Asymptotically Efficient Multi-Channel Estimation for Opportunistic Spectrum Access

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Abstract

The problem of estimating the parameters of multiple independent continuous-time Markov on-off processes is considered. The objective is to minimize the total mean square error (MSE) under a constraint on the total sensing time. The Fisher information matrix for the primary traffic model and the maximum likelihood estimator are obtained. A sequential estimation strategy is proposed which operates under an epoch structure with growing epoch length. It is shown that this sequential estimation strategy is asymptotically efficient as the total sensing time increases. This result finds application in opportunistic spectrum access where secondary users need to estimate the channel occupancy model of the primary system for efficient exploitation of spectrum opportunities.

Index Terms

Channel estimation, continuous-time Markov process, sequential estimation, cognitive radio, opportunistic spectrum access (OSA).

I. Introduction

In opportunistic spectrum access, secondary users sense and access temporally unused channels in the spectrum without causing unacceptable interference to primary users [1]. An accurate stochastic modeling of the primary system channel occupancy plays a crucial role in designing

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the optimal algorithms for sensing, tracking, and exploiting spectrum opportunities. For instance, in [2]–[4], the channel occupancy of the primary system is modeled as a continuous-time Markov on-off process. Given these parameters, optimal sensing and access strategies of the secondary users are designed. In practice, however, the stochastic model of the primary traffic may not be known *a priori*; such a model must be learned through channel sensing.

In this paper, we consider the problem of estimating the parameters of multiple independent continuous-time Markov on-off processes. The objective is to minimize the total mean squared error (MSE) across all channels under a constraint on the total sensing time. To this end, we obtain the Fisher information matrix and the maximum likelihood estimator (MLE). Given that the optimal allocation of the total sensing time to multiple channels depends on the unknown parameters, a sequential estimation strategy is proposed which dynamically adjusts the allocation of sensing time based on the partial learning results obtained up to the current time. Referred to as SEINE (Sequence Estimation with Increasing Nested Epochs), the proposed sequential estimation policy operates under an epoch structure. Within each epoch, channels are sensed in turn, each for a fraction of the epoch length with the fraction determined by the current estimate of the channel parameters. The epoch length grows over time to take advantage of the increasing accuracy of the estimates. It is shown that SEINE is asymptotically efficient, *i.e.,* it achieves the Cramér-Rao Bound (CRB) as the total sensing time grows.

Learning the stochastic models of primary channel occupancy has received relatively little attention. There exist a few published results, all focusing on a single channel and discrete sampling. For example, in [5], [6], Maximum Likelihood and Bayesian estimation of channel parameters under a uniform sampling strategy were studied. In [7], relationship between estimation accuracy, the number of samples taken, and the channel state transition probabilities was analyzed by using the sampling and estimation framework proposed in [5]. Park *et al.* in [8] proposed a channel state predictor based on the reinforcement learning techniques where the channel model is assume to be a hidden Markov process. In [9], a wavelet transform based channel estimator was proposed. In [10], the performance of the single channel MLE of the uniform and random discrete-time sampling strategies were compared. It is demonstrated that when the samples are sparse enough, the random sampling outperforms the uniform sampling. The analysis of [10] assumes that the utilization factor of the channel is known which reduces the problem to a single (scaler) parameter estimation problem. A dynamic programming approach is proposed to obtain the best and the worst sampling scheme which can be solved numerically. For the time-varying channel parameters, an adaptive random sensing scheme is proposed and shown to outperform its counterpart using uniform sensing.

II. Problem Statement and Fundamental statistics

Consider a network that consists of M channels. These M channels are licensed to an unslotted primary network. The spectrum occupancy of channel m is modeled as a continuous-time Markov process with two states: $S_m(t) = 1$ (busy) and $S_m(t) = 0$ (idle). These M Markov processes are jointly independent. In particular, for channel m, the sojourn times in the busy and idle states are exponentially distributed with rates $\lambda_{1,m}$ and $\lambda_{0,m}$, respectively. Theses parameters are unknown to the secondary system. A secondary user's objective is to learn the primary network occupancy model. It aims to estimate the set of channel parameters $\{\lambda_{0,m}, \lambda_{1,m}\}_{m=1}^{M}$ by sensing these M channels.

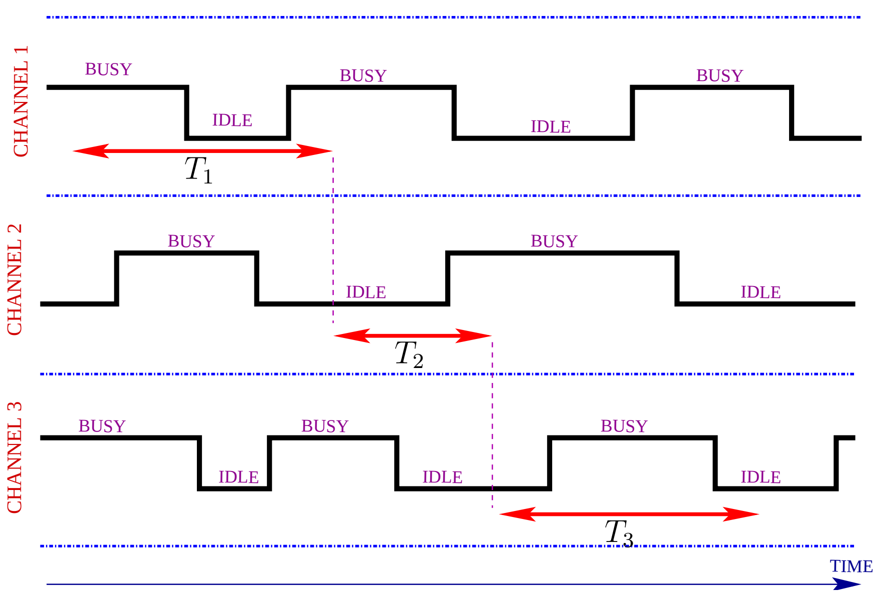

Fig. 1. Channel Sensing Model

We assume that the secondary user can only sense one channel at a time, and there is a budget for the total sensing time. An illustration of a particular sensing scenario is shown in Fig 1 where the secondary user monitors a particular channel continuously for a period of time before switching to a different channel. It is intuitive that a channel with greater statistical variance requires longer total sensing time to achieve the same level of accuracy as that for a channel with smaller variance. The main challenge here is to design a sensing policy that allocates time spent on each channel optimally.

While the continuous-time on-off Markovian abstraction of the primary channels is widely accepted, it has several subtle modeling complications. Specifically, the transitions of the primary traffic are instantaneous. From a signal theoretic point of view, such a process has infinite bandwidth and no discrete-time sensing can be made without loss of information. In this paper, we adopt a continuous sensing model where it is assumed that the secondary user can observe the channel continuously. This of course can only be an approximation of a practical sensing mechanism, but it has the significant theoretical benefit that the sensing process does not lead to information loss.

We also have to make a few additional assumptions on sensing. The on-off model requires a hypothesis test to decide whether a particular channel is idle or busy. When the observation is noisy, such hypothesis testing suffers from miss detection and false alarm. The modeling of sensing errors leads to significant complications and is not considered in this paper. In addition, switching from one channel to another also takes time in practice. This too will be ignored in our development.

A. *Likelihood Function, Sufficient Statistics, and Fisher Information: A Single Interval Analysis*

In this section we focus on a single interval sensed from a given channel and derive fundamental statistics. These statistics are extended to the multi-interval multi-channel case in the next section. In the following, the channel index is dropped for the ease of presentation.

The information observed in $[t_1, t_1+T_1]$ from a given channel is illustrated in Fig. 2; it consists of the state S_{t_1} of the channel at the beginning of the sensed interval and the realizations of the consecutive observed idle/busy periods $\{(z_i)_{i=1}^n\}$, where n is the number of such periods. Hence the observation model for the interval $[t_1, t_1+T_1]$ will be $\{S_{t_1}, (z_i)_{i=1}^n\}$. Denote by Z_i's the exponential random variables where z_i's are respective realizations of what the secondary user has observed. From Fig. 2, we have $Z_1 \geq z_1$, $Z_n \geq z_n$, and $Z_i = z_i$ for $i = 2, \ldots, n-1$. In the following we refer to z_i's with $i \in \{2, \ldots, n-1\}$ as *complete* periods since $Z_i = z_i$ for them which is not true for $i = 1, n$.

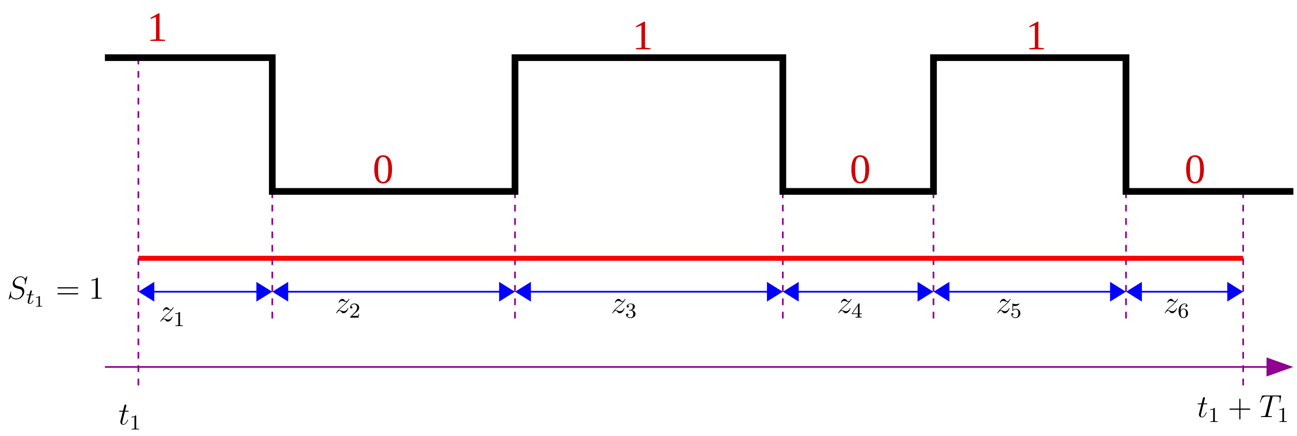

Fig. 2. Observation Model

Now assume that a realization of observation for the interval $[t_1, t_1 + T_1]$ is given to be $\{S_{t_1} = s, Z_1 \geq z_1, (Z_i = z_i)_{i=2}^{n-1}, Z_n \geq z_n\}$. Since the state of the channel during Z_1 is given to be s, the state of channel during Z_i will be $s_i = s + (\frac{1+(-1)^i}{2})(1-2s)$. Therefore the likelihood function given that idle/busy periods for the given channel are exponentially distributed with parameters λ_0, and λ_1 is

$$\begin{aligned} f(s,(z_i)_{i=1}^n|\lambda_0,\lambda_1) &= \Pr\{S_{t_0}=s|\lambda_0,\lambda_1\}\Pr\{Z_1\geq z_1, Z_n\geq z_n\} f_{Z_2,\ldots,Z_{n-1}}\{z_2,\ldots,z_{n-1}|\lambda_0,\lambda_1,S_{t_0}=s\} \\ &= (\frac{\lambda_0}{\lambda_0+\lambda_1})^s(\frac{\lambda_1}{\lambda_0+\lambda_1})^{1-s}\ e^{-\lambda_{s_1}z_1}\Pi_{i=2}^{n-1}(\lambda_{s_i}e^{-\lambda_{s_i}z_i})\ e^{-\lambda_{s_n}z_n}. \end{aligned}$$

