Modeling of the CML FD noise-to-jitter conversion as an LPTV process

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0. Revision history

Version	*Date*	*Comments*
1.0	12/02/2004	First version – merged two documents: 1. Cyclostationary Noise and Application to CML Frequency Divider Jitter/Phase Noise Analysis (file *CycloNoiseAndFDjitter_v1.0.doc*) 2. Frequency Analysis of LPTV Systems and Application to the FD noise-to-jitter generation (file *LPTVandFDjitter_v1.1.doc*)

1. Theory of LPTV systems and cyclostationary noise processes

1.1 Formulation of the system function of an LPTV system

Assume an LPTV system with an impulse response function $h(t,\tau)$, where t represents the current time (time at which the system is observed), and τ represents the arrival time of the input impulse. Obviously, $t \geq \tau$ (it is assumed that the system is causal) and, since the system is periodic, $h(t,\tau) = h(t+T,\tau+T)$. A *system function* can be defined, which represents the time-varying transfer function of the system [Zadeh 1950]:

$$H(j\omega,t) = \int_{-\infty}^{+\infty} h(t,\tau)e^{-j\omega(t-\tau)}d\tau = \int_{-\infty}^{t} h(t,\tau)e^{-j\omega(t-\tau)}d\tau \qquad (1)$$

Since the system is periodically-time varying, $H(j\omega,t)$ can be represented by its Fourier series:

$$H(j\omega,t) = \sum_{n=-\infty}^{+\infty} H_n(j\omega)e^{jn\omega_0 t} \qquad (2)$$

where the Fourier coefficients $H_n(j\omega)$ are time-independent and are called *harmonic transfer functions*:

$$H_n(j\omega) = \frac{1}{T}\int_{-T/2}^{T/2} H(j\omega,t)e^{-jn\omega_0 t}dt \qquad (3)$$

In (3), $\omega_0 = 2\pi/T$. Substituting $H(j\omega,t)$ from (1) into (3), (3) becomes:

$$H_n(j\omega) = \frac{1}{T}\int_{-T/2}^{T/2}\left\{e^{-j\omega t}\int_{-\infty}^{t} h(t,\tau)e^{j\omega\tau}d\tau\right\}e^{-jn\omega_0 t}dt \tag{4}$$

and finally:

$$H_n(j\omega) = \frac{1}{T}\int_{-T/2}^{T/2}\left\{\int_{-\infty}^{t} h(t,\tau)e^{j\omega\tau}d\tau\right\}e^{-j(\omega+n\omega_0)t}dt \tag{5}$$

Lower limit in the inner integral in (5) is actually $-T/2$, since only one period of the impulse response function is observed.

Another way of finding the harmonic transfer function follows [Gardner 1990]. If $h(t,\tau)$ is the impulse response of an LPTV system, then $h(t+\xi,t)$ is periodic with T. Here, ξ represents the time difference between the occurrence of the impulse at the input of the system, and the time at which the impulse response of the system is observed. Since $h(t+\xi,t)$ is periodic, it can be represented by its Fourier series:

$$h(t+\xi,t) = \sum_{n=-\infty}^{+\infty} h_n(\xi)e^{jn\omega_0 t} \tag{6}$$

where $h_n(\xi)$ is:

$$h_n(\xi) = \frac{1}{T}\int_{-T/2}^{T/2} h(t+\xi)e^{-jn\omega_0 t}dt \tag{7}$$

The system can again be represented by the system function:

$$H(j\omega,t) = \int_{-\infty}^{\infty} h(t,t-\xi)e^{-j\omega\xi}d\xi = \int_{0}^{\infty} h(t,t-\xi)e^{-j\omega\xi}d\xi \tag{8}$$

The definition (8) of the system function is identical to the one given in (1), when τ in (1) is substituted by $t-\xi$. Function $h(t,t-\xi)$ is actually $h(t+\xi,t)$, delayed by ξ. Therefore, its Fourier series is:

$$h(t,t-\xi) = \sum_{n=-\infty}^{+\infty} h_n(\xi)e^{jn\omega_0 t}e^{-jn\omega_0\xi} \tag{9}$$

The system function now becomes:

$$H(j\omega,t)=\int_{0}^{\infty}\sum_{n=-\infty}^{\infty}\left\{\left[\frac{1}{T}\int_{-T/2}^{T/2}h(z+\xi,z)e^{-jn\omega_0 z}dz\right]e^{-jn\omega_0\xi}e^{jn\omega_0 t}\right\}e^{-j\omega\xi}d\xi \qquad (10)$$

