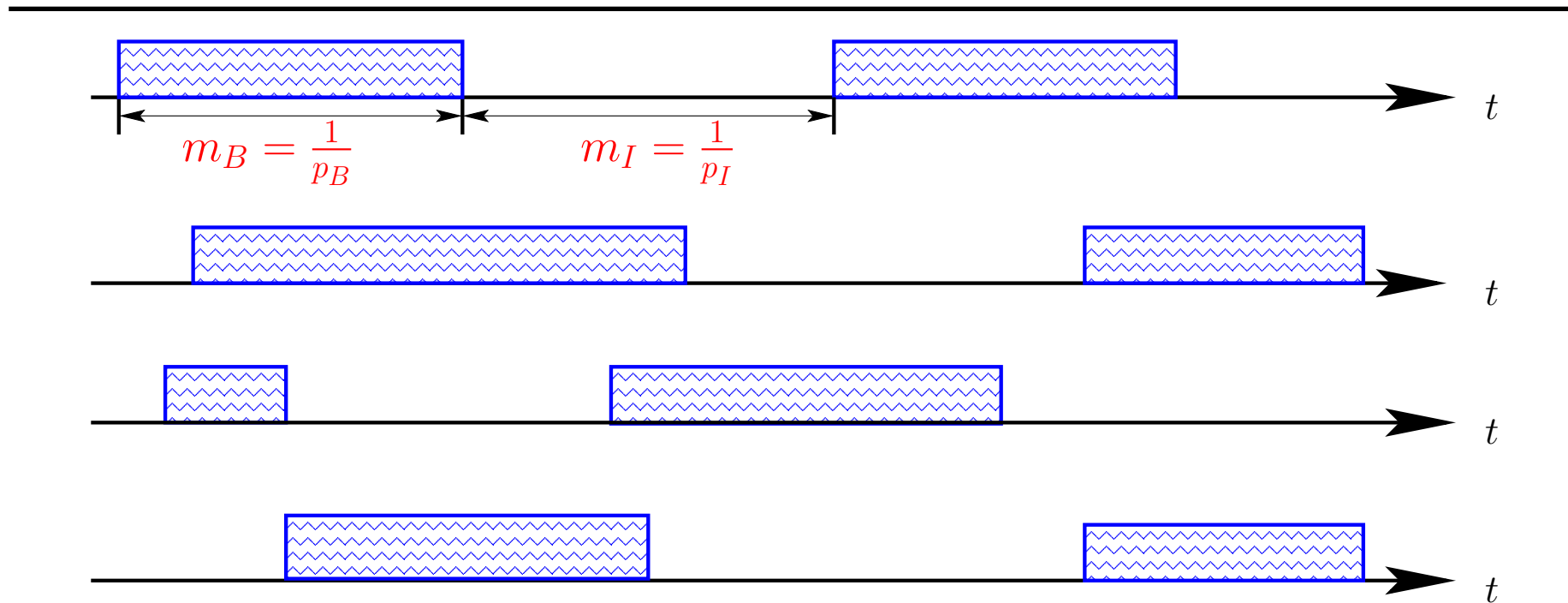
$$\lambda_t \triangleq \Pr[T_0 \leq t | X_1, X_2, \dots, X_t].$$

- ▶ Shiryaev's detection rule:

$$T_d = \inf\{t : \lambda_t \geq \eta_d\}$$

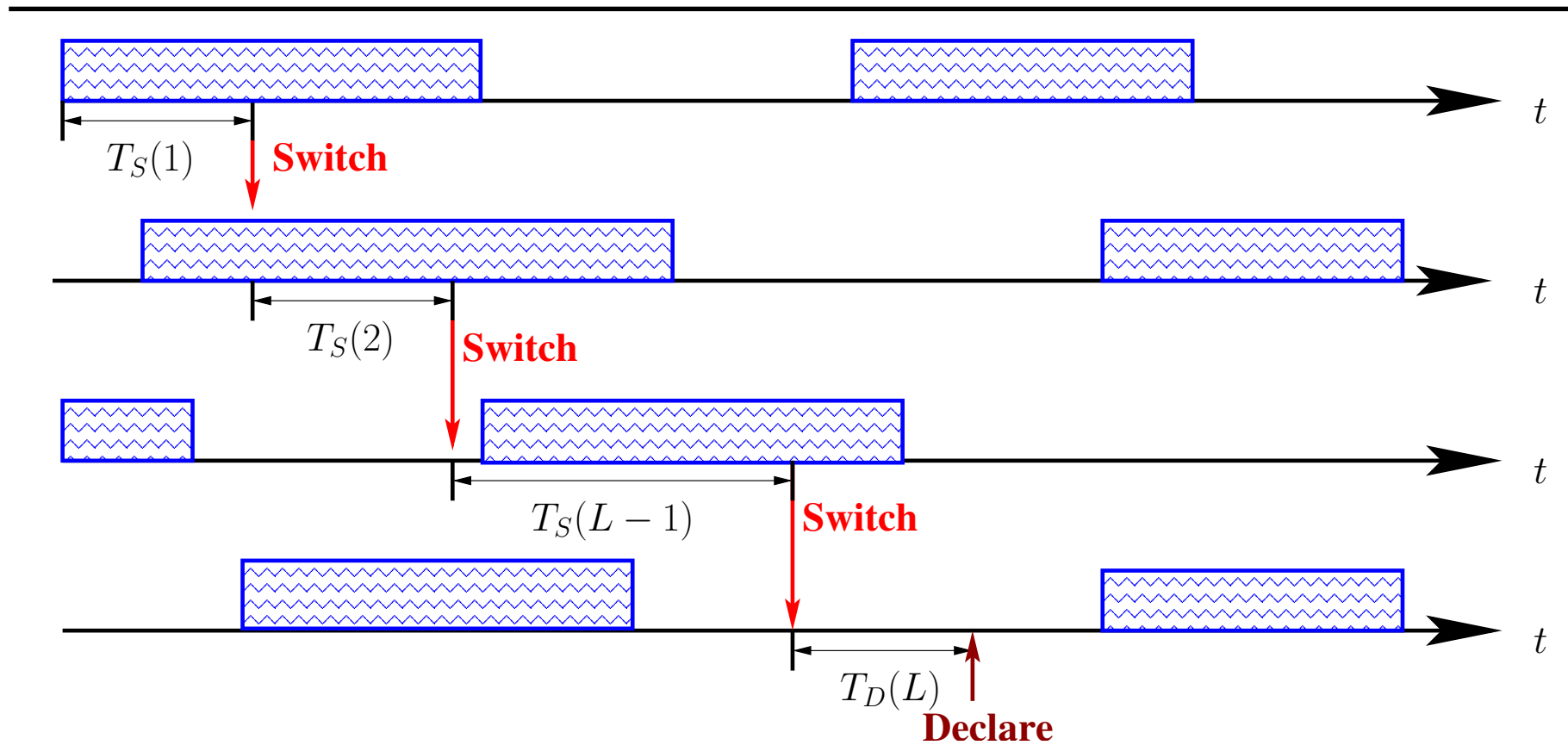
- ▶ Detection threshold η_d : determined by the reliability constraint ζ .
- ▶ Setting $\eta_d = 1 - \zeta$ is asymptotically optimal as $\zeta \rightarrow 0$.

Quickest Detection In Multiple On-Off Processes



- ▶ A large number of independent homogeneous on-off processes.
- ▶ Busy period: geometrically distributed with mean $m_B = \frac{1}{p_B}$.
- ▶ Idle period: geometrically distributed with mean $m_I = \frac{1}{p_I}$.
- ▶ Fraction of idle time: $\lambda_0 = \frac{m_I}{m_I + m_B}$.