Let n_0, and n_1 be the number of *complete* idle and busy states (note that $i = 1, n$ are excluded) respectively. Thus we have

$$n_s = \begin{cases} (\frac{n}{2}-1)^+ & n \text{ is even} \\ (\frac{n-3}{2})^+ & n \text{ is odd.} \end{cases} \tag{1}$$

$$n_{1-s} = (n-2-n_s)^+. \tag{2}$$

Therefore the likelihood function can be rewritten as

$$f(s,(z_i)_{i=1}^n|\lambda_0,\lambda_1) = (\frac{\lambda_{1-s}}{\lambda_s+\lambda_{1-s}})\ \lambda_s^{n_s}e^{-\lambda_s\sum_{i=1}^{\lfloor\frac{n+1}{2}\rfloor}z_{2i-1}}\ \lambda_{1-s}^{n_{1-s}}e^{-\lambda_{1-s}\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}z_{2i}}. \tag{3}$$

The likelihood function in (3) shows that the *sufficient statistics* for a single interval is

$$(s, n, \sum_{i=1}^{\lfloor\frac{n+1}{2}\rfloor} z_{2i-1}, \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor} z_{2i}). \tag{4}$$

The Fisher information is the negative of the expectation of the second derivative of the log likelihood function, $L(\lambda_0,\lambda_1)=\log f(s,(z_i)_{i=1}^n|\lambda_0,\lambda_1)$, with respect to the unknown parameters, namely,

$$I_{[t_1,\ t_1+T_1]}=\mathbb{E}(-\nabla^2_{\lambda_0,\lambda_1}L(\lambda_0,\lambda_1))$$

where

$$-\nabla^2_{\lambda_0,\lambda_1}L(\lambda_0,\lambda_1)=\begin{bmatrix}\frac{n_0+1_{\{s=1\}}}{\lambda_0^2}-\frac{1}{(\lambda_0+\lambda_1)^2} & -\frac{1}{(\lambda_0+\lambda_1)^2}\\ -\frac{1}{(\lambda_0+\lambda_1)^2} & \frac{n_1+1_{\{s=0\}}}{\lambda_1^2}-\frac{1}{(\lambda_0+\lambda_1)^2}\end{bmatrix}, \tag{5}$$

where $1_{s=j}=1$ if $s=j$ and zero otherwise. The Fisher information can be expressed in closed form given in the following Theorem.

Theorem 1: The Fisher information matrix for a single channel which is sensed for a continuous-time interval $[t_1,\ t_1+T]$ can be written as,

$$I_{[t_1,\ t_1+T]}=\begin{bmatrix}\frac{T}{\lambda_0^2(\frac{1}{\lambda_0}+\frac{1}{\lambda_1})}(1+o(T)) & -\frac{1}{(\lambda_0+\lambda_1)^2}\\ -\frac{1}{(\lambda_0+\lambda_1)^2} & \frac{T}{\lambda_1^2(\frac{1}{\lambda_0}+\frac{1}{\lambda_1})}(1+o(T))\end{bmatrix}.$$

Proof: See the Appendix A. ∎

From theorem 1 for the inverse of the Fisher information matrix we get,

$$I^{-1}_{[t_1,\ t_1+T]}=\begin{bmatrix}\frac{\lambda_0^2(\frac{1}{\lambda_0}+\frac{1}{\lambda_1})}{T}(1+o(T)) & o(T)\\ o(T) & \frac{\lambda_1^2(\frac{1}{\lambda_0}+\frac{1}{\lambda_1})}{T}(1+o(T))\end{bmatrix}. \tag{6}$$

B. Likelihood Function and Fisher Information for Multiple Intervals

Since the channels are independent, the likelihood function of the network will be the product of the likelihood functions of each channel. Similarly, the Fisher information of the network will be the sum of the Fisher information matrices of individual channels. Therefore without loss of generality, we derive the likelihood function and the Fisher information matrix for multiple intervals for a given channel i.

Assume that the secondary user has sensed K intervals $\{[t_j,t_j+T_j]\}_{j=1}^K$ where the jth and $(j+1)$th intervals are separated by $\Delta t_j=t_{j+1}-(t_j+T_j)$. Fig. 3 shows an example where $K=4$.

Using the same notation introduced in section II-A and adding index j indicating the jth interval, the likelihood function for K intervals is $f(\{s_j,(z_{h,j})_{h=1}^{n_j}\}_{j=1}^K|\lambda_{0,i},\lambda_{1,i})$. Based on the

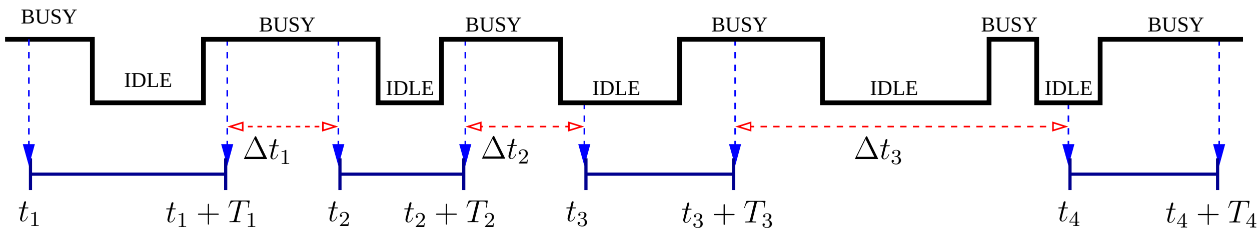

Fig. 3. Multiple Intervals

Markov property the likelihood function can be reduced to (derivations for two intervals are given in Appendix B),

$$f(\{s_j,(z_{h,j})_{h=1}^{n_j}\}_{j=1}^{K}|\lambda_{0,i},\lambda_{1,i}) = \Pi_{j=1}^{K} f(s_j,(z_{h,j})_{h=1}^{n_j}|\lambda_{0,i},\lambda_{1,i}) \times \Pi_{j=1}^{K-1} \frac{\Pr\{S_{t_{j+1}}=s_{j+1}|S_{t_j+T_j}=s_{n_j,j}\}}{\Pr\{S_{t_{j+1}}=s_{j+1}\}}. \quad (7)$$

Denote the continuous and discrete parts of the likelihood function by

$$\begin{aligned} L_j^C &= f(s_j,(z_{h,j})_{h=1}^{n_j}|\lambda_{0,i},\lambda_{1,i}), \\ L_j^D &= \frac{\Pr\{S_{t_{j+1}}=s_{j+1}|S_{t_j+T_j}=s_{n_j,j}\}}{\Pr\{S_{t_{j+1}}=s_{j+1}\}}. \end{aligned}$$

The likelihood function can be written as,

$$f(\{s_j,(z_{h,j})_{h=1}^{n_j}\}_{j=1}^{K}|\lambda_{0,i},\lambda_{1,i}) = \Pi_{j=1}^{K}\ [L_j^C]\ \ \Pi_{j=1}^{K-1}\ [L_j^D]. \quad (8)$$

Note that L_j^D (stands for discrete likelihood) only depends on the state of the channels at two points t_j+T_j (end point of the jth interval) and t_{j+1} (the starting point of the $(j+1)$th interval). Therefore the term L_j^D can be regarded as the factor in the likelihood function coming from the discrete sampling at single points t_j+T_j, and t_j. L_j^C (stands for continuous likelihood) is the likelihood function for the single interval $[t_j,\ t_j+T_j]$ (computed in section II-A).

Therefore the likelihood function for multiple intervals consists of two type of factors, the likelihood functions of individual intervals (L_j^C's) and the factors from the discrete sampling (L_j^D's) at the end points of the neighboring intervals. Fig. 4 shows continuous sampling (individual intervals) and discrete sampling for two intervals. The conditional probability can be written as

$$\Pr\{S_{t_{j+1}}=s_{j+1}|S_{t_j+T_j}=s_{n_j,j}\} = \frac{\lambda_{1-s_{j+1},i}}{\lambda_{s_{j+1},i}+\lambda_{1-s_{j+1},i}} + (-1)^{s_{n_j,j}+s_{j+1}} \frac{\lambda_{1-s_{n_j,j},i}}{\lambda_{s_{n_j,j},i}+\lambda_{1-s_{n_j,j},i}} e^{-(\lambda_{0,i}+\lambda_{1,i})\Delta t_j}$$

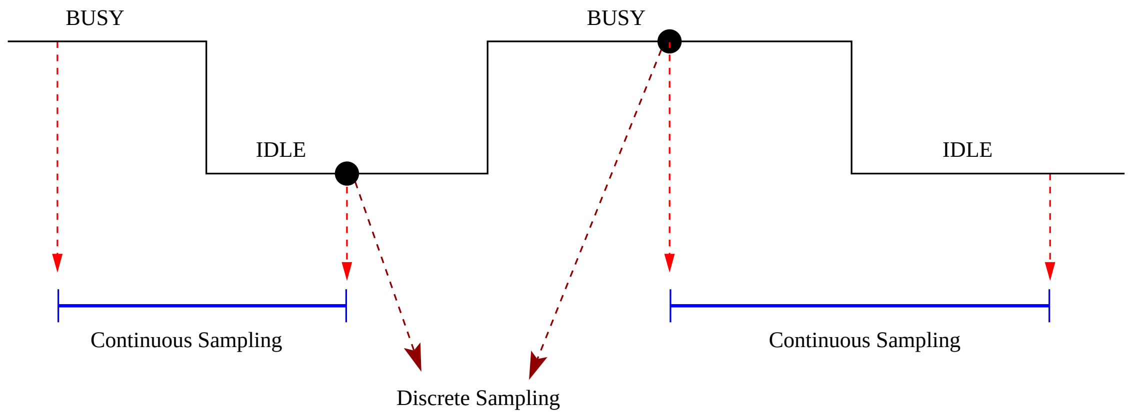

Fig. 4. Continuous and Discrete Sampling

We also know

$$\Pr\{S_{t_{j+1}} = s_{j+1}\} = \frac{\lambda_{1-s_{j+1},i}}{\lambda_{s_{j+1},i} + \lambda_{1-s_{j+1},i}}.$$

