Strategic Communications in Opinion Diffusion

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***Abstract*—We propose a strategic communication model, in which social agents play a central role in determining how much effort and time they want to invest in interacting with others. The starting point for this paper is the so called bounded confidence model in which agents update their opinions only when they are like-minded (i.e., their opinion distance is smaller than a threshold). In our model, in addition to the existence of trust between interacting agents, the agents also individually determine the expected time and effort in interacting with their neighboring agents. The strategic communication thus refers to the process that allows individuals to select neighbors, with whom interaction produces a maximization of the local utility functions. Our goal is to analyze the dynamics of opinion formation under the proposed communication strategy, with a focus on understanding how and under what conditions clustering patterns emerge in opinion space.**

I. Introduction

How societies steer individual beliefs and values is, to a large extent, closely related to the process of information diffusion in social networks. To understand this process, we begin with a brief overview of various mathematical models that attempt to capture the impact of social interactions on opinion formation. In particular, we restrict our attention to non-Bayesian models that emerged from the field of statistical physics [1]. The basic idea is to capture the dynamics of agents' opinions by postulating a simple, but still natural, opinion updating rule: agents exchange and update their opinions by taking a weighted average, if certain conditions are met. (See [2]–[6].) Interactions between agents are often random and local while the learning rule is designed to approximate the resulting change in agents' beliefs. Models under this framework have the benefit of giving insights on more complex network structures, while providing explicit answers to the dynamics of opinion formation in a social group.

One of the early models in this class was studied by DeGroot [2]. In the DeGroot model, individuals start with an initial opinion profile represented by a vector of probabilities. The update process is captured by a fixed stochastic matrix T. Beliefs of individuals are updated linearly by taking a weighted average of their neighbors' beliefs (T_{ij} are the weights representing the relative trust that agent i places on agent j's belief). Some generalizations of the Degroot model were investigated in [3]–[7] in which a bounded confidence was introduced to capture the trust that may exist between like-minded agents; the resulting belief update is nonlinear.

We next consider an example of nonlinear belief update. In the *Deffuant-Weisbuch* (DW) model of pairwise interaction [5], [6], let $\mathcal{V} = \{1, 2, \cdots, n\}$ be a set of social agents in a *fixed*, *undirected*, and *connected* communications graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is the set of edges. Denote by $\mathcal{N}_i$ the set of agents (also called neighbors) connected to agent i in $\mathcal{G}$, i.e.,

$$\mathcal{N}_i = \{j \in \mathcal{V} \setminus i \mid (i,j) \in \mathcal{E}\}.$$

Each individual's opinion is represented by a real number x_j that lies in a bounded interval. Agents i and j are randomly selected for interaction. Let $I_i[k;\tau_0] = \{j \in \mathcal{N}_i : |x_i[k] - x_j[k]| < \tau_0\}$. If $j \in I_i[k;\tau_0]$ and thus $i \in I_j[k;\tau_0]$, then after the interaction, opinions are updated pairwise as follows

$$\begin{aligned} x_i[k+1] &= x_i[k] + \bar{\mu}(x_j[k] - x_i[k]) \\ x_j[k+1] &= x_j[k] + \bar{\mu}(x_i[k] - x_j[k]), \end{aligned} \tag{1}$$

where $\bar{\mu} \in (0, 0.5]$ is called the mixing parameter. Deffuant et al. in [5] explored this system over a square grid in which individuals are only connected with their four immediate neighbors. Weisbuch in [6] extended this simple lattice topology to a scale free network topology.

An extension of the DW model to multi-alternative decision making (decision between multiple alternatives) is proposed in [8]–[10]. Rather than restricting agents' opinions to lie in a bounded (real) interval, each agent's opinion is treated as a vector of probabilities $\mathbf{x}_j[0] = [x_{j1}[0], \cdots, x_{jq}[0]]$, in a probability simplex of q dimension

$$\mathcal{X} = \left\{\mathbf{x} = [x_1, \cdots, x_q]^T \mid \sum_{\ell=1}^{q} x_\ell = 1 \text{ and } x_\ell \in [0,1]\right\};$$

each element of the opinion vector represents the probability that a certain alternative is true. Another generalization of the DW model is the introduction of a state-dependent trust function $\mu(d)$. Although it is similar in spirit to the parameter μ_0 defined in [3]–[7], the trust function $\mu(d)$ studied in [9], [10] is more general and it varies with the squared opinion distance d between the interacting agents. Clearly, the effect of $\mu(d)$ is time varying since agents' opinions evolve over time and its value depends on how distance is defined.