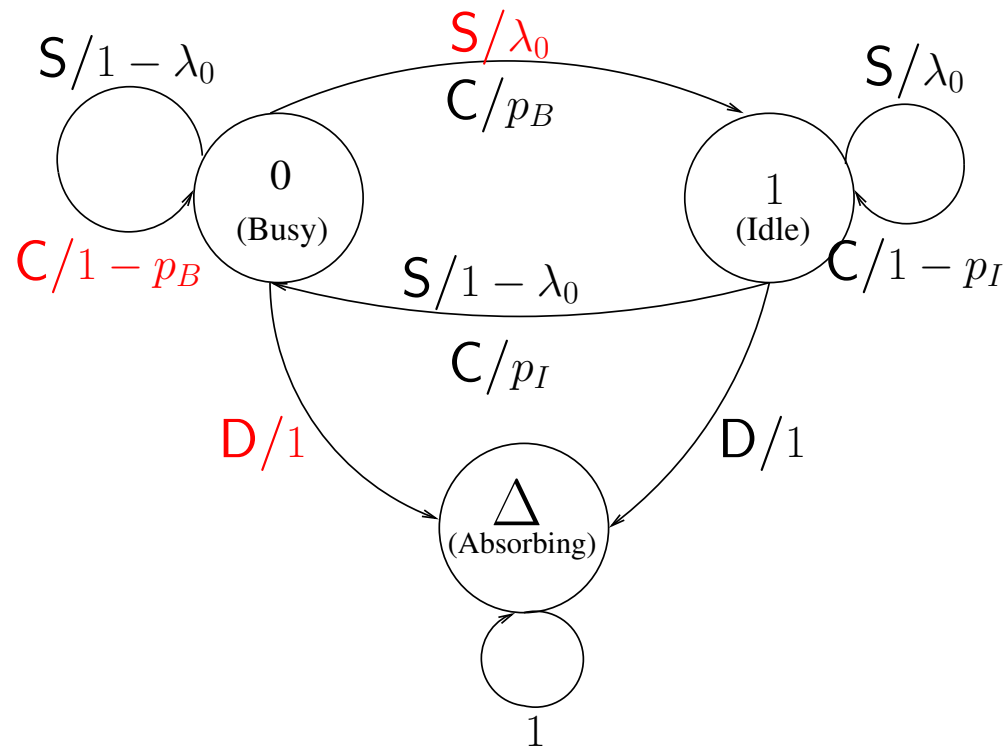
Quickest Detection In Multiple On-Off Processes



$$\min \underbrace{\mathbb{E}\left[\sum_{l=1}^{L-1} T_s(l) + T_d(L)\right]}_{\text{Detection Time}} \quad s.t. \quad \underbrace{\Pr\left[Z_L\left(\sum_{l=1}^{L-1} T_s(l) + T_d(L)\right) = \text{busy}\right]}_{\text{Reliability Constraint}} \leq \zeta$$

A POMDP Formulation

- ▶ State Space: 0 (busy), 1 (idle), Δ (absorbing state)
- ▶ Action Space: S (Switch), C (Continue), D(Declare)
- ▶ State Transition:



- ▶ Cost:

- Switch or Continue: 1
- Declare during a busy period: γ

A POMDP Formulation

- ▶ **A Sufficient Statistic:** the information state (belief)

$$\lambda_t = \Pr[Z_t = \text{idle} | X_1, X_2, \dots, X_t]$$

$$\lambda_0 = \frac{m_I}{m_I + m_B}$$

- ▶ **Update of the Information State**

$$\lambda_t = \begin{cases} \mathcal{T}(\lambda_0|x) & a(t-1) = \mathbf{S}, X_t = x \\ \mathcal{T}(\lambda_{t-1}|x) & a(t-1) = \mathbf{C}, X_t = x \end{cases}.$$

- ▶ $\mathcal{T}(\lambda|x)$: updated information state based on the new measurement x .

$$\mathcal{T}(\lambda|x) \triangleq \frac{(\lambda\bar{p}_I + \bar{\lambda}p_B)f_1(x)}{(\lambda\bar{p}_I + \bar{\lambda}p_B)f_1(x) + (\lambda p_I + \bar{\lambda}\bar{p}_B)f_0(x)}.$$

A POMDP Formulation

- ▶ Channel switching and change detection policy π :

$$\lambda_t \in [0, 1] \implies a(t) \in \{S, C, D\}, \text{ for each time } t.$$