Therefore the likelihood function is

$$\begin{aligned} f(\{s_j, (z_{h,j})_{h=1}^{n_j}\}_{j=1}^{K} | \lambda_{0,i}, \lambda_{1,i}) &= \Pi_{j=1}^{K}[(\frac{\lambda_{1-s_j,i}}{\lambda_{s_j,i} + \lambda_{1-s_j,i}}) \lambda_{s_j,i}^{n_{s_j,j}} e^{-\lambda_{s,i} \sum_{h=1}^{\lfloor \frac{n_j+1}{2} \rfloor} z_{2h-1,j}} \lambda_{1-s_j}^{n_{1-s_j}} e^{-\lambda_{1-s_j} \sum_{h=1}^{\lfloor \frac{n_j}{2} \rfloor} z_{2h,j}}] \\ &\times \Pi_{j=1}^{K-1}[\frac{\frac{\lambda_{1-s_{j+1},i}}{\lambda_{s_{j+1},i}+\lambda_{1-s_{j+1},i}} + (-1)^{s_{n_j,j}+s_{j+1}} \frac{\lambda_{1-s_{n_j,j},i}}{\lambda_{s_{n_j,j},i}+\lambda_{1-s_{n_j,j},i}} e^{-(\lambda_{0,i}+\lambda_{1,i})\Delta t_j}}{\frac{\lambda_{1-s_{j+1},i}}{\lambda_{s_{j+1},i}+\lambda_{1-s_{j+1},i}}}]. \end{aligned}$$

The Fisher information is derived from the logarithm of the likelihood function. Therefore by the same argument given for (8), the Fisher information of the multiple intervals will be the sum of the Fisher information of the individual intervals plus the sum of the information obtained from the discrete samplings at the end points of neighboring intervals ,$t_j + T_j$'s and t_{j+1}'s. Here we have dropped the channel index i for the ease of presentation.

$$I_{\cup_{j=1}^{K}\{[t_j,\ t_j+T_j]\}} = \sum_{j=1}^{K} I_{[t_j,\ t_j+T_j]} + \sum_{j=1}^{K-1} I_j^D(\Delta t_j). \tag{9}$$

where

$$I_j^D(\Delta t_j) = \mathbb{E}(-\nabla^2_{\lambda_{0,i},\lambda_{1,i}} \log(\frac{\Pr\{S_{t_{j+1}} = s_{j+1} | S_{t_j+T_j} = s_{n_j,j}\}}{\Pr\{S_{t_{j+1}} = s_{j+1}\}})), \tag{10}$$

is the information obtained from the discrete sampling at $t_j + T_j$ and t_{j+1} which is a function of Δt_j (and also the unknown parameters $\lambda_{0,i}$, and $\lambda_{1,i}$).

In the following theorem we show the optimality of single interval versus multiple intervals with the same total length under a sparsity condition.

Theorem 2: There exists a finite value $\tau > 0$ such that the Fisher information of a single interval $[t_0,\ t_0+T]$ is larger than the Fisher information of K intervals $\{[t_j,\ t_j+T_j]\}_{j=1}^{K}$ where $\Delta t_j \geq \tau\ \forall j \in \{1,\ldots,K-1\}$, and $\sum_{j=1}^{K} T_j = T$, *i.e.*,

$$I_{[t_0,\ t_0+T]} \geq I_{\cup_{j=1}^{K}[t_j,\ t_j+T_j]}. \tag{11}$$

Proof: See Appendix B. ■

Therefore if the distance between intervals (Δt_j's) is quite large (bigger than a constant τ), sensing one big interval with the length $T = \sum_{j=1}^{K} T_j$ has higher Fisher information. Based on this theorem, a sensing policy is called *sparse* if it satisfies the condition in Theorem 2, namely, $\min_{j=1}^{K-1} \Delta t_j > \tau$.

III. Maximum Likelihood and Moment Estimators

A. The Maximum Likelihood Estimator

The MLE (maximum likelihood estimator) is obtained by $\frac{\partial L}{\partial \lambda_j} = 0, \quad j = s, 1-s$, where

$$L = \log f(\{s_j, (z_{h,j})_{h=1}^{n_j}\}_{j=1}^{K} | \lambda_{0,i}, \lambda_{1,i})$$

is the log likelihood function, and $f(\{s_j, (z_{h,j})_{h=1}^{n_j}\}_{j=1}^{K} | \lambda_{0,i}, \lambda_{1,i})$ is given in section II-B. It is apparent that the MLE depends on the number of intervals K and also Δt_j's and maximizing the likelihood function does not lead to a convenient form that can be used as a framework for sensing policy design. It is worth noting, however, that if there is only a single period ($K = 1$), a case that can be optimal for some cases, the ML estimator can be obtained in a closed-form as shown in the following Lemma.

Lemma 1: When there is only a single period ($K = 1$), the MLE can be obtained in a closed from (3) in section II-A to be,

$$\begin{aligned}
\hat{\lambda}_{1-s} &= \frac{1}{\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} z_{2i}} \left(\frac{C}{C+1} + n_{1-s}\right), \\
\hat{\lambda}_s &= C\hat{\lambda}_{1-s},
\end{aligned}$$

where $C = \sqrt{A^2 + B} - A$, and

$$A = \frac{1}{2}\left(1 - \frac{\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} z_{2i}}{\sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} z_{2i-1}}\right)\left(1 - \frac{1}{1 + n_{1-s}}\right),$$

$$B = \frac{\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} z_{2i}}{\sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} z_{2i-1}} \frac{n_s}{1 + n_{1-s}}.$$

The MLE is simply obtained by putting the derivatives of the logarithm of (3) with respect to λ_s, and λ_{1-s} equal to zero and solving two equations with two unknowns.

B. A Moment Estimator

We propose a simple heuristic estimator and we prove that it is asymptotically efficient. This heuristic estimator is based on the sample mean of the *complete* idle and busy periods and is independent of the sensing structure. It means that the definition of the estimator does not depend on how many intervals (one single interval or many) are sensed on each channel. Denote by $n_i^i(t)$, and $n_i^b(t)$ the number of complete idle and busy periods sensed on channel i up to the time instant t (which includes the observed periods from all time intervals up to time t), and let $z_{i,j}^i$, and $z_{i,j}^b$ be the the duration of jth such periods. Then the estimators are

$$\hat{\lambda}_{0,i}(t) = \frac{n_i^i(t) - 1}{\sum_{j=1}^{n_i^i(t)} z_{i,j}^i}, \tag{12}$$

$$\hat{\lambda}_{1,i}(t) = \frac{n_i^b(t) - 1}{\sum_{j=1}^{n_i^b(t)} z_{i,j}^b}. \tag{13}$$

Note that the numerators of the estimators are $n_i^i(t) - 1$, and $n_i^b(t) - 1$ instead of $n_i^i(t)$ and $n_i^b(t)$. This is to make sure that these estimators are unbiased.

When a limited period of time T is given to the secondary user, it wants to optimally distribute it among the channels to minimize the total MSE (the sum of the MSE of all parameters). In the following we show that the heuristic estimators given in (12), and (13) can achieve optimal efficiency.

IV. SEQUENTIAL POLICY AND ASYMPTOTIC PROPERTIES

A. Optimal Sensing Policy

In this section we consider an optimal sensing policy that distributes total sensing budget to a set of M channels. A sensing policy π is defined by $\mathcal{T}^\pi = \{T_{ij}^\pi\}$ where T_{ij}^π is the jth sensing interval on channel i under policy π. Let $T_i^\pi = \sum_j T_{ij}^\pi$ and the total sensing time is given by $T = |\mathcal{T}^\pi| = \sum_i T_i^\pi$.

We first consider the problem of allocating the total amount of sensing time to different channels in a single installment, *i.e.*, we consider the class of policies Π_1 where each channel has a *single* interval of duration T_i. From (6), we have

$$I_i^{-1} = \begin{bmatrix} \frac{\lambda_{0,i}^2(\frac{1}{\lambda_{0,i}}+\frac{1}{\lambda_{1,i}})}{T_i}(1+o(T_i)) & o(T_i) \\ o(T_i) & \frac{\lambda_{1,i}^2(\frac{1}{\lambda_{0,i}}+\frac{1}{\lambda_{1,i}})}{T_i}(1+o(T_i)) \end{bmatrix}.$$

Since channels are jointly independent, we have

$$I^{-1} = \begin{bmatrix} I_1^{-1} & & & \\ & I_2^{-1} & & \\ & & \ddots & \\ & & & I_M^{-1} \end{bmatrix}.$$

Since Cramer-Rao bound (CRB) is a lower bound on the MSE, first we minimize the lower bound (Cramer-Rao Bound), then later we show that we can achieve the optimal lower bound.

The total variance (Cramer-Rao bound) is given by

$$\sum_{i=1}^{M} \frac{(\lambda_{0,i}^2 + \lambda_{1,i}^2)(\frac{1}{\lambda_{0,i}} + \frac{1}{\lambda_{1,i}})}{T_i}. \tag{14}$$

The problem is to minimize (14) under the condition

$$\sum_{i=1}^{M} T_i = T. \tag{15}$$

This problem can be solved by Lagrangian multiplier which gives the following solution, $\forall 1 \leq i, j \leq M$,

$$\frac{T_i}{\sqrt{(\lambda_{0,i}^2 + \lambda_{1,i}^2)(\frac{1}{\lambda_{0,i}} + \frac{1}{\lambda_{1,i}})}} = \frac{T_j}{\sqrt{(\lambda_{0,j}^2 + \lambda_{1,j}^2)(\frac{1}{\lambda_{0,j}} + \frac{1}{\lambda_{0,j}})}},$$

which results in

$$T_i = \alpha_i T, \tag{16}$$

where

$$\alpha_i = \frac{\sqrt{(\lambda_{0,i}^2 + \lambda_{1,i}^2)(\frac{1}{\lambda_{0,i}} + \frac{1}{\lambda_{1,i}})}}{\sum_{i=1}^{M}\sqrt{(\lambda_{0,i}^2 + \lambda_{1,i}^2)(\frac{1}{\lambda_{0,i}} + \frac{1}{\lambda_{1,i}})}}. \tag{17}$$

Therefore the optimal (minimum) CRB for all policies $\pi \in \Pi_1$ where each channel has a single interval is

$$\min_{\pi\in\Pi_1} \text{CRB}(\mathcal{T}^\pi) = \text{CRB}^*(T) = \frac{1}{T}\sum_{i=1}^{M} \frac{(\lambda_{0,i}^2+\lambda_{1,i}^2)(\frac{1}{\lambda_{0,i}}+\frac{1}{\lambda_{1,i}})}{\alpha_i}. \quad (18)$$

Here we denote by $\text{CRB}^*(T)$ the optimal CRB when $T = |\mathcal{T}^\pi|$ under the class of policies Π_1 with single installment per channel.