After rearranging (10) becomes:

$$H(j\omega,t)=\sum_{n=-\infty}^{\infty}\left\{\int_{0}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2}h(z+\xi,z)e^{-jn\omega_0 z}dz\right]e^{-j(\omega+n\omega_0)\xi}d\xi\right\}e^{-jn\omega_0 t} \qquad (11)$$

In (11), $H(j\omega,t)$ is again broken into its Fourier series, and the coefficients of the series are the harmonic transfer functions $H_n(j\omega)$:

$$H_n(j\omega)=\int_{0}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2}h(z+\xi,z)e^{-jn\omega_0 z}dz\right]e^{-j(\omega+n\omega_0)\xi}d\xi \qquad (12)$$

The upper limit in the outer integral in (12) should be $T/2$, since only one period of the impulse response function is observed.

Compact form of the expression for the harmonic transfer function is:

$$H_n(j\omega)=\int_{0}^{\infty}h_n(\xi)e^{-j(\omega+n\omega_0)\xi}d\xi \qquad (13)$$

1.2 Characteristics of cyclostationary noise

Cyclostationary random processes are described in [Gardner 1990]. A random process is said to be cyclostationary in the wide sense if its mean value and autocorrelation function are periodic. For the analysis of propagation of (cyclostationary) noise through a linear (time variant or invariant) system, autocorrelation function and its Fourier transform (i.e. power spectral density) are of great interest.

The autocorrelation function of a cyclostationary process $X(t)$, $R_X(t_1,t_2)$, satisfies:

$$R_X(t_1,t_2)=R_X(t_1+T,t_2+T) \qquad (14)$$

where t_1 and t_2 are the time instances at which the process is observed and T is the period. (14) can also be written as

$$R\left(t+\frac{\tau}{2},t-\frac{\tau}{2}\right)=R\left(t+\frac{\tau}{2}+T,t-\frac{\tau}{2}+T\right) \qquad (15)$$

where τ is the difference between t_1 and t_2 (subscript 'X' is omitted in (15)).

Since autocorrelation function is periodic, it can be represented as a Fourier series:

$$R\left(t+\frac{\tau}{2},t-\frac{\tau}{2}\right)=\sum_{n=-\infty}^{\infty}R_n(\tau)e^{jn\omega_0 t} \tag{16}$$

where $\omega_0=2\pi/T$ and

$$R_n(\tau)=\frac{1}{T}\int_{-T/2}^{T/2}R\left(t+\frac{\tau}{2},t-\frac{\tau}{2}\right)e^{-jn\omega_0 t}dt \tag{17}$$

A time-variant power spectral density can be defined as the Fourier transform of the autocorrelation function:

$$S(\omega,t)=\int_{-\infty}^{\infty}R\left(t+\frac{\tau}{2},t-\frac{\tau}{2}\right)e^{-j\omega\tau}d\tau \tag{18}$$

In case of a cyclostationary noise process, $S(\omega,t)$ can be represented by the Fourier series:

$$S(\omega,t)=\sum_{n=-\infty}^{\infty}S_n(\omega)e^{jn\omega_0 t} \tag{19}$$

where

$$S_n(\omega)=\frac{1}{T}\int_{-T/2}^{T/2}S(\omega,t)e^{-jn\omega_0 t}dt \tag{20}$$

Coefficients $S_n(\omega)$ are called *cyclic spectra* and it can be shown that they are Fourier transforms of the coefficients $R_n(\tau)$ from (3):

$$S_n(\omega)=\int_{-\infty}^{\infty}R_n(\tau)e^{-j\omega\tau}d\tau \tag{21}$$

1.3 Filtering of cyclostationary noise through an LPTV system

When cyclostationary noise, with a known power spectral density $S_X(\omega,t)$ (i.e. cyclic spectra $S^n{}_X(\omega)$), is passed through an LPTV system whose harmonic transfer functions $H_n(\omega)$ are also known, the resulting output noise power spectral density can be found by applying the matrix equation from [Roychowdhury 1998] :

$$\underline{S}_Y(\omega) = \underline{H}(\omega)\underline{S}_X(\omega)\underline{H}^H(\omega) \tag{22}$$