A. Problem Statement

Consider the following squared distance function

$$d(\mathbf{x}_i, \mathbf{x}_j) = \|\mathbf{x}_i - \mathbf{x}_j\|_A^2 = (\mathbf{x}_i - \mathbf{x}_j)^T A (\mathbf{x}_i - \mathbf{x}_j)$$

where $A \in \mathbb{R}^{q \times q}$ is a positive definite matrix. The opinion space $\mathcal{X}$ is bounded with respect to the *norm* $\|\mathbf{x}_j\|_A = \sqrt{d(\mathbf{x}_j, 0)}$, i.e., $\max_j \|\mathbf{x}_j\|_A < \infty$ for $\forall \mathbf{x}_j \in \mathcal{X}$ and $\max_j$

denotes the maximum among all $j \in \mathcal{V}$. Hence, the triangular inequality implies $\sqrt{d(\mathbf{x}_i, \mathbf{x}_j)} \leq 2\max_i \|\mathbf{x}_i\|_A := \sqrt{d_{\max}}$. For ease of notation, let $d_{ij}[k] = d(\mathbf{x}_i[k], \mathbf{x}_j[k])$ denote the squared opinion distance after k network-wide interactions have occurred. Hence, given that agents i and j are interacting at the $(k+1)^{\text{th}}$ time step, the opinions are updated as follows:

$$\mathbf{x}_i[k+1] = \mathbf{x}_i[k] + \mu\left(d_{ij}[k]\right)\left(\mathbf{x}_j[k] - \mathbf{x}_i[k]\right) \quad (2)$$

$$\mathbf{x}_j[k+1] = \mathbf{x}_j[k] + \mu(d_{ij}[k])(\mathbf{x}_i[k] - \mathbf{x}_j[k]) \quad (3)$$

where $\mu(d)$ is called the trust function and it satisfies the following assumption:

Assumption 1:

(a) The trust function $\mu(d) \in [0, 0.5]$ is a *non-increasing* function of the squared opinion distance d.
(b) There exists a *threshold* τ : $\forall d \geq \tau \rightarrow \mu(d) = 0$ and $\frac{\mu(0)}{\mu(\tau^-)} \leq \beta < \infty$.
(c) The trust function $\mu(d)$ is *concave* and C^2-differentiable for $\forall d \in (0, \tau)$.

Note that $\mu(d)$ models the trust that may exist between like-minded agents (Assumption 1-a) to reflect the amount of risk one is willing to take in social interactions. Assumption 1-b is a variation of the bounded confidence model [3]–[7], i.e., agents have no influence over each other when they are not like-minded[1]. Although $\mu(d)$ is not restricted to the step-function in (1), the condition $\frac{\mu(0)}{\mu(\tau^-)} < \infty$ implies that there has to be a discontinuity of $\mu(d)$ at $d = \tau$. The regularity condition imposed in Assumption 1-c is needed for analytical reasons (See [9], [10]).

The rate of communication between agents is defined as follows. Suppose that p_i represents the probability of agent $i \in \mathcal{V}$ initiating an interaction and it is assumed to be time-invariant and uniformly distributed, i.e., $p_i = \frac{1}{n}$. Let $P_{ij}[k]$ be the probability that agent i chooses to interact with agent j at the k^{th} time step. Our goal is to design a strategy that maximizes the local utility function of agent i. In particular, the strategy considered in this paper is in the form of probabilities $P_{ij}[k]$ for $\forall j \in \mathcal{N}_i$. Based on this strategy, agent i interacts with one of the neighbors in $\mathcal{N}_i$, followed by the opinion update rule given in (2) and (3). At the next time step, the same procedure is applied. It is to be noted that the rate of communication $P_{ij}[k]$ initiated by agent i is time-varying since the optimal strategy changes with time. For notational simplicity, we omit the time variable k whenever it does not cause confusion.