Derivations above show that the optimal time distribution among the channels depends on the values of the unknown parameters $\{\lambda_{0,i}, \lambda_{1,i}\}_{i=1}^M$. Since channel parameters are unknown, to achieve asymptotic optimality, we consider the class of sequential policies that involves multiple rounds of estimation, each round is based on data collected up to the previous round. For example, within each round, the sensing time spent on each channel can be allocated based on the most current channel estimates according to (17). Note that the use of data in sensing makes the sensing policy random, which complicates the MSE analysis presented in Section IV-E.

B. SEINE: Sequential Estimation with Increasing Nested Epochs

We describe here policy SEINE—an adaptive estimation scheme that adjust data collection strategies based on current estimates. In Section IV-E, we will establish the asymptotic optimality of SEINE. The key of SEINE is to involve multiple rounds with increasing duration. Within each round, sensing time of each channel is allocated according to (17) based on the estimated channel parameters using all data collected up to the previous round.

Policy SEINE (Sequence Estimation with Increasing Nested Epochs):

(1) Round 0: Pick a constant $c_0 = C$ large enough with respect to the average idle busy times of all channels. Sense all channels equally by $\frac{C}{M}$ time.

(2) Round n ($n \geq 1$): Use all the observations from rounds $0, \ldots, n-1$ to obtain the channel parameter estimates $\{\hat{\lambda}_{0,i}(n), \hat{\lambda}_{1,i}(n)\}_{i=1}^M$ using (12) and (13) where $t = \sum_{j=0}^{n-1} c_j$. Substitute these estimated parameters in (17) to get the estimates $\hat{\alpha}_i(n)$s to distribute time among channels. Pick the total round time c_n and spend $\hat{\alpha}_i(n)c_n$ amount of time on channel i, where $\{c_n\}_{n=1}^\infty$ is an increasing sequence such that $\lim_{n\to\infty} \frac{\sum_{i=1}^n c_i}{n} = \infty$.

In what follows we consider three asymptotic properties of the estimator and policy proposed above.

C. Asymptotic Statistics Properties: Strong Consistency

A sequence of estimators $\hat{\lambda}(T)$ for parameter λ is strongly consistent if $\hat{\lambda}(T)$ converges *almost surely* to λ.

$$\hat{\lambda}(T) \xrightarrow{T\to\infty} \lambda \quad \text{a.s.} \tag{19}$$

The first step to show consistency is to make sure the number of samples required for the estimators in (12) and (13) tends to infinity as time grows. This fact is shown in the following lemma which will be used later too.

Lemma 2: Define the event

$$E_n^i \triangleq \{\text{in the } nth \text{ round, at least one complete idle busy period is sensed on channel } i\}$$

, then

$$\Pr\{E_n^i \ i.o.\} = 1.$$

Proof: Let $\gamma_i = \min\{\lambda_{0,i}, \lambda_{1,i}\}$, then

$$\Pr\{E_n^i\} \geq F_{Erlang}(c_n, 4, \gamma_i) = 1 - \sum_{k=0}^{3} e^{-\gamma_i c_n} \frac{(\gamma_i c_n)^k}{k!}, \tag{20}$$

where $F_{Erlang}(x, 4, \gamma_i)$ is the CDF of the Erlang distribution of order 4 with parameter γ_i.

Based on the definition of the policy SEINE,

$$\lim_{n\to\infty} c_n = \infty. \tag{21}$$

Therefore

$$\lim_{n\to\infty} \Pr\{E_n^i\} = 1, \tag{22}$$

so that we can conclude

$$\Pr\{E_n^i \ i.o.\} = \Pr\{\limsup E_n^i\} \geq \lim_{n\to\infty} \Pr\{E_n^i\} = 1. \tag{23}$$

■

Theorem 3: The estimators defined in (12) and (13) are consistent.

Proof: Based on lemma 2, we know

$$\lim_{T\to\infty} [\hat{\lambda}_{0,i}(T), \hat{\lambda}_{1,i}(T)] = [\lim_{n_0\to\infty} \hat{\lambda}_{0,i}(n_0), \lim_{n_1\to\infty} \hat{\lambda}_{1,i}(n_1)]. \tag{24}$$

Since the complete periods are i.i.d random variables, based on the strong Law of Large Numbers

$$[\lim_{n_0\to\infty}\frac{1}{\hat{\lambda}_{0,i}(n_0)},\lim_{n_1\to\infty}\frac{1}{\hat{\lambda}_{1,i}(n_1)}] = [\lim_{n_0\to\infty}(\frac{n_0-1}{n_0}\frac{1}{\hat{\lambda}_{0,i}(n_0)}),\lim_{n_1\to\infty}(\frac{n_1-1}{n_1}\frac{1}{\hat{\lambda}_{1,i}(n_1)})] \quad (25)$$
$$= [\frac{1}{\lambda_{0,i}},\frac{1}{\lambda_{1,i}}] \quad a.s.$$

From probability theory we know that If

$$\lim_{n\to\infty} X_n = C \qquad a.s. \tag{26}$$

where $C > 0$, then

$$\lim_{n\to\infty}\frac{1}{X_n} = \frac{1}{C} \qquad a.s. \tag{27}$$

Since all $\lambda_{j,i}$'s are strictly positive, by choosing $X_n = \frac{1}{\hat{\lambda}_{j,i}}$, *consistency* follows from (27).

■

D. Asymptotic Statistics Properties: Asymptotic Distribution

Perhaps the most common distribution to arise as an asymptotic distribution is the normal distribution. In particular, the central limit theorem provides an example where the asymptotic distribution is the normal distribution.

Consider the reciprocal of the estimators in (12) and (13)

$$\frac{1}{\hat{\lambda}_{j,i}(n)} = \frac{n}{n-1}\frac{\sum_{k=1}^{n} z_{j,i}^{k}}{n}, \tag{28}$$

where $z_{j,i}^{k}$'s are i.i.d with exponential distribution and parameter $\lambda_{j,i}$. From CLT (Central Limit Theorem) we have

$$\frac{n-1}{\sqrt{n}}(\frac{1}{\hat{\lambda}_{j,i}(n)}-\frac{1}{\lambda_{j,i}}) \xrightarrow{d} \mathcal{N}(0,\frac{1}{\lambda_{j,i}^2}). \tag{29}$$

In general if $Y_n = \frac{1}{X_n}$, the distribution of the reciprocal of the non-negative random variable X_n is

$$F_{Y_n}(y) = \Pr\{Y_n \le y\} = \Pr\{\frac{1}{X_n} \le y\} = \Pr\{X_n \ge \frac{1}{y}\} = 1 - F_{X_n}(\frac{1}{y}), \tag{30}$$

and for the pdf we have

$$f_{Y_n}(y) = \frac{1}{y^2} f_{X_n}(\frac{1}{y}). \tag{31}$$

Therefore since $\frac{1}{\hat{\lambda}_{j,i}(n)}$ has asymptotic distribution given in (29), the asymptotic distribution of the estimator $\hat{\lambda}_{j,i}(n)$ can be obtained in closed-form by (30), and (31).

E. Asymptotic Statistics Properties: Asymptotic Efficiency

We now establish the asymptotic efficiency of SEINE for the class of sparse sensing policies. The reason to consider only sparse sensing policies is that, in practice, when switching cost is not negligible, each channel should be observed at least for some minimum amount of time, and if there are many channels to consider, the gaps between observation intervals are sufficiently large. Recall that a sensing policy π is defined by the allocation of sensing time $\mathcal{T}^\pi = \{T_{ij}^\pi\}$ where T_{ij}^π is the duration of the jth sensing interval on channel i. We denote the total sensing time by $T = |\mathcal{T}^\pi| = \sum_{ij} T_{ij}^\pi$. The MSE of a channel estimator associated with sensing policy π is given by

$$\text{MSE}(\mathcal{T}^\pi) = \sum_{i=1}^{M} \{(\hat{\lambda}_{0,i} - \lambda_{0,i})^2 + (\hat{\lambda}_{1,i} - \lambda_{1,i})^2\}. \tag{32}$$

For the class of sparse sensing policies (as defined in Theorem 2), Theorem 2 shows that the Fisher information $I(\mathcal{T}^\pi)$ associated with $\mathcal{T}^\pi$ is upper bounded by the Fisher information when each channel is allocated with a single sensing interval

$$I(\mathcal{T}^\pi) \leq I_1(\mathcal{T}^\pi), \tag{33}$$

where $I_1(\mathcal{T}^\pi)$ is the Fisher information when each channel is allocated with a single sensing interval with total sensing time $|\mathcal{T}^\pi|$. Therefore, for a sparse sensing policy π, the optimal CRB is obtained under the class of policies Π_1 (with single installment per channel) in (18). Hence we define the asymptotic efficiency by

$$\eta^\pi = \lim_{|\mathcal{T}^\pi| \to \infty} \frac{\text{CRB}^*(|\mathcal{T}^\pi|)}{\text{MSE}(\mathcal{T}^\pi)}, \tag{34}$$

where $\text{CRB}^*(|\mathcal{T}^\pi|) = \min_{\pi \in \Pi_1} \text{CRB}(\mathcal{T}^\pi)$ defined in (18) is the optimal CRB for the class of policies Π_1 which is optimal for the class of sparse policies as well based on (33).

When the sensing policy is randomized, the resulting allocation of sensing time $\mathcal{T}^\pi$ is also random. We use stopping times (the time instants when a certain number of complete idle and busy periods are at hand) in the calculation of MSE to deal with the randomness, and we define the relative efficiency the same way as (34). Note, however, that the relative efficiency is also random. A sensing and estimation policy π is almost surely asymptotically efficient if

$$\lim_{|\mathcal{T}^\pi| \to \infty} \frac{\text{CRB}^*(|\mathcal{T}^\pi|)}{\text{MSE}(\mathcal{T}^\pi)} = 1 \quad a.s. \tag{35}$$

The following theorem establishes the a.s. asymptotic efficiency.