where

$$\underline{S}_Y(\omega) = \begin{bmatrix} \cdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & S_Y^0(-\omega+\omega_0) & S_Y^1(-\omega+\omega_0) & S_Y^2(-\omega+\omega_0) & \cdots \\ \cdots & S_Y^{-1}(-\omega) & S_Y^0(-\omega) & S_Y^1(-\omega) & \cdots \\ \cdots & S_Y^{-2}(-\omega-\omega_0) & S_Y^{-1}(-\omega-\omega_0) & S_Y^0(-\omega-\omega_0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\underline{S}_X(\omega) = \begin{bmatrix} \cdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & S_X^0(-\omega+\omega_0) & S_Y^1(-\omega+\omega_0) & S_X^2(-\omega+\omega_0) & \cdots \\ \cdots & S_X^{-1}(-\omega) & S_X^0(-\omega) & S_X^1(-\omega) & \cdots \\ \cdots & S_X^{-2}(-\omega-\omega_0) & S_X^{-1}(-\omega-\omega_0) & S_X^0(-\omega-\omega_0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\underline{H}(\omega) = \begin{bmatrix} \cdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & H_0(\omega+\omega_0) & H_{-1}(\omega) & H_2(\omega-\omega_0) & \cdots \\ \cdots & H_{-1}(\omega+\omega_0) & H_0(\omega) & H_1(\omega-\omega_0) & \cdots \\ \cdots & H_{-2}(\omega+\omega_0) & H_1(\omega) & H_0(\omega-\omega_0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

and ^{H} signifies the Hermitian matrix (transposed matrix of complex conjugates).

As an example, assume the LPTV system and the input cyclostationary noise process have nonzero harmonic transfer functions and cyclic spectra, respectively, for $n = -1$, 0 and 1. Then the output noise cyclic spectrum for n=0 (the stationary component) is given by:

$$\begin{aligned} S_Y^0(-\omega) &= H_{-1}(\omega+\omega_0)S_X^0(-\omega+\omega_0)H_{-1}^*(\omega+\omega_0) + H_0(\omega)S_X^{-1}(-\omega)H_{-1}^*(\omega+\omega_0) + \\ &+ H_{-1}(\omega+\omega_0)S_X^1(-\omega+\omega_0)H_0^*(\omega) + H_0(\omega)S_X^0(-\omega)H_0^*(\omega) + \\ &+ H_1(\omega-\omega_0)S_X^{-1}(-\omega-\omega_0)H_0^*(\omega) + H_0(\omega)S_X^1(-\omega)H_1^*(\omega-\omega_0) + \\ &+ H_{-1}(\omega-\omega_0)S_X^0(-\omega-\omega_0)H_1^*(\omega-\omega_0) \end{aligned} \tag{23}$$

where * signifies the complex conjugate.

As can be seen in the equation above, the LPTV system introduces down and up conversion of the input noise process. To see this even more clearly, assume that the input noise process is just a sine wave at frequency ω_S, i.e. $S^0{}_X(-\omega)=\delta(-\omega_S)$ and $S^n{}_X(-\omega)=0$ for $n\neq0$. The input process is stationary in this case. Then, the stationary component of the output noise is:

$$S_Y^0(-\omega) = H_{-1}(\omega_S + \omega_0)\delta(-\omega_S + \omega_0)H_{-1}^*(\omega_S + \omega_0) + H_0(\omega_S)S_X^0(-\omega_S)H_0^*(\omega_S) + \\ + H_1(\omega_S - \omega_0)S_X^0(-\omega_S - \omega_0)H_1^*(\omega_S - \omega_0) \quad (24)$$

As can be seen from (24), the output noise spectrum now contains components at ω_S and $\omega_S \pm \omega_0$. Therefore, when a noise process (stationary or cyclostationary) is passed through an LPTV system, the output noise is cyclostationary.

2. Application of the theory on jitter generation of the CML frequency divider

2.1 FD jitter generation as an LPTV system

The noise-to-jitter impulse response function of a frequency divider (or any other switching circuit) can be determined either through simulation or analytically. Here, we will assume that noise-to-jitter impulse response function is already known and can be represented as follows:

$$h(t,\tau) = x(\tau)\left[u(t-\tau) - u(t-T)\right] \quad (25)$$

where τ is the arrival time of the current impulse, t is the moment at which the system is observed and $u(t)$ is the unity step function. The time instant τ=0 corresponds to the nominal arrival of the clock signal, which retimes the system. $x(\tau)$ represents the crossing time variation caused by the noise impulse at τ (corresponds to the impulse sensitivity function, ISF, from [Hajimiri 1998]. $x(\tau)$ can be obtained through simulation or determined analytically. The plot of $x(\tau)$ for a FD circuit is given in Fig. 1.