B. Organization

In Section II, we begin with a general definition of the utility function for each agent. The basic idea is that agents obtain utility from its expected net benefit (i.e., reward minus cost) of interacting with their neighbors. As there can be many ways to describe the cost and benefit for interaction, to get an understanding of how the utility function may affect the opinion formation process, we introduce two specific cost functions and analyze their convergence properties. In Section III, we present simulation results to validate our theoretical findings.

[1]The system will not change if $\mathbf{x}_i[k] = \mathbf{x}_j[k]$, i.e., agents are already in agreement.

II. Strategic Opinion Formation

There are two steps in modeling strategic interactions in a network of agents. First, one needs to explicitly model the incentives that each agent receives to interact more or less often with its neighboring agents. Second, the strategic model should be tractable so that it can provide insights or predictions on the formation of the asymptotic opinion profile.

A. Cost and Benefit Functions

We define the following local utility function for agent i:

$$U_i\left([P_{ij}]_{j\in\mathcal{N}_i}\right) = \sum_{j\in\mathcal{N}_i} P_{ij}\left[r(d_{ij}) - c(d_{ij}, P_{ij})\right] \quad (4)$$

where $r(d_{ij})$ is the benefit or reward agent i receives from interacting with agent j, $c(d_{ij}, P_{ij})$ represents the cost of interacting with agent j and $[P_{ij}]_{j\in\mathcal{N}_i}$ defines the *strategy* for agent i.

Let $D^{ij}[k+1] := \sum_{r=1}^{n}\sum_{m=r+1}^{n} d_{rm}[k+1]$ be the sum of the squared distances for all possible pairs of agents in the network, given that agents i and j interacted at $k+1$. Let $D[k] = \sum_{r=1}^{n}\sum_{m=r+1}^{n} d_{rm}[k]$ be the sum of squared distances at time k which is prior to the interaction between agents i and j. It can be easily checked from (2) and (3) that the change in D after the interaction equals

$$D^{ij}[k+1] - D[k] = -2n\rho(d_{ij}[k])d_{ij}[k]. \quad (5)$$

where $\rho(d_{ij}) := \mu(d_{ij})\left[1 - \mu(d_{ij})\right]$. We call this decrease in the overall sum of squared distances the *social marginal benefit* caused by the interaction between agents i and j at $k+1$ and it is denoted by $MB_s[k+1]$. The change in d_{ij} of the interacting pair the agent's *private marginal benefit*, i.e.,

$$MB_p[k+1] := d_{ij}[k+1] - d_{ij}[k] = -4\rho(d_{ij}[k])d_{ij}[k] \ .$$

Notice that the social marginal benefit $MB_s[k+1]$ depends entirely on the squared opinion distance $d_{ij}[k]$ between the two interacting agents and it is $\frac{n}{2}$ times as large as the private marginal benefit $MB_p[k+1]$. Motivated by this, we set the reward function to be proportional to the agent's private marginal benefit, i.e.,

$$r(d_{ij}) = 4\alpha\mu(d_{ij})(1 - \mu(d_{ij}))d_{ij} = 4\alpha\rho(d_{ij})d_{ij},$$

where $\alpha > 0$ is called the *reward coefficient*.

Let us now consider the cost of interaction $c(d_{ij}, P_{ij})$, which should capture costs in terms of both time and energy. Formally, we define this cost as follows.

$$c(d_{ij}, P_{ij}) = P_{ij} + \xi(d_{ij})$$

The first term describes the fraction of time agent i is expected to spend in interacting with agent j, while the second term specifies the energy for interacting with agent j whose opinion is $\sqrt{d_{ij}}$ away from agent i. Here we present two interesting constructions of the energy function $\xi(d_{ij})$:

(i) $\xi(d_{ij}) = \gamma_1 d_{ij}$ if $d_{ij} < \tau$ and $\xi(d_{ij}) = +\infty$ if $d_{ij} \geq \tau$