- ▶ Quickest change detection:

$$\pi^* = \arg \min_{\pi} \mathbb{E}_{\pi} \left[\underbrace{\sum_{t=0}^{\infty} R_{\pi}(\lambda_t)}_{\text{Cost}} \mid \lambda_0 = \frac{m_I}{m_B + m_I} \right],$$

Quickest Change Detection: Value Functions

- ▶ $V(\lambda_t)$: the minimum expected total cost-to-go when the current belief is λ_t .

$$V(\lambda_t) = \min\{ \underbrace{V_C(\lambda_t)}_{\text{Continue}}, \underbrace{V_S(\lambda_t)}_{\text{Switch}}, \underbrace{V_D(\lambda_t)}_{\text{Declare}} \}.$$

- ▶ $V_C(\lambda_t)$: the minimum expected total cost-to-go if continue at t .

$$V_C(\lambda_t) = 1 + \int_x \underbrace{P(x; \lambda_t)}_{\text{Pr[observe } x \text{ under } \lambda_t]} V(\mathcal{T}(\lambda_t|x)) dx$$

- ▶ $V_S(\lambda_t)$: the minimum expected total cost-to-go if switch at t .

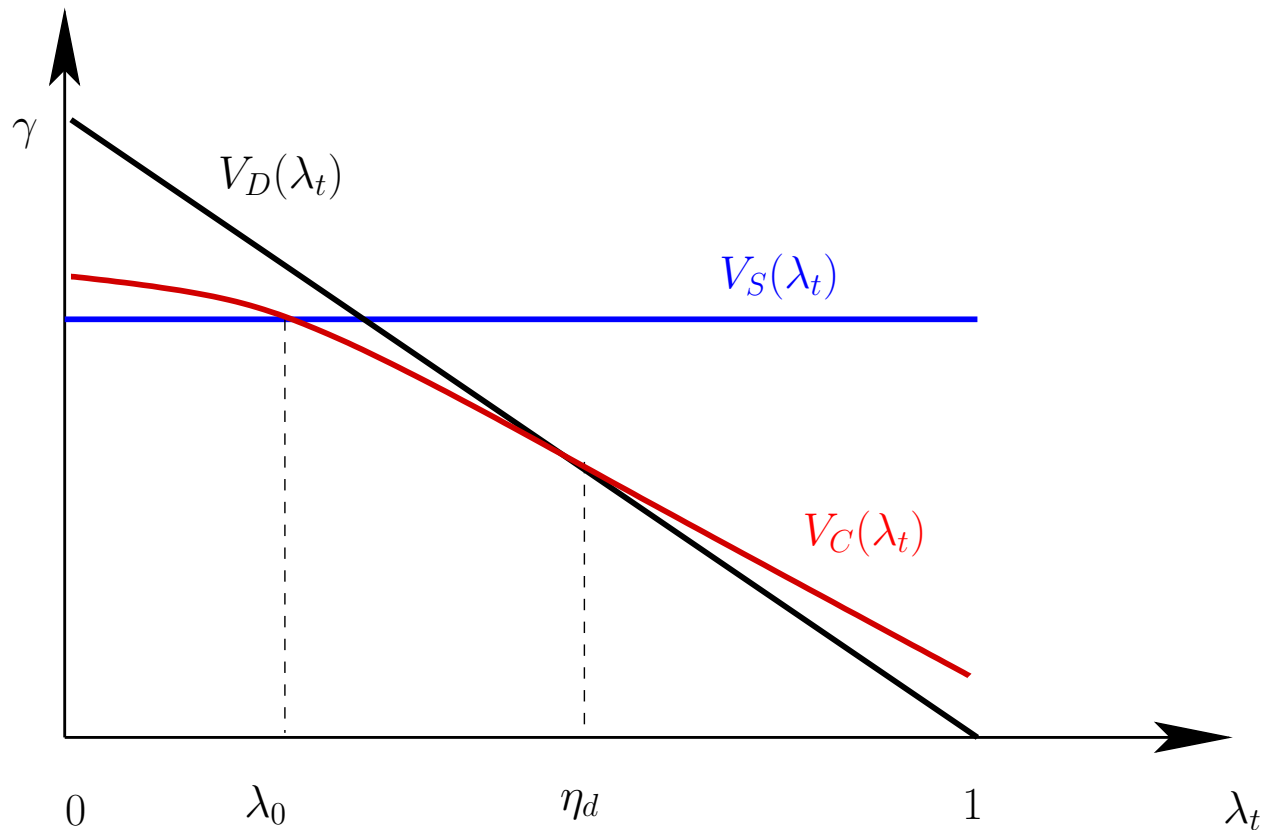
$$V_S(\lambda_t) = 1 + \int_x \underbrace{P(x; \lambda_0)}_{\text{Pr[observe } x \text{ under } \lambda_0]} V(\mathcal{T}(\lambda_0|x)) dx = V_C(\lambda_0)$$