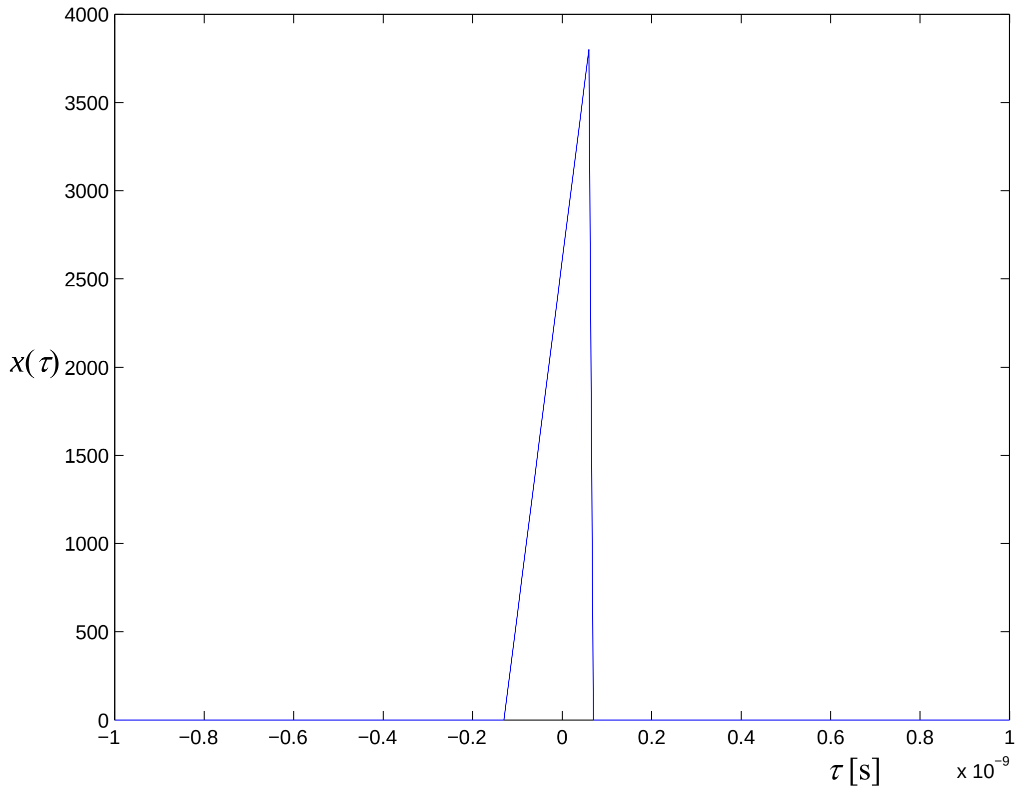

Fig. 1. Typical $x(\tau)$ plot.

The impulse response function defined in (25) is presented in Fig. 2 for two values of τ, together with $x(\tau)$. In this example, T=1ns.

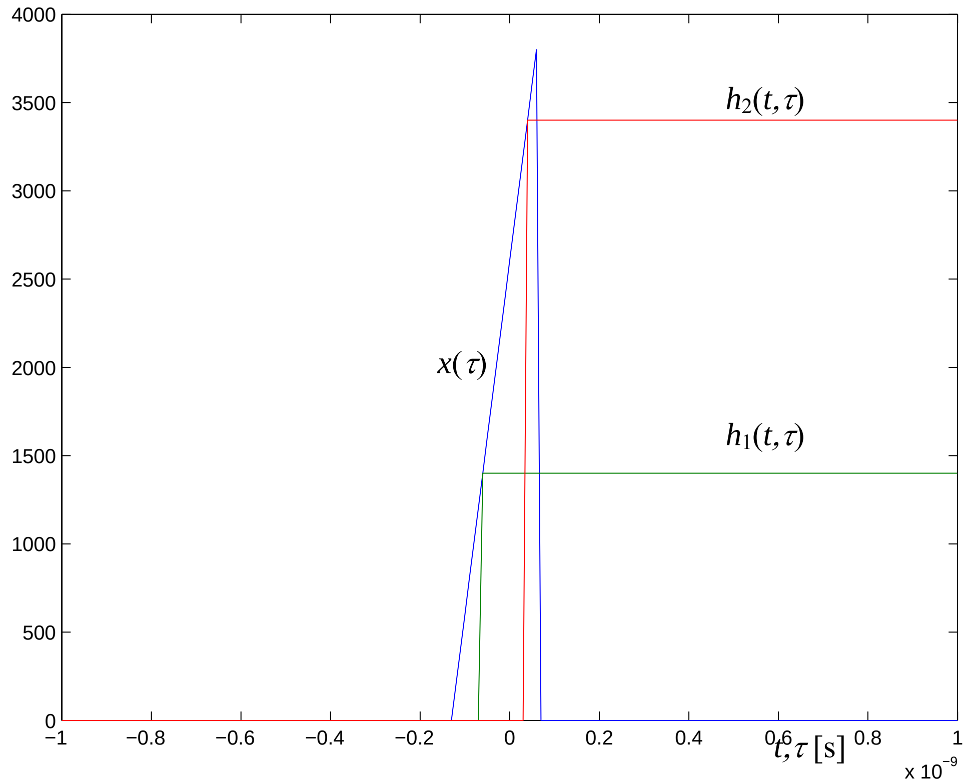

Fig. 2. Impulse response function for two different values of τ

Substituting the impulse response function defined $h(t,\tau)$ in (25) into (5) or (12), we get the harmonic transfer functions $H_n(j\omega)$. Fig. 3 shows single-sided $|H_n(j\omega)|$ for n = 0,..,4 as functions of frequency f=2$\pi\omega$ rather than ω.