(ii) $\xi(d_{ij}) = \gamma_2 \mu^2(d_{ij}) d_{ij}$ if $d_{ij} < \tau$ and $\xi(d_{ij}) = +\infty$ if $d_{ij} \geq \tau$

where γ_1 and γ_2 are called the *cost coefficients* associated with the two energy functions, respectively. The rationale for the first energy function is that it takes more effort to convince someone farther away in opinion. Rather than choosing a linear cost function, we could have chosen a function of the form $c(d_{ij}, P_{ij}) = P_{ij} + \gamma_1 f(d_{ij}) d_{ij}$ where $f(d)$ is any positive non-decreasing function of d. As will become evident from the analysis in the next section, the choice $f(d) = 1$ does not lead to any loss of generality. In contrast, the second energy function implies that the amount of energy spent in interacting with agent j is proportional to agent j's squared change in opinion after the interaction, i.e., $d(x_j[k], x_j[k+1]) = \mu^2(d_{ij}[k]) d_{ij}[k]$, given that d_{ij} is less than the threshold. In both cases, agents spend infinite nervy in trying to convince neighbors who are not like-minded. While detailed analyses will be presented shortly, each of the two cases will lead to different opinion formation processes.

B. Local Strategy

Suppose that agent i is chosen at time k. Recall that the first step in the strategic interaction model is to determine P_{ij} for $\forall j \in \mathcal{N}_i$ maximizing the utility function in (4), i.e.,

$$\max_{P_{ij}} \sum_{j \in \mathcal{N}_i} P_{ij} \left[4\alpha\rho(d_{ij}) d_{ij} - \xi(d_{ij}[k])\right] - P_{ij}^2.$$

under the constraint that $\sum_{j \in \mathcal{N}_i \cup \{i\}} P_{ij} = 1$ and $P_{ij} \geq 0$. Solving the above optimization with respect to P_{ij} yields

$$P_{ij}[k] = \frac{1}{S_i} [4\alpha\rho(d_{ij}) d_{ij} - \xi(d_{ij}[k])]^+ \tag{6}$$

for $\forall j \in \mathcal{N}_i$ where $S_i := \sum_{m \in \mathcal{N}_i} [4\alpha\rho(d_{im}) d_{im} - \xi(d_{im})]^+$ is the scaling factor, and $[a]^+ = a$ if $a > 0$ and 0 otherwise. Hence, it follows from the constructions of the energy functions that $P_{ij} = 0$ for $\forall d_{ij} \geq \tau$. Moreover,

$$P_{ii}[k] = \begin{cases} 1 & \text{if } P_{ij}[k] = 0 \text{ for } \forall j \in \mathcal{N}_i \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

That is, if interacting with any of its neighbors will bring zero or negative net benefit (i.e., $r(d_{ij}) - c(d_{ij}, P_{ij}) \leq 0$) to agent i, then agent i will not interact with anyone for the moment.

C. ODE Approximation of the Distance Dynamic

Let $\overline{d}(k) = \mathbb{E}\{d(k)\}$ denote the expected squared opinion distance with respect to the average distribution $f_k(d)$ of d. Applying Euler's approximation to (5), the following *ordinary differential equation* (ODE) can be derived

$$\dot{\overline{d}}(t) = -\sum_{(i,j) \in \mathcal{E}} \frac{1}{n} (P_{ij} + P_{ji}) \rho(d_{ij}(t)) d_{ij}(t). \tag{8}$$

See the Appendix.

1) **Case Study (i):** Consider the case when the energy function is defined as $\xi(d) = \gamma_1 d$ for $d < \tau$ and $+\infty$ otherwise.

Assumption 2: The reward coefficient and the cost coefficient satisfy the relation $\frac{\gamma_1}{\alpha} < 4\rho(0)$.