- ▶ $V_D(\lambda_t)$: the minimum expected total cost-to-go if declare at t .

$$V_D(\lambda_t) = (1 - \lambda_t)\gamma.$$

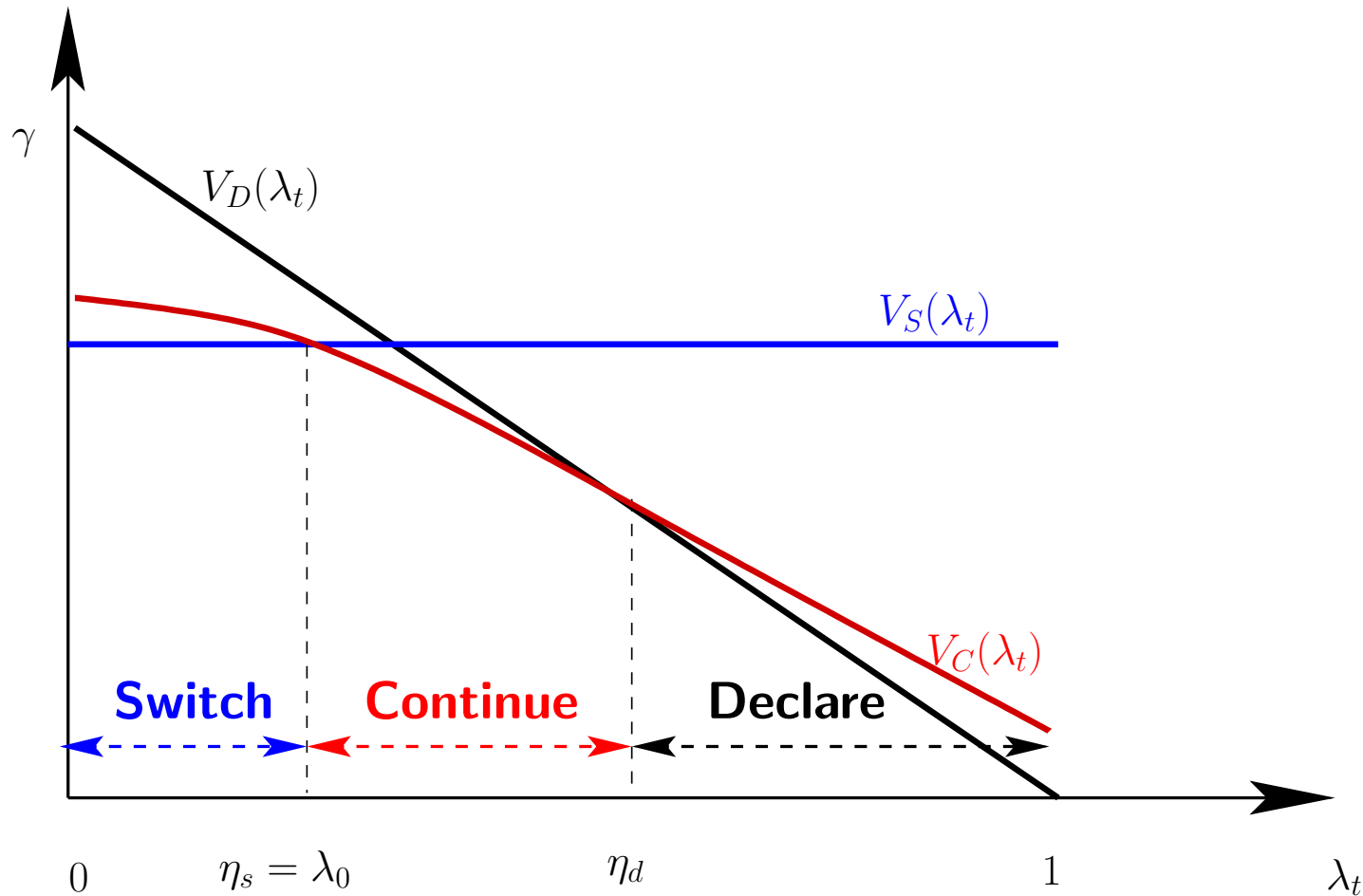
Quickest Change Detection: A Threshold Policy