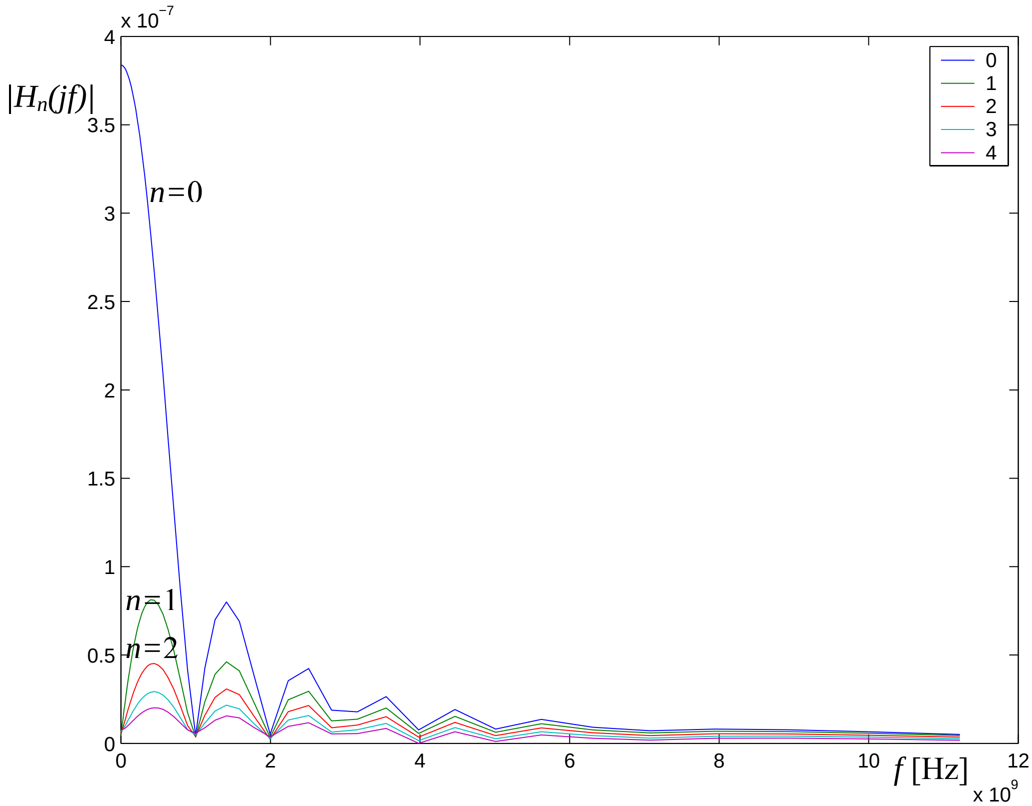

Fig. 3. Harmonic transfer functions of the noise-to-jitter conversion system

2.2 A method for obtaining the S(ω,t) of a CML frequency divider

Assume that all the noise at the frequency divider output originates from the level shifter (according to BDASIM simulations, devices in the level shifter contribute to the FD output noise by more than ~60%). Also, we will assume that the noise is white, and autocorrelation function is given by

$$R(t) = \sigma^2(t)\delta(t) \tag{26}$$

where $\sigma^2(t)$ is the time-varying power of the noise process. Then, $S(\omega,t)$ of a switching circuit can be determined using the following method: At each time instant t, power spectral density $S(\omega,t)$ of the switching circuit can be found by applying the traditional noise analysis on a linear time-invariant circuit, where DC operating point of the new circuit are currents and voltages that correspond to those of the switching circuit at time t. In other words, currents and voltages of the switching circuit are "frozen" at time t and noise analysis is then performed on that time invariant circuit. As an example, Fig. 4 shows $S(0,t)$ of a CML level shifter, over one period of the output signal (T=1ns).