Note that under Assumption 1-a, the term $\rho(d) = \mu(d)(1 - \mu(d))$ is a non-increasing function of d, with maximum value $\rho(0)$. When $d_{ij} < \tau$, replacing $\xi(d_{ij})$ with $\gamma_1 d_{ij}$ in (6), we observe that, if the ratio $\frac{\gamma_1}{\alpha}$ is greater than or equal to the product $4\rho(0)$, then $\frac{\gamma_1}{\alpha} \geq 4\rho(d_{ij})$ for $\forall d_{ij}$. Hence, it follows from equation (6) that $P_{ij} = 0$ for $\forall j \in \mathcal{N}_i$. This situation implies that the agents have insufficient incentives to interact with each other and will always choose to remain inactive, i.e., $P_{ii}[k] = 1$ for $\forall k$. Thus, Assumption 2 provides a *necessary* condition for agents to interact.

A direct result of Assumption 2 is that the rate of interaction P_{ij} changes over time, depending on the relative distances between agent i and its neighbors. If the cost of interacting with an agent exceeds the benefit, then the agent who is to initiate an interaction will impose a zero rate of interaction with the agent that causes a negative utility. On the contrary, more probability weight will be put on the neighbors yielding higher (positive) utilities. Hence, the probability distribution of pairwise interactions $\overline{P}_{ij} = \frac{1}{n}(P_{ij} + P_{ji})$ is also dependent on the opinion distances between the agents. It follows from (8) and (6) that the dynamic of the expected squared distance $\dot{\overline{d}}$ equals

$$\dot{\overline{d}} = -\frac{1}{N} \sum_{(i,j) \in \mathcal{E}} \left(\frac{1}{S_i} + \frac{1}{S_j}\right) [\eta(d_{ij})]^+ \rho(d_{ij}) d_{ij}^2, \tag{9}$$

where the scaling factor equals $S_i = 1$ if $P_{ii} = 1$ and

$$\eta(d) = \begin{cases} 4\alpha\rho(d) - \gamma_1 & \text{if } d < \tau \\ -\infty & \text{otherwise.} \end{cases}$$

From (9), one can clearly see that the system stops evolving (i.e., $\dot{\overline{d}} = 0$) if d_{ij} for $\forall (i,j) \in \mathcal{E}$ satisfies one of the two conditions: (i) $d_{ij} = 0$; (ii) $\eta(d_{ij}) \leq 0$. The first condition is satisfied if the interacting agents are in consensus. The second condition implies that agents will not interact if their squared opinion distance d_{ij} belongs to the union $\mathcal{D}_1 \cup [\tau, d_{\max})$ where

$$\mathcal{D}_1 = \left\{ d \in (0, \tau) \mid 4\rho(d) \leq \frac{\gamma_1}{\alpha} \right\}. \tag{10}$$

Note that $\mathcal{D}_1$ is an empty set when the ratio $\frac{\gamma_1}{\alpha} < 4\rho(\tau^-)$. In this case, agents will not update their opinions if their squared opinion distance $d_{ij} > \tau$ for $\forall (i,j) \in \mathcal{E}$.

On the other hand, as shown in Fig. 1, when the ratio $\frac{\gamma_1}{\alpha} \geq 4\rho(\tau^-)$, the set is nonempty and the threshold $\tau > \inf(\mathcal{D}_1)$, where inf denotes the infimum. Since $\rho(d)$ is an non-increasing function of d, the range of the set $\mathcal{D}_1$ goes from $\inf(\mathcal{D}_1)$ to τ. Clearly, for $\forall d_{ij} \geq \inf(\mathcal{D}_1)$, the associated agents will not update their opinions. The opinion diffusion in this case will not converge to a consensus not only because the agents may not be sufficiently like-minded (i.e., $d_{ij} \geq \tau$), but also because $\frac{\gamma_1}{\alpha}$ is too big to warrant sufficient

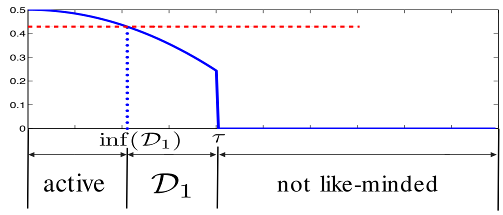

Fig. 1: The solid line represents the trust function $\mu(d)$ and the vertical line maps the value of μ such that $\frac{\gamma_1}{\alpha} = 4\mu(d)(1-\mu(d))$ to the point $d = \inf(\mathcal{D}_1)$. The "active" region means that agents will interact if their squared opinion distance d_{ij} falls in this region.