Theorem 4: The asymptotic efficiency of the Policy SEINE converges to 1 asymptotically almost surely. Namely

$$\eta^{\text{SEINE}} = \lim_{|\mathcal{T}^{\text{SEINE}}|\to\infty} \frac{\text{CRB}^*(|\mathcal{T}^{\text{SEINE}}|)}{\text{MSE}(\mathcal{T}^{\text{SEINE}})} = 1 \quad a.s. \tag{36}$$

Proof: See Appendix C. ■

V. Simulation Results

In this section the simulation results are presented in three subsections. First we present the simulation results showing the consistency of the estimator, then we present the simulation results regarding the asymptotic distribution of the estimator, and at the end we show the asymptotic convergence of the MSE of policy SEINE to the Cramer-Rao bound (asymptotic convergence of efficiency to one). For simulation a network consisting of 4 channels occupied by primary system is considered with the following parameter set

$$\begin{aligned} \vec{\lambda_0} &= [0.1,\ 0.6,\ 0.2,\ 0.5], \\ \vec{\lambda_1} &= [0.7,\ 0.1,\ 0.8,\ 0.9]. \end{aligned}$$

The sequence of round lengths used in the policy SEINE, $\{c_n\}_{n=0}^{\infty}$, is picked to be exponentially increasing with n. The policy SEINE is run for 22 rounds for 200 monte-carlos. We refer to this setting as Case 1.

A. Consistency

In this section we illustrate the consistency of our estimator by simulation. In Fig. 5 the estimated λ_0 and λ_1 (averaged over all monte-carlo runs) for all channels are presented. It is shown that all of the estimated parameters converge to their true values.

In Fig. 6 two parameters $\lambda_{0,1}$ (λ_0 for channel 1), and $\lambda_{0,3}$ (λ_0 for channel 3) are chosen and the estimated sequence is plotted for all the Monte-Carlo runs. As shown in the figure the estimated sequence converges to the true values for *all* the Monte-Carlos verifying the almost surely convergence (strong consistency).

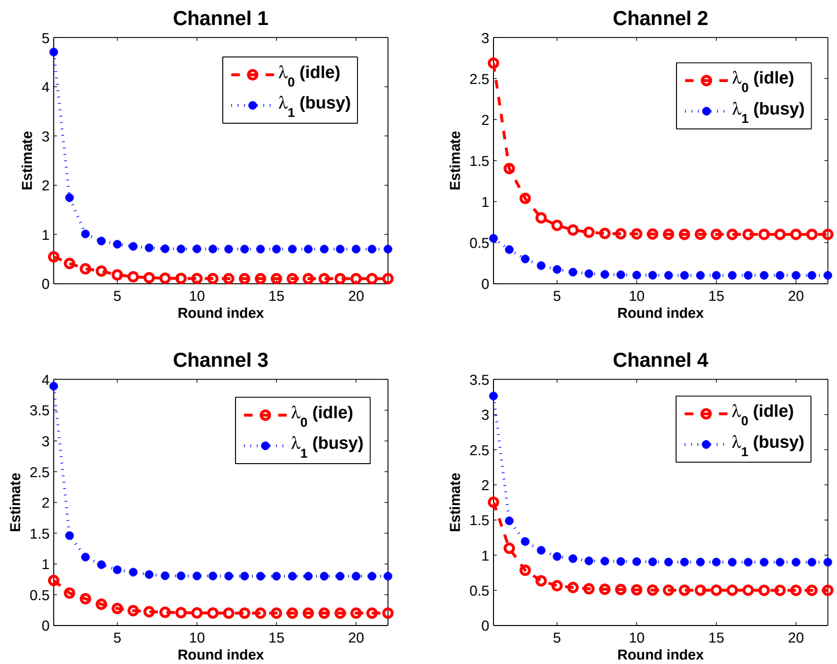

Fig. 5. Convergence of estimated parameters for all channels

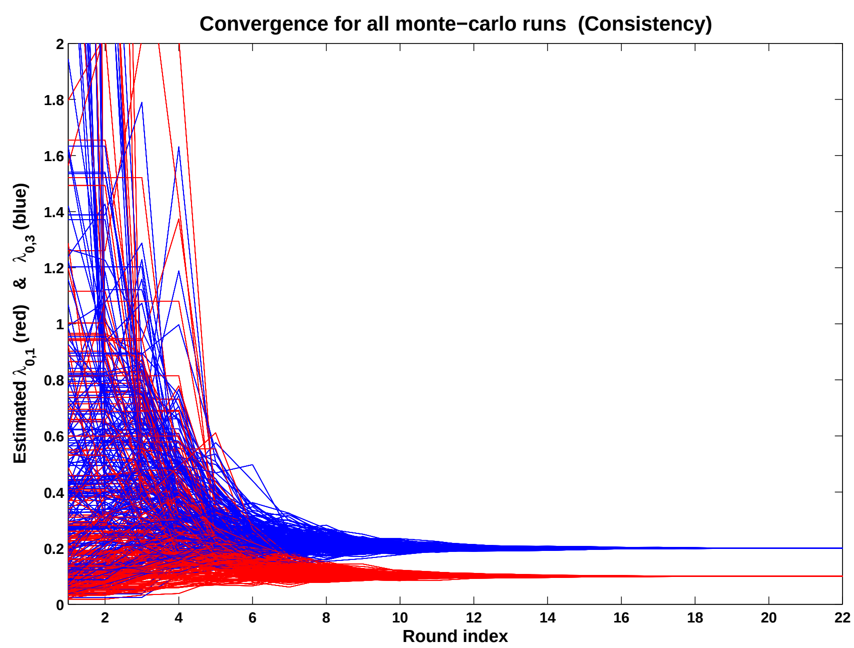

Fig. 6. Convergence of estimated parameters for all monte-carlos

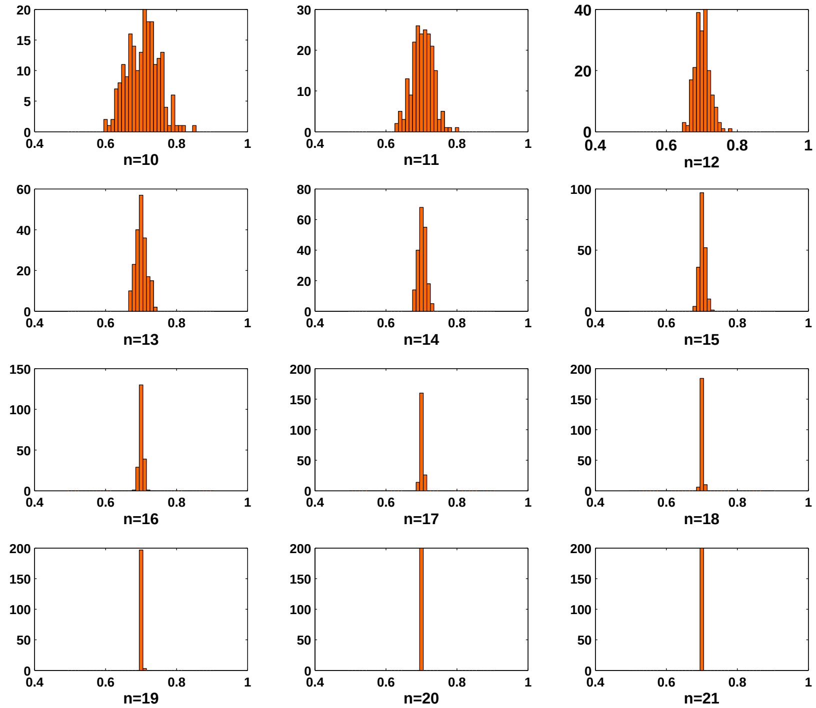

Fig. 7. Histogram of the $\hat{\lambda}_{1,1}(n)$

B. Asymptotic Distribution

In this section we plot the histogram of the estimated sequence for $\lambda_{1,1}$ (λ_1 for channel 1). Fig. 7 shows the distribution of the $\hat{\lambda}_{1,1}(n)$ for different round indexes n, and how it is distributed around and converges to $\lambda_{1,1} = 0.7$ as the round index n grows.

C. Asymptotic Efficiency

In this section we plot the MSE of the policy SEINE vs. the optimal Cramer-Rao bound for the sparse policies (CRB*). Here we consider another network setting with the same number of channels but different parameter set

$$\vec{\lambda_0} = [0.3,\ 0.2,\ 0.3,\ 0.4],$$
$$\vec{\lambda_1} = [0.3,\ 0.3,\ 0.2,\ 0.15].$$