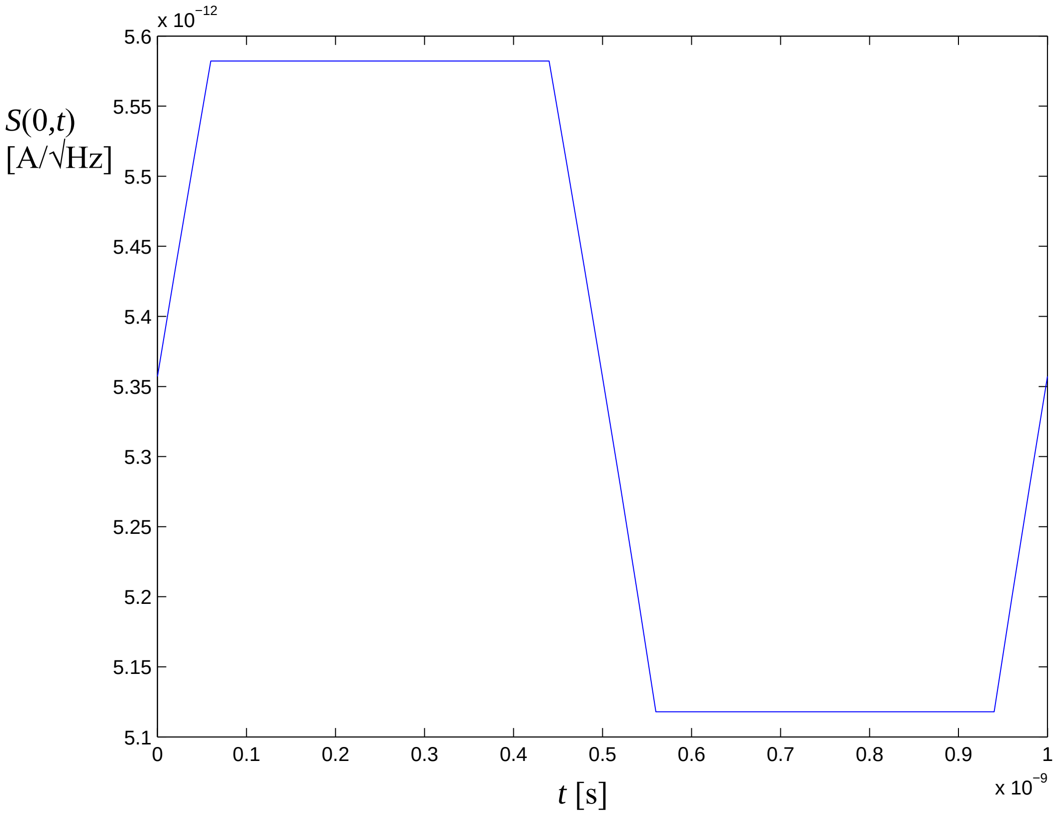

Fig. 4. $S(0,t)$ of a CML level shifter

Fig. 5 shows how noise power spectral density varies for different output voltage levels. To relate plots in Fig. 4 and 5, output voltage takes the value of 0.9V at times 560ps through 940ps and value 1.2V from 60ps to 440ps in Fig. 4.