- ▶ $V_D(\lambda_t)$ is linear.
- ▶ $V_C(\lambda_t)$ is monotonically decreasing (if $\frac{1}{m_B} + \frac{1}{m_I} \leq 1$) and concave.
- ▶ $V_S(\lambda_t) = V_C(\lambda_0)$, where $\lambda_0 = \frac{m_I}{m_I + m_B}$.



Quickest Change Detection: A Threshold Policy

$$V(\lambda_t) = \min\left\{ \underbrace{V_S(\lambda_t)}_{\text{Switch}}, \underbrace{V_C(\lambda_t)}_{\text{Continue}}, \underbrace{V_D(\lambda_t)}_{\text{Declare}} \right\}.$$

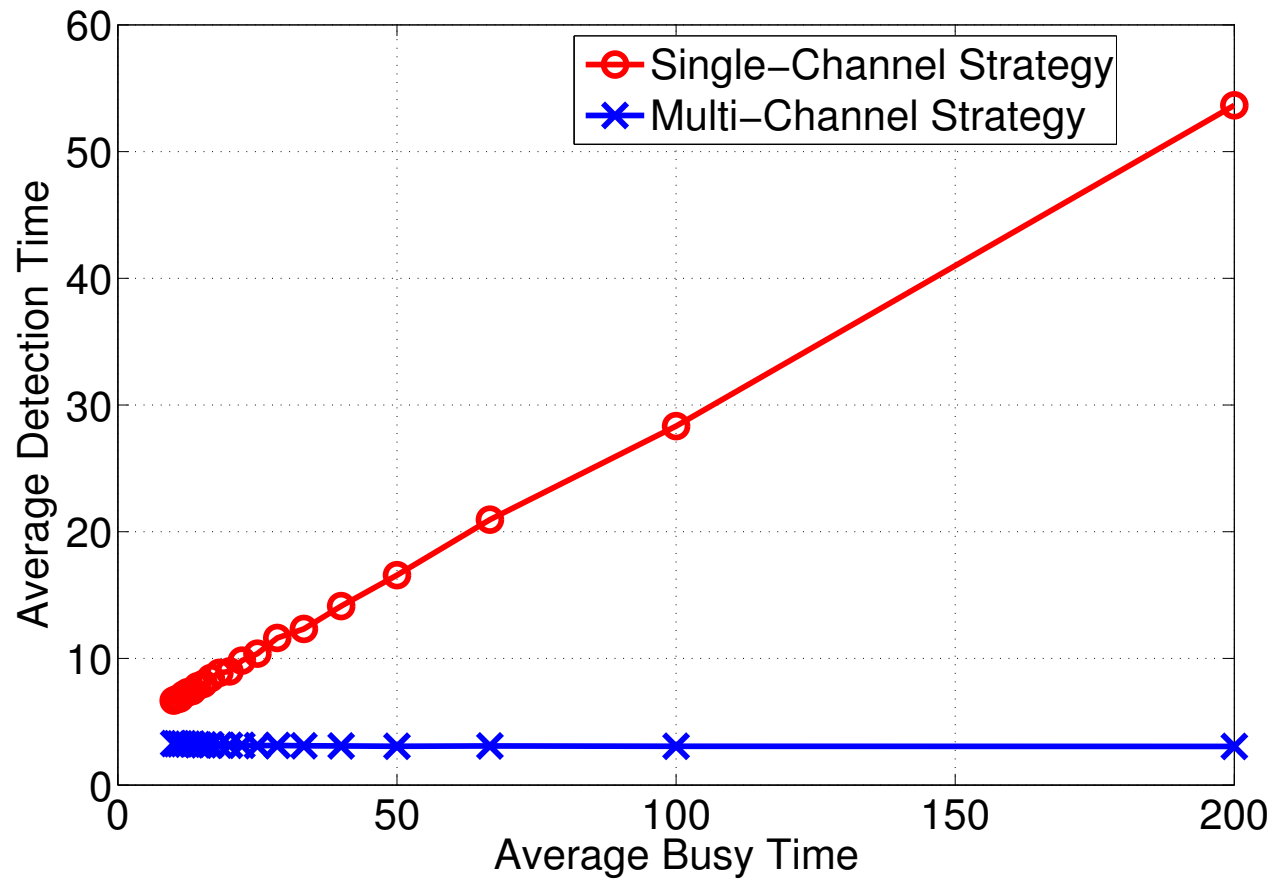


Simulation Examples

- ▶ $f_0(x), f_1(x)$: Gaussian with zero mean and different variances.
- ▶ $SNR = 10dB$.
- ▶ $\eta_d = 1 - \zeta$.

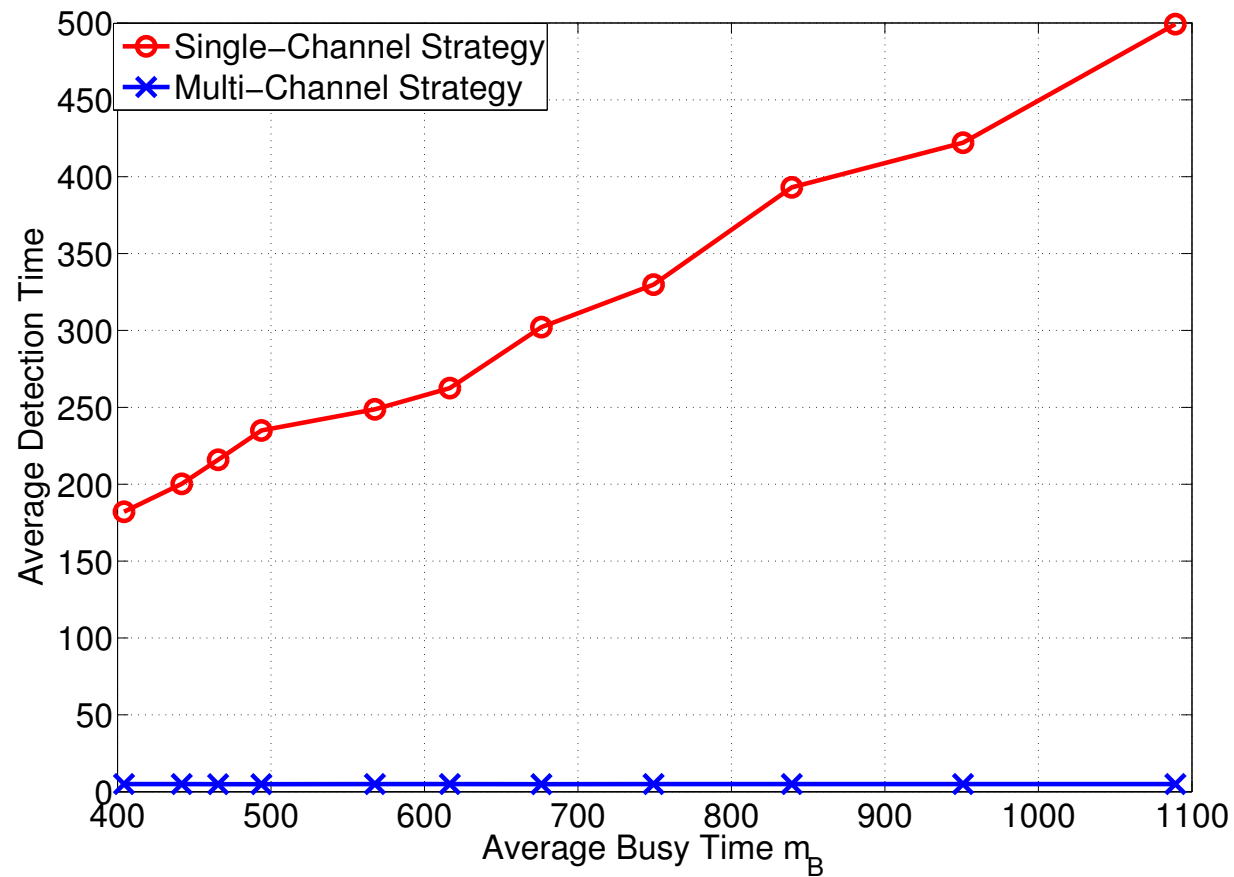
Simulation Example: Geometric Distribution