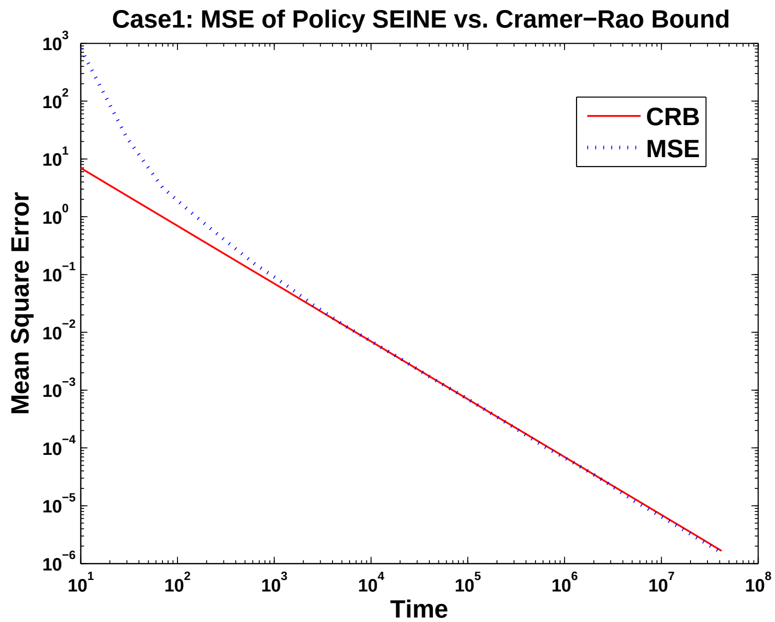

Fig. 8. MSE vs. CRB*: Case 1

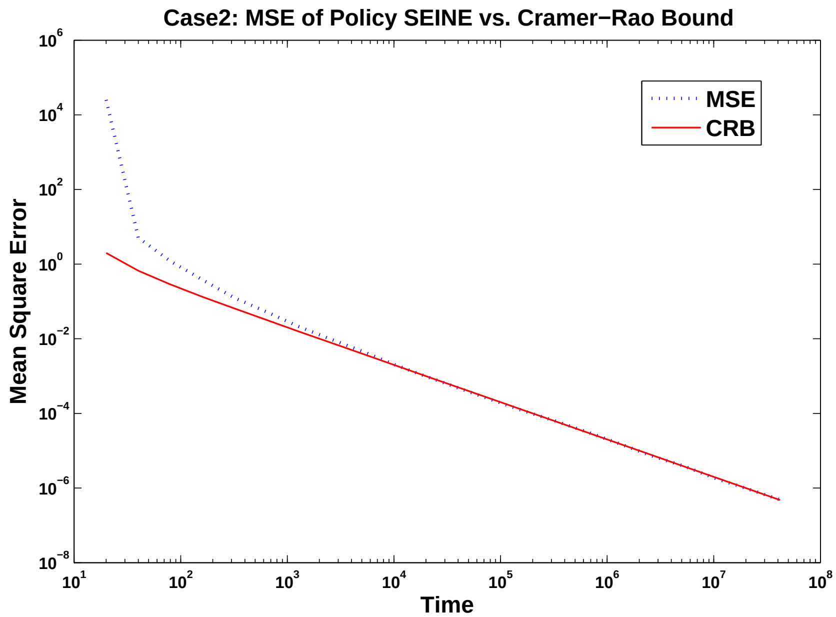

Fig. 9. MSE vs. CRB* : Case 2

The policy SEINE is run the same as the case 1 for 22 rounds with 200 monte-carlo runs. We refer to this new setting as case 2. The MSE of policy SEINE vs. CRB* is plotted for both case 1, and case 2 in Fig. 8 and Fig. 9. As you can see the MSE asymptotically converges to CRB* indicating the asymptotic efficiency of the policy SEINE for the estimator within sparse policies.

VI. Conclusion

In this paper, we have considered the problem of multichannel estimation in OSA. The channel occupations of the primary system are modeled by two-states Markov processes. Secondary system's objective is to learn the channel parameters in the optimal way in order to maximize the estimator's efficiency. A sequential policy is proposed that achieves the optimal efficiency asymptotically.

Appendix A: Proof of Theorem 1

From (5)

$$-\nabla^2_{\lambda_0,\lambda_1} L(\lambda_0,\lambda_1) = \begin{bmatrix} \frac{n_0+1_{\{s=1\}}}{\lambda_0^2} - \frac{1}{(\lambda_0+\lambda_1)^2} & -\frac{1}{(\lambda_0+\lambda_1)^2} \\ -\frac{1}{(\lambda_0+\lambda_1)^2} & \frac{n_1+1_{\{s=0\}}}{\lambda_1^2} - \frac{1}{(\lambda_0+\lambda_1)^2} \end{bmatrix},$$

we get

$$\begin{aligned}
\mathbb{E}\{\frac{\partial^2 L}{\partial \lambda_0^2}\} &= \frac{\lambda_1}{\lambda_0+\lambda_1}\mathbb{E}\{\frac{n_s}{\lambda_s^2}|s=0\} + \frac{\lambda_0}{\lambda_0+\lambda_1}\mathbb{E}\{\frac{n_{1-s}+1}{\lambda_{1-s}^2}|s=1\} - \frac{1}{(\lambda_0+\lambda_1)^2} \\
&= \Pr\{\text{n is even}\}\frac{\lambda_1}{\lambda_0+\lambda_1}\mathbb{E}\{\frac{n_0-c_e^0}{\lambda_s^2}|s=0,\text{n is even}\} + \Pr\{\text{n is odd}\}\frac{\lambda_1}{\lambda_0+\lambda_1}\mathbb{E}\{\frac{n_0-c_o^0}{\lambda_s^2}|s=0,\text{n is odd}\} \\
&+ \Pr\{\text{n is even}\}\frac{\lambda_0}{\lambda_0+\lambda_1}\mathbb{E}\{\frac{n_0-c_e^1}{\lambda_{1-s}^2}|s=1,\text{n is even}\} \\
&+ \Pr\{\text{n is odd}\}\frac{\lambda_0}{\lambda_0+\lambda_1}\mathbb{E}\{\frac{n_0-c_o^1}{\lambda_{1-s}^2}|s=1,\text{n is odd}\} - \frac{1}{(\lambda_0+\lambda_1)^2} \\
&= \mathbb{E}\{\frac{n_0}{\lambda_0^2}\} + \Pr\{\text{n is even}\}(\frac{\lambda_1 c_e^0+\lambda_0 c_e^1}{\lambda_0+\lambda_1}) + \Pr\{\text{n is odd}\}(\frac{\lambda_1 c_o^0+\lambda_0 c_o^1}{\lambda_0+\lambda_1}) \\
&= \mathbb{E}\{\frac{n_0}{\lambda_0^2}\} + f_0,
\end{aligned}$$

where

$$f_0 = \Pr\{\text{n is even}\}(\frac{\lambda_1 c_e^0+\lambda_0 c_e^1}{\lambda_0+\lambda_1}) + \Pr\{\text{n is odd}\}(\frac{\lambda_1 c_o^0+\lambda_0 c_o^1}{\lambda_0+\lambda_1}) \tag{37}$$

is a bounded function of parameters λ_0, λ_1, and T_1, and c_o^j, and c_e^j are some constants depending on n being even or odd and also s; but since they are constants, their exact value does not affect our final results. Similarly we get

$$\mathbb{E}\{\frac{\partial^2 L}{\partial \lambda_1^2}\} = \mathbb{E}\{\frac{n_1}{\lambda_1^2}\} + f_1.$$

And clearly for the cross terms $\mathbb{E}\{\frac{\partial^2 L}{\partial\lambda_1 \partial\lambda_0}\} = -\frac{1}{(\lambda_0+\lambda_1)^2}$. Note that

$$\mathbb{E}\{n_0\} = \mathbb{E}\{n_1\} = \frac{T}{\frac{1}{\lambda_0} + \frac{1}{\lambda_1}}. \tag{38}$$

Therefore

$$I_{[t_1,\ t_1+T]} = \begin{bmatrix} \frac{T}{\lambda_0^2(\frac{1}{\lambda_0}+\frac{1}{\lambda_1})} + f_0 - \frac{1}{(\lambda_0+\lambda_1)^2} & -\frac{1}{(\lambda_0+\lambda_1)^2} \\ -\frac{1}{(\lambda_0+\lambda_1)^2} & \frac{T}{\lambda_1^2(\frac{1}{\lambda_0}+\frac{1}{\lambda_1})} + f_1 - \frac{1}{(\lambda_0+\lambda_1)^2} \end{bmatrix}.$$

From definition of f_0, and f_1 we can get

$$I_{[t_1,\ t_1+T]} = \begin{bmatrix} \frac{T}{\lambda_0^2(\frac{1}{\lambda_0}+\frac{1}{\lambda_1})}(1+o(T_1)) & -\frac{1}{(\lambda_0+\lambda_1)^2} \\ -\frac{1}{(\lambda_0+\lambda_1)^2} & \frac{T}{\lambda_1^2(\frac{1}{\lambda_0}+\frac{1}{\lambda_1})}(1+o(T_1)) \end{bmatrix}.$$

Appendix B: Proof of Theorem 2

Let first consider the Likelihood function for the case we have two disjoint intervals. Assume that two intervals $[t_1, t_1+T_1] \cup [t_2, t_2+T_2]$ are observed. Therefore the likelihood function becomes

$$\begin{aligned} & f(\{s_j, (z_{h,j})_{h=1}^{n_j}\}_{j=1}^{2} | \lambda_0, \lambda_1) \\ = \;& f(s_1, (z_{h,1})_{h=1}^{n_1} | \lambda_0, \lambda_1) f(s_2, (z_{h,2})_{h=1}^{n_2} | \lambda_0, \lambda_1, S_{t_1+T_1} = s') \\ = \;& f(s_1, (z_{h,1})_{h=1}^{n_1} | \lambda_0, \lambda_1) \Pr\{S_{t_2} = s_2 | S_{t_1+T_1} = s'\} f((z_{h,2})_{h=1}^{n_2} | \lambda_0, \lambda_1, S_{t_2} = s_2) \\ = \;& f(s_1, (z_{h,1})_{h=1}^{n_1} | \lambda_0, \lambda_1) \frac{\Pr\{S_{t_2} = s_2 | S_{t_1+T_1} = s'\}}{\Pr\{S_{t_2} = s_2\}} f(s_2, (z_{h,2})_{h=1}^{n_2} | \lambda_0, \lambda_1) \\ = \;& f_{T_1} \frac{\Pr\{S_{t_2} = s_2 | S_{t_1+T_1} = s'\}}{\Pr\{S_{t_2} = s_2\}} f_{T_2}. \end{aligned}$$

Here we denote the likelihood function of a single interval of length T_j as obtained in previous section by f_{T_j}. It is seen that the extra information provided by the discrete sampling is

$$\frac{\Pr\{S_{t_2} = s_2 | S_{t_1+T_1} = s'\}}{\Pr\{S_{t_2} = s_2\}}. \tag{39}$$

It is apparent that when these intervals are far enough so that S_{t_2} and $S_{t_1+T_1}$ are independent, this term becomes

$$\frac{\Pr\{S_{t_2} = s_2 | S_{t_1+T_1} = s'\}}{\Pr\{S_{t_2} = s_2\}} = \frac{\Pr\{S_{t_2} = s_2\}}{\Pr\{S_{t_2} = s_2\}} = 1. \tag{40}$$

Thus the discrete sampling part will not provide any extra information and the Fisher information will be the sum of the Fisher information for these two intervals.

The Fisher information matrix for $[t_1, t_1+T_1] \cup [t_2, t_2+T_2]$ is

$$I_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} = I_{[t_1,t_1+T_1]} + I_{[t_1,t_1+T_2]} + I^D_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]}, \tag{41}$$

where

$$I^D_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} = \mathbb{E}(-\nabla^2_{\lambda_0,\lambda_1} \log(\frac{\Pr\{S_{t_2}=s_2|S_{t_1+T_1}=s'\}}{\Pr\{S_{t_2}=s_2\}})). \tag{42}$$

Now based on the above argument and (5), we compare the Fisher information matrixes for these two cases of single and two intervals.