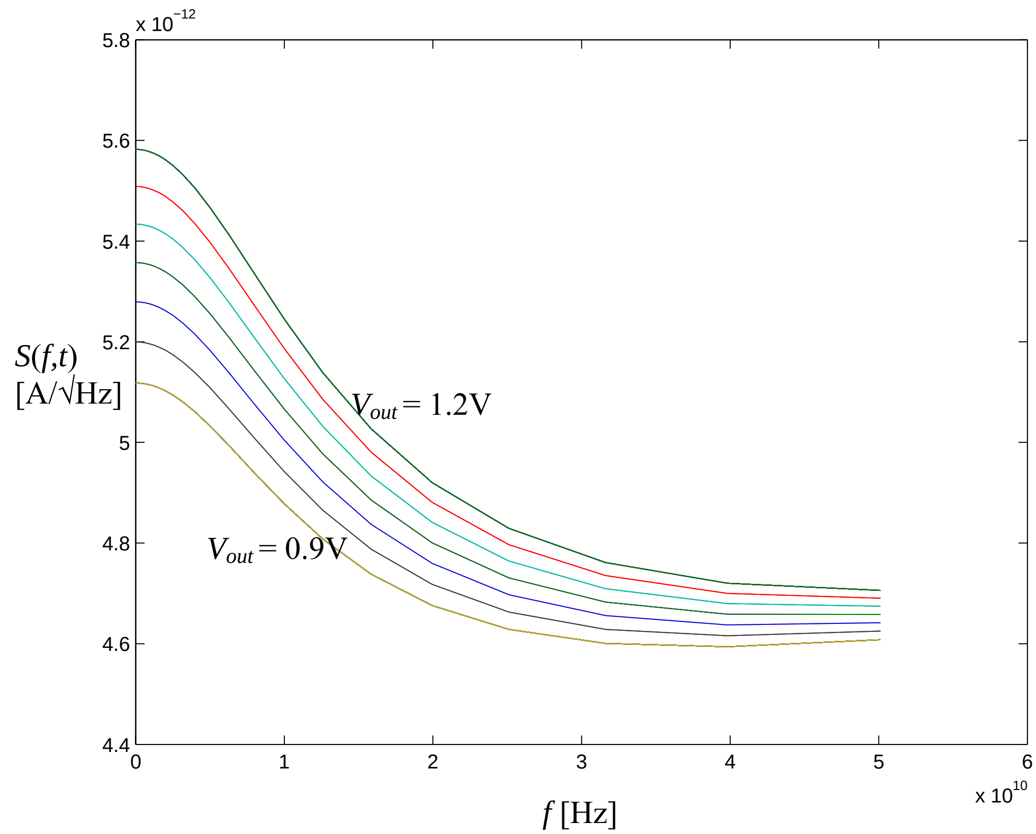

Fig. 5. Power spectral density for different output voltage levels

Now, cyclic spectra can be easily found using (7) and they are shown in Fig. 6. It can be seen from Fig. 6 that the most significant cyclic spectrum is the one for n=0. This can be explained by the fact that, since the output voltage swing of the CML level shifter is small (only 300mV in this example), power spectral density does not vary much with time, and its most significant harmonic is the one at n=0 (DC).

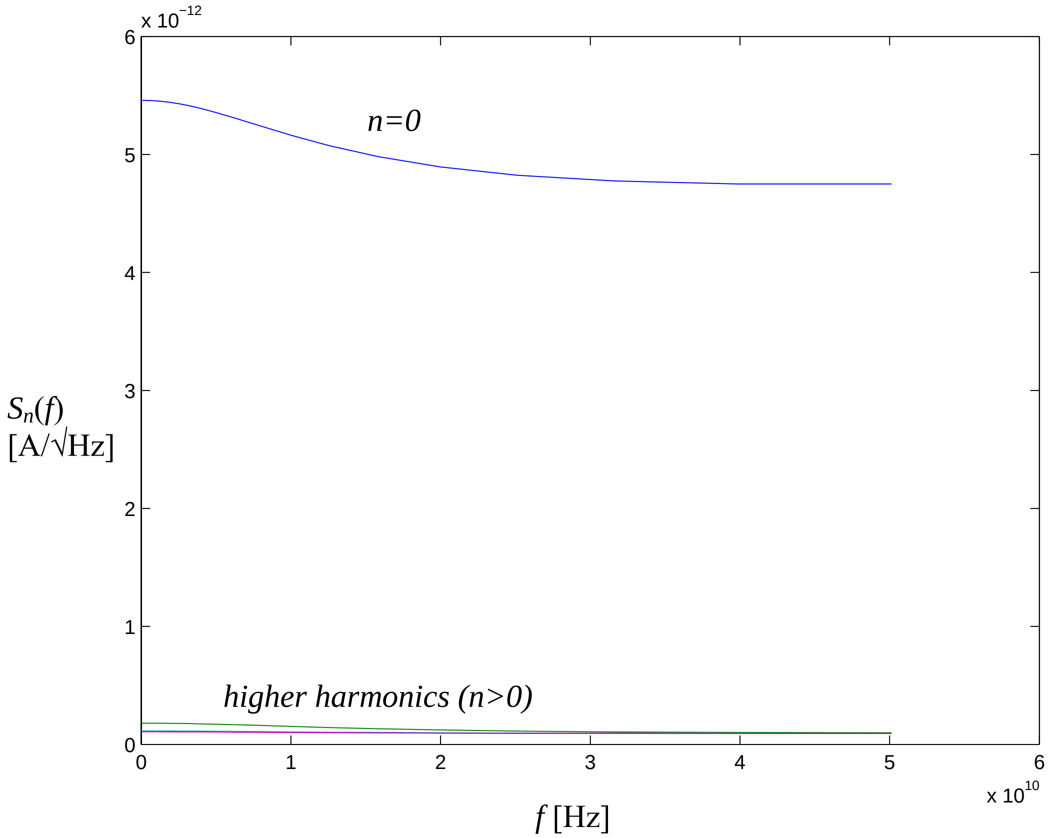

Fig. 6. Cyclic spectra of the CML level shifter time varying power spectral density

Finally, the above results are compared with results of the BDASIM periodic noise analysis. Fig. 7 shows the cyclic spectra determined according to the analysis described above, and output noise spectrum determined by BDASIM (green line). It can be seen from Fig. 7 that the analysis presented here matches the periodic noise simulation results.