- ▶ Increase both m_B and m_I while keeping λ_0 fixed



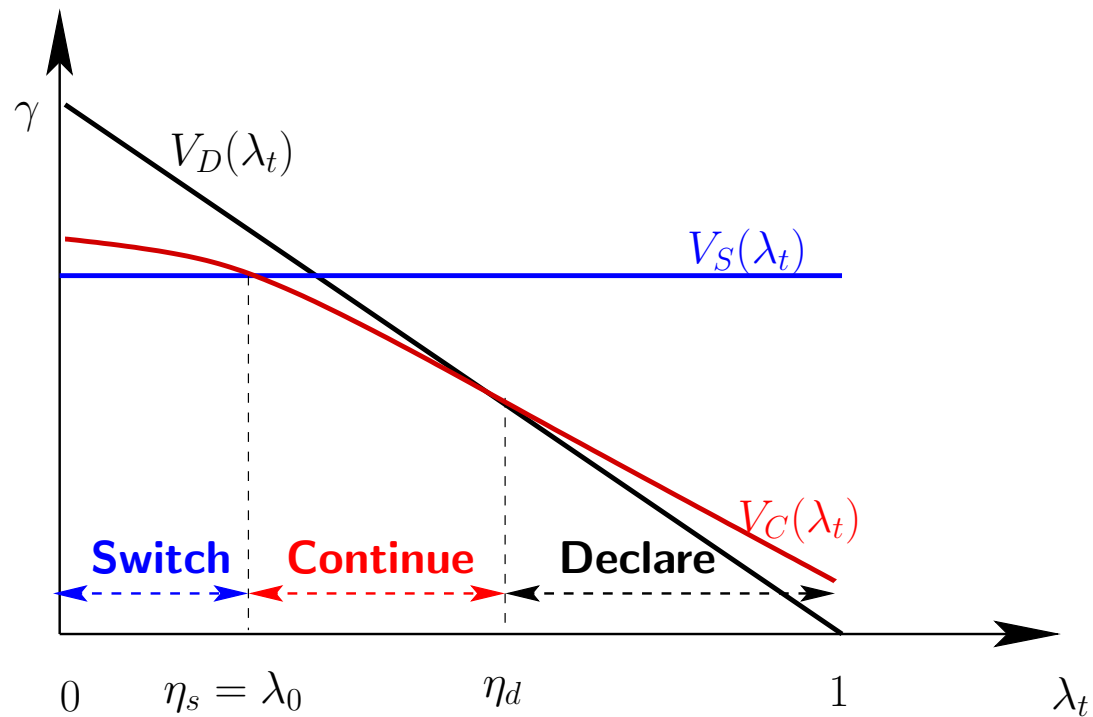
Simulation Example: Arbitrary Distributions

- ▶ Busy period: Pareto distribution with increasing tail index



Conclusion and Work in Progress

Quickest Detection in Multiple On-Off Processes:



Work in Progress:

- Asymptotic optimality for arbitrary distributions and non-i.i.d. data.
- Minimax formulation.