For the single interval $[t_0,\ t_0+T]$, the Fisher information matrix is

$$I_{[t_0,\ t_0+T]} = \mathbb{E}\begin{bmatrix} \frac{n_0+1_{\{s_0=1\}}}{\lambda_0^2} - \frac{1}{(\lambda_0+\lambda_1)^2} & -\frac{1}{(\lambda_0+\lambda_1)^2} \\ -\frac{1}{(\lambda_0+\lambda_1)^2} & \frac{n_1+1_{\{s_0=0\}}}{\lambda_1^2} - \frac{1}{(\lambda_0+\lambda_1)^2} \end{bmatrix}.$$

The Fisher information matrix for $[t_1, t_1+T_1] \cup [t_2, t_2+T_2]$ is

$$I_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} = I_{[t_1,t_1+T_1]} + I_{[t_1,t_1+T_2]} + I^D_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]}. \tag{43}$$

It can be rewritten as

$$\begin{aligned} I_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} &= \mathbb{E}\begin{bmatrix} \frac{n_0^1+1_{\{s_1=1\}}}{\lambda_0^2} - \frac{1}{(\lambda_0+\lambda_1)^2} & -\frac{1}{(\lambda_0+\lambda_1)^2} \\ -\frac{1}{(\lambda_0+\lambda_1)^2} & \frac{n_1^1+1_{\{s_1=0\}}}{\lambda_1^2} - \frac{1}{(\lambda_0+\lambda_1)^2} \end{bmatrix} \\ &+ \mathbb{E}\begin{bmatrix} \frac{n_0^2+1_{\{s_2=1\}}}{\lambda_0^2} - \frac{1}{(\lambda_0+\lambda_1)^2} & -\frac{1}{(\lambda_0+\lambda_1)^2} \\ -\frac{1}{(\lambda_0+\lambda_1)^2} & \frac{n_1^2+1_{\{s_2=0\}}}{\lambda_1^2} - \frac{1}{(\lambda_0+\lambda_1)^2} \end{bmatrix} + I^D_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} \\ &= \mathbb{E}\begin{bmatrix} \frac{n_0^1+n_0^2+1_{\{s_1=1\}}+1_{\{s_2=1\}}}{\lambda_0^2} - \frac{2}{(\lambda_0+\lambda_1)^2} & -\frac{2}{(\lambda_0+\lambda_1)^2} \\ -\frac{2}{(\lambda_0+\lambda_1)^2} & \frac{n_1^1+n_1^2+1_{\{s_1=0\}}+1_{\{s_2=0\}}}{\lambda_1^2} - \frac{2}{(\lambda_0+\lambda_1)^2} \end{bmatrix} \\ &+ I^D_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]}, \end{aligned} \tag{44}$$

where n_s^j, $j=1,2$, and $s=0,1$ is the corresponding value of n_s for the jth interval. Note that since chopping the interval $[t_0,\ t_0+T]$ into two pieces gives smaller number of *complete* idle and busy periods in expectation, we have

$$\mathbb{E}[n_s^1 + n_s^2 + 1_{\{s_1=1-s\}} + 1_{\{s_2=1-s\}}] \ \leq\ \mathbb{E}[n_s + 1_{\{s_0=1-s\}}]. \tag{45}$$

Using this inequality we get

$$\begin{aligned} I_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} &= \mathbb{E}\begin{bmatrix} \frac{n_0^1+n_0^2+1_{\{s_1=1\}}+1_{\{s_2=1\}}}{\lambda_0^2} - \frac{2}{(\lambda_0+\lambda_1)^2} & -\frac{2}{(\lambda_0+\lambda_1)^2} \\ -\frac{2}{(\lambda_0+\lambda_1)^2} & \frac{n_1^1+n_1^2+1_{\{s_1=0\}}+1_{\{s_2=0\}}}{\lambda_1^2} - \frac{2}{(\lambda_0+\lambda_1)^2} \end{bmatrix} \\ &+ I^D_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} \\ &\leq \mathbb{E}\begin{bmatrix} \frac{n_0+1_{\{s_0=1\}}}{\lambda_0^2} - \frac{2}{(\lambda_0+\lambda_1)^2} & -\frac{2}{(\lambda_0+\lambda_1)^2} \\ -\frac{2}{(\lambda_0+\lambda_1)^2} & \frac{n_1+1_{\{s_0=0\}}}{\lambda_1^2} - \frac{2}{(\lambda_0+\lambda_1)^2} \end{bmatrix} + I^D_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} \\ &= I_{[t_0,\ t_0+T]} - \frac{1}{(\lambda_0+\lambda_1)^2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + I^D_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]}. \end{aligned}$$

From (40) it is clear that when $\Delta t_1 \to \infty$ the discrete sampling gives no extra information. Therefore

$$\lim_{\Delta t_1 \to \infty} I^D_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{46}$$

Also note that the matrix 1 is positive definite

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} > 0. \tag{47}$$

Therefore when $\Delta t_1 \to \infty$ (which makes the matrix $I^D_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]}$ zero) we have

$$I_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} \leq I_{[t_0,\ t_0+T]} - \frac{1}{(\lambda_0+\lambda_1)^2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} < I_{[t_0,\ t_0+T]}. \tag{48}$$

Thus from (46) we know that $\exists \tau > 0$ such that

$$\tau = \inf\{\Delta t_1 > 0 \mid I^D_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} \leq \frac{1}{(\lambda_0+\lambda_1)^2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\}. \tag{49}$$

For every $\Delta t_1 \geq \tau$

$$I_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} \leq I_{[t_0,\ t_0+T]} - \frac{1}{(\lambda_0+\lambda_1)^2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + I^D_{[t_1,t_1+T_1]\cup[t_2,t_2+T_2]} \leq I_{[t_0,\ t_0+T]}. \tag{50}$$

It means that sensing one continuous interval has higher Fisher information than chopping it into two pieces.

For K intervals with $\min_{j=1}^{K-1}\{\Delta t_j\} > \tau$ we can use induction. We take the first two intervals and replace them by one interval to get $K-1$ intervals. We do this procedure for $K-1$ steps to get a single interval with length $T = \sum_{j=1}^{K} T_j$.

Appendix C: Proof of Theorem 4

Since the MSE is obtained under the policy SEINE, in what follows we drop the superscript SEINE for simplicity.

Lemma 3:

$$\lim_{n\to\infty}\hat{\alpha}_i(n)=\lim_{\bar{k}\to\infty}\hat{\alpha}_i^{\bar{k}}=\alpha_i,\quad \forall i\in\{1,\dots,M\}\quad a.s. \tag{51}$$

where $\bar{k}=(k_i)_{i=1}^M$, $\hat{\alpha}_i(n)$ is the estimated α_i at the end of round n, and $\hat{\alpha}_i^{\bar{k}}$ is the estimated α_i when $\bar{k}$ complete samples of idle and busy periods is at hand for all channels.

Proof: $\lim_{\bar{k}\to\infty}\hat{\alpha}_i^{\bar{k}}=\alpha_i$ is based on the strong Law of Large Numbers

$$\lim_{(k_i)_{i=1}^M\to\infty}(\hat{\lambda}_{0,i},\hat{\lambda}_{1,i})\to(\lambda_{0,i},\lambda_{1,i})\quad a.s.\quad \forall i\in\{1,\dots M\}, \tag{52}$$

and the fact that α_i is a continuous function of $(\lambda_{0,j},\lambda_{1,j}),\quad j\in\{1,\dots M\}$.
$\lim_{n\to\infty}\hat{\alpha}_i(n)=\lim_{\bar{k}\to\infty}\hat{\alpha}_i^{\bar{k}}\quad a.s.$ is a direct consequence of lemma 2. ($\lim_{n\to\infty}\hat{\alpha}_i(n)=\lim_{\bar{k}\to\infty}\hat{\alpha}_i^{\bar{k}}$ on the event $\{E_n^i\ i.o.\}$ and this event has probability 1). ■
Define the stopping times

$$\tau_{k_i}\ \triangleq\ \inf\{\text{time}\mid\text{exactly } k_i \text{ complete idle and busy periods is sensed on channel } i\}. \tag{53}$$

Consider the time sequences $\{\tau_{k_i}\}_{k=1}^{\infty}$. $\mathrm{MSE}_{\tau_{k_i}}^i$ is the sum of MSEs for the parameters $\lambda_{0,i}$ and $\lambda_{1,i}$ at the time instant τ_{k_i}. It is computed for the estimators in (12), and (13) using the complete idle or busy periods of channel i which can be rewritten as $\hat{\lambda}_{j,i}(k_i)=\frac{k_i-1}{\sum_{h=1}^{k_i}z_h}$ where z_h's are i.i.d exponential random variables with parameter $\lambda_{j,i}$.

In general if $X=\frac{\sum_i^n\rho_i}{n-1}$ where ρ_is are i.i.d exponentially distributed random variables with parameter λ, X has Erlang distribution of degree n with parameter $(n-1)\lambda$. Therefore for the MSE of $\frac{1}{X}$ we have

$$\begin{aligned}
\mathrm{MSE} &= \mathbb{E}\{(\frac{1}{X}-\lambda)^2\}=\int_0^\infty(\frac{1}{t^2}-2\frac{\lambda}{t}+\lambda^2)\frac{((n-1)\lambda)^n t^{n-1}e^{-\lambda(n-1)t}}{(n-1)!}dt\\
&= \lambda^2\int_0^\infty\frac{((n-1)\lambda)^n t^{n-1}e^{-\lambda(n-1)t}}{(n-1)!}dt+\frac{((n-1)\lambda)^2}{(n-1)(n-2)}\int_0^\infty\frac{((n-1)\lambda)^{n-2}t^{n-3}e^{-\lambda(n-1)t}}{(n-3)!}dt\\
&\quad -2\lambda\frac{((n-1)\lambda)}{n-1}\int_0^\infty\frac{((n-1)\lambda)^{n-1}t^{n-2}e^{-\lambda(n-1)t}}{(n-2)!}dt\\
&= \frac{\lambda^2}{n-2}.
\end{aligned}$$

Therefore for our problem we get

$$\text{MSE}^i_{\tau_{k_i}} = \frac{\lambda_{0,i}^2 + \lambda_{1,i}^2}{k_i - 2}. \tag{54}$$