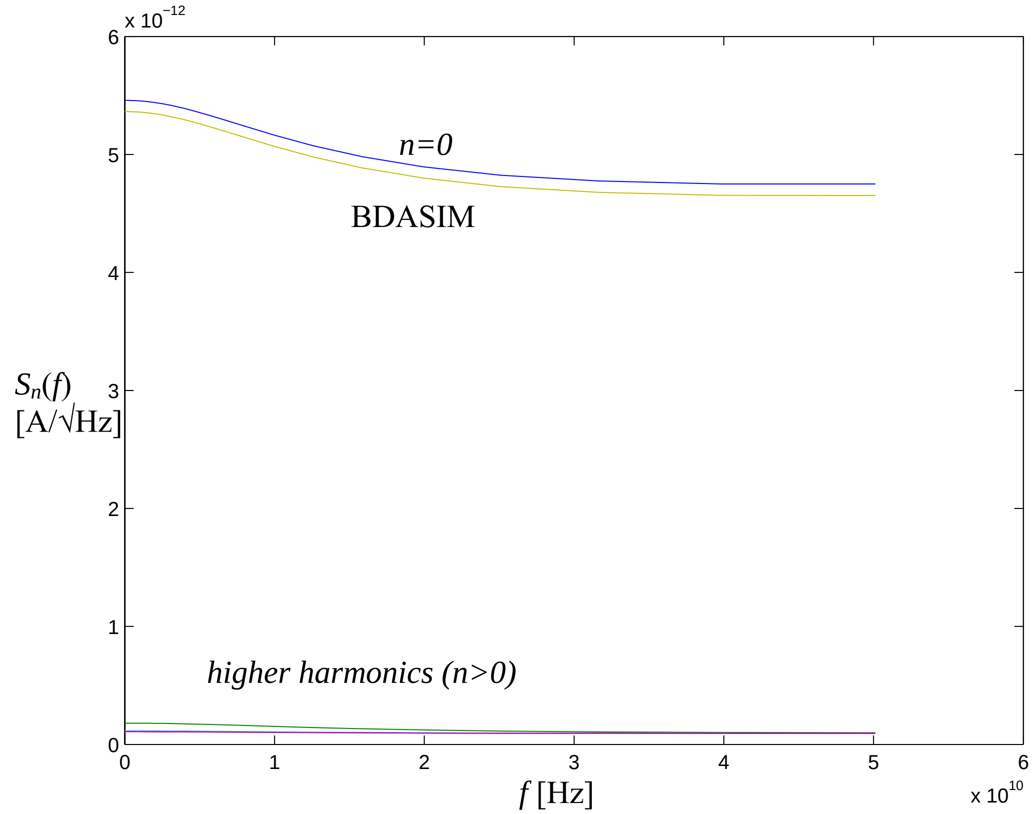

Fig. 7. Comparison with BDASIM

The analysis above shows that, if all the noise sources in the circuit are white and output swing of the circuit is significantly small, equivalent cyclostationary output noise is represented by only one cyclic spectrum (for n=0), and can be substituted by a stationary output noise source whose power spectral density is equal to the power spectral density in the operating point (when no AC sources are applied to the switching circuit).

2.3 FD jitter results

Jitter of a CML frequency divider can now be estimated by applying (22) with harmonic transfer functions and cyclic spectra calculated above. As seen, the only significant harmonic transfer functions and cyclic spectra are those for n=0. For that reason, (22) reduces to

$$S_{jitter}(\omega) = \left|H_0(j\omega)\right|^2 S_0(\omega) \tag{27}$$

Jitter power can then be found as

$$\sigma^2_{jitter} = \int_0^\infty S_{jitter}(\omega)d\omega = \int_0^\infty \left|H_0(j\omega)\right|^2 S_0(\omega)d\omega \tag{28}$$

After evaluating (28) for the example above, estimated RMS jitter is found to be

$$\sigma_{jitter} = 45.9\text{fs}.$$

When this result is compared to the results obtained using the FD jitter model described in [Levantino 2004] which yields σ_{jitter} = 152.8fs, it can be seen that the method presented here underestimates jitter (assuming that the model presented in [Levantino 2004] is valid).

One noticeable drawback of the model presented here is that jitter value varies with the output frequency of the frequency divider. Whether jitter does depend on output frequency should be confirmed experimentally, but that is not expected and is in contradiction with other published models. However, this method may be useful for finding the phase noise of the frequency divider.

References

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