Denote

$$\begin{aligned}
T(n) &\triangleq \text{total time spent on the system at the end of round } n. \\
T_i(n) &\triangleq \text{total time spent on channel } i \text{ at the end of round } n. \\
T_i^*(n) &\triangleq \text{total time spent on channel } i \text{ at the end of round } n \\
&\quad\ \text{that only includes complete idle/busy periods.} \\
T_i(\tau_{k_i}) &\triangleq \text{total time spent on channel } i \text{ up to time } \tau_{k_i}. \\
T_i^*(\tau_{k_i}) &\triangleq \text{total time spent on channel } i \text{ up to time } \tau_{k_i} \text{ that} \\
&\quad\ \text{only includes complete idle/busy periods.}
\end{aligned}$$

Lemma 4:

$$\lim_{k_i \to \infty} \frac{T_i^*(\tau_{k_i})}{T_i(\tau_{k_i})} = 1 \qquad a.s.$$

Proof: Note that $T_i(\tau_{k_i})$, and $T_i^*(\tau_{k_i})$ differ only at the incomplete idle/busy periods that only happens at the beginning and the end of each round. Therefore

$$\frac{T_i^*(\tau_{k_i})}{T_i(\tau_{k_i})} - 1 = -\frac{\sum \text{incomplete periods}}{T_i(\tau_{k_i})}.$$

Denote

$$n_* \triangleq \sup\{n : T_i(n) \leq T_i(\tau_{k_i})\},$$

then the number of incomplete periods is at most $2n_* + 1$. Thus

$$\frac{\sum \text{incomplete periods}}{T_i(\tau_{k_i})} \leq \frac{\sum \text{incomplete periods}}{T_i(n_*)}.$$

Note that

$$\begin{aligned}
0 &\leq \lim_{n_* \to \infty} \frac{\sum \text{incomplete periods}}{T_i(n_*)} \\
&= \lim_{n_* \to \infty} \frac{((m^b + o(m^b))\frac{1}{\lambda_{1,i}} + (m^i + o(m^i))\frac{1}{\lambda_{0,i}})}{\sum_{j=1}^{n_*} c_j \hat{\alpha}_i(j)} \quad a.s. \\
&\leq \lim_{n_* \to \infty} \frac{(2(n_* + o(n_*)) + 1)\frac{1}{\gamma_i}}{\sum_{j=1}^{n_*} c_j \hat{\alpha}_i(j)},
\end{aligned}$$

where $m^i + m^b = 2n_* + 1$, and,$1/\gamma_i = \max\{1/\lambda_{0,i}, 1/\lambda_{1,i}\}$. Since $\alpha_i > 0$,

$$\exists \epsilon_i > 0 \text{s.t. } \alpha_i - \epsilon_i > 0.$$

From lemma 3,

$$\exists n_{\epsilon_i} s.t.\ \forall n > n_{\epsilon_i},\ |\hat{\alpha}_i(n) - \alpha_i| < \epsilon_i.$$

Therefore

$$\sum_{j=1}^{n_*} c_j \hat{\alpha}_i(j) \geq \alpha_i^m \sum_{j=1}^{n_*} c_j.$$

where $\alpha_i^m = \min\{\min_{j=1,\ldots n_{\epsilon_i}}\{\hat{\alpha}_i(j)\}, \alpha_i - \epsilon_i\} > 0$. Thus

$$\begin{aligned} 0 &\leq \lim_{n_* \to \infty} \frac{\sum \text{incomplete periods}}{T_i(n_*)} \\ &\leq \lim_{n_* \to \infty} \frac{(2(n_* + o(n_*)) + 1)\frac{1}{\gamma_i}}{\sum_{j=1}^{n_*} c_j \hat{\alpha}_i(j)} \\ &\leq \lim_{n_* \to \infty} \frac{(2(n_* + o(n_*)) + 1)\frac{1}{\gamma_i}}{\alpha_i^m \sum_{j=1}^{n_*} c_j}. \end{aligned}$$

By definition $\lim_{n_* \to \infty} \frac{\sum_{j=1}^{n_*} c_j}{n_*} = \infty$, therefore

$$0 \leq \lim_{n_* \to \infty} \frac{\sum \text{incomplete periods}}{T_i(n_*)} \leq 0.$$

Then the lemma follows. ■

Lemma 5:

$$\lim_{k_i \to \infty} \frac{T_i(\tau_{k_i})}{k_i} = \frac{1}{\lambda_{0,i}} + \frac{1}{\lambda_{1,i}} \qquad a.s.$$

Proof: From lemma 4 we have

$$\lim_{k_i \to \infty} \frac{T_i(\tau_{k_i})}{k_i} = \lim_{k_i \to \infty} \frac{T_i^*(\tau_{k_i})}{k_i} \qquad a.s.$$

The equality

$$\lim_{k_i \to \infty} \frac{T_i^*(\tau_{k_i})}{k_i} = \frac{1}{\lambda_{0,i}} + \frac{1}{\lambda_{1,i}} \qquad a.s.$$

is based on the strong Law of large numbers since $T_i^*(\tau_{k_i})$ is exactly equal to sum of k_i complete idle, and busy realizations divided by k_i. ■

We have

$$\text{MSE}^i_{\tau_{k_i}} = \frac{\lambda^2_{0,i} + \lambda^2_{1,i}}{k_i - 2}$$

therefore from Lemma 5 we get

$$\lim_{k_i \to \infty} T_i(\tau_{k_i})\text{MSE}^i_{\tau_{k_i}} = (\lambda^2_{0,i} + \lambda^2_{1,i})(\frac{1}{\lambda_{0,i}} + \frac{1}{\lambda_{1,i}}) \quad a.s.$$

Note that

$$\forall n \in \mathbb{N},\ \exists \bar{\tilde{k}} = (\tilde{k}_i)^M_{i=1},\ \text{s.t. } T_i(\tau_{\tilde{k}_i}) \le T_i(n) \le T_i(\tau_{\tilde{k}_i+1}).$$

$$\lim_{\bar{\tilde{k}} \to \infty} T_i(\tau_{\tilde{k}_i})\text{MSE}^i_{\tau_{\tilde{k}_i}} \le \lim_{n \to \infty} T_i(n)\text{MSE}^i_{\tau_{\tilde{k}_i}} \le \lim_{\bar{\tilde{k}} \to \infty} T_i(\tau_{\tilde{k}_i+1})\text{MSE}^i_{\tau_{\tilde{k}_i}}$$

Also

$$\text{MSE}^i(T(n)) = \text{MSE}^i_n = \text{MSE}^i_{\tau_{\tilde{k}_i}} \quad a.s.$$

Therefore we have

$$\lim_{n \to \infty} T_i(n)\text{MSE}^i(T(n)) = (\lambda^2_{0,i} + \lambda^2_{1,i})(\frac{1}{\lambda_{0,i}} + \frac{1}{\lambda_{1,i}}) \quad a.s. \tag{55}$$

Lemma 6:

$$\lim_{n \to \infty} \frac{T_i(n)}{T(n)} = \alpha_i \qquad a.s.$$

Proof: From lemma 3 we have

$$\lim_{n \to \infty} \hat{\alpha}_i(n) = \alpha_i \quad a.s.$$

Therefore

$$\forall \epsilon > 0,\ \exists n_1 \text{ s.t. } \forall n \ge n_1, \quad |\hat{\alpha}_i(n) - \alpha_i| < \frac{\epsilon}{2} \quad a.s.$$

For this given n_1, $\exists n_2$ such that $\forall n \ge n_2$,

$$\frac{\sum_{k=1}^{n_1} c_k}{\sum_{k=1}^{n} c_k} < \frac{\epsilon}{2}.$$

Then $\forall n > n_2$,

$$\frac{T_i(n)}{T(n)} = \frac{\sum_{k=1}^{n} c_k \hat{\alpha}_i(k)}{\sum_{k=1}^{n} c_k} = \frac{\sum_{k=1}^{n_1} c_k(\hat{\alpha}_i(k) - \alpha_i)}{\sum_{k=1}^{n} c_k} + \frac{\sum_{k=1}^{n_1} c_k\alpha_i + \sum_{k=n_1+1}^{n} c_k \hat{\alpha}_i(k)}{\sum_{k=1}^{n} c_k}. \tag{56}$$

Since $0 < \hat{\alpha}_i(k) < 1$, and $0 < \alpha_i < 1$,

$$-\frac{\epsilon}{2} < -\frac{\sum_{k=1}^{n_1} c_k}{\sum_{k=1}^{n} c_k} < \frac{\sum_{k=1}^{n_1} c_k(\hat{\alpha}_i(k) - \alpha_i)}{\sum_{k=1}^{n} c_k} < \frac{\sum_{k=1}^{n_1} c_k}{\sum_{k=1}^{n} c_k} < \frac{\epsilon}{2}. \tag{57}$$

Also

$$\alpha_i - \frac{\epsilon}{2} < \alpha_i - \frac{\epsilon}{2}\frac{\sum_{k=n_1+1}^{n} c_k}{\sum_{k=1}^{n} c_k} < \frac{\sum_{k=1}^{n_1} c_k\alpha_i + \sum_{k=n_1+1}^{n} c_k\hat{\alpha}_i(k)}{\sum_{k=1}^{n} c_k} < \alpha_i + \frac{\epsilon}{2}\frac{\sum_{k=n_1+1}^{n} c_k}{\sum_{k=1}^{n} c_k} < \alpha_i + \frac{\epsilon}{2} \quad a.s.$$

Combining it with (56), and (57), we get $\forall \epsilon > 0, \exists n_2,$ s.t. $\forall n > n_2$,

$$\alpha_i - \epsilon < \frac{T_i(n)}{T(n)} < \alpha_i + \epsilon. \quad a.s.$$

Thus

$$\lim_{n\to\infty} \frac{T_i(n)}{T(n)} = \alpha_i \qquad a.s.$$

■

From lemma 6 and (55) we get

$$\lim_{n\to\infty} T(n)\text{MSE}^i(T(n)) = \frac{(\lambda_{0,i}^2 + \lambda_{1,i}^2)(\frac{1}{\lambda_{0,i}} + \frac{1}{\lambda_{1,i}})}{\alpha_i} \quad a.s.$$

Therefore

$$\lim_{n\to\infty} T(n)\sum_{i=1}^{M}\text{MSE}^i(T(n)) = \sum_{i=1}^{M}\frac{(\lambda_{0,i}^2 + \lambda_{1,i}^2)(\frac{1}{\lambda_{0,i}} + \frac{1}{\lambda_{1,i}})}{\alpha_i} \quad a.s.,$$

or equivalently for the efficiency

$$\lim_{T\to\infty} \frac{\text{CRB}^*(T)}{\text{MSE}(T)} = 1 \quad a.s.$$

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