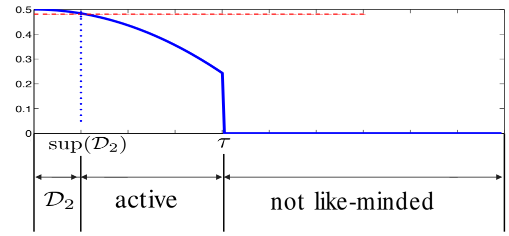

Fig. 2: The solid line represents the trust function $\mu(d)$ and the vertical line maps the value of μ such that $\frac{\gamma_2}{\alpha} = 4(1-\mu(d))/\mu(d)$ to the point $d = \sup(\mathcal{D}_2)$.

incentives to interact between certain pairs of like-minded agents whose squared opinion distances belong to the interval $d_{ij} \in [\inf(\mathcal{D}_1), \tau)$. Hence, $\inf(\mathcal{D}_1)$ represents the *effective threshold* under this scenario.

2) **Case Study (ii):** We now examine the case when the energy function is defined as $\xi(d) = \gamma_2 \mu^2(d) d$ for $d < \tau$ and $+\infty$ otherwise. Again, the *necessary* condition for interaction is given in the following assumption.

Assumption 3: The reward coefficient and the cost coefficient satisfy the relation $\frac{\gamma_2}{\alpha} \geq \frac{4(1-\mu(0))}{\mu(0)}$.

It follows from Assumption 1-a that the quotient $\frac{1-\mu(d)}{\mu(d)}$ is a non-decreasing function of d, with a minimum value $\frac{1-\mu(0)}{\mu(0)}$. Hence, if the ratio $\frac{\gamma_2}{\alpha} < \frac{4(1-\mu(0))}{\mu(0)}$, then $\frac{\gamma_2}{\alpha} < \frac{4(1-\mu(d_{ij}))}{\mu(d_{ij})}$ for $\forall d_{ij}$. Replacing $\xi(d_{ij})$ with $\gamma_2 \mu^2(d_{ij}) d_{ij}$ in equation (6), we have $P_{ij} = 0$ for $\forall j \in \{\zeta \mid \zeta \in \mathcal{N}_i \text{ and } d_{i\zeta} < \tau\}$. This result, together with the fact that $P_{ij} = 0$ whenever $d_{ij} \geq \tau$, justifies Assumption 3, which states a *necessary* condition for interaction.

We now consider the opinion evolution of a system satisfying Assumption 3. Since $\xi(d) = \gamma_2 \mu^2(d) d$ for $d < \tau$, the dynamic of the expected squared distance $\bar{d}$ has the same expression as (9) except that

$$\eta(d_{ij}) = \begin{cases} 4\alpha\mu(d_{ij}) - (4\alpha + \gamma_2)\mu^2(d_{ij}) & \text{if } d_{ij} < \tau \\ -\infty & \text{otherwise.} \end{cases}$$

The system reaches a fixed point if d_{ij} for $\forall (i,j) \in \mathcal{E}$ also satisfies one of the two conditions: (i) $d_{ij} = 0$; (ii) $\eta(d_{ij}) \leq 0$. Thus agents with different opinions will not interact if d_{ij} belongs to the union $\mathcal{D}_2 \cup [\tau, d_{\max})$ where

$$\mathcal{D}_2 = \left\{ d \in (0, \tau) \;\middle|\; \frac{4(1-\mu(d))}{\mu(d)} \geq \frac{\gamma_2}{\alpha} \right\}. \qquad (11)$$

Or equivalently, $d_{ij} \in (0, \sup(\mathcal{D}_2)] \cup [0, d_{\max})$.

Clearly, if the threshold $\tau \leq \sup(\mathcal{D}_2)$ is small relative to the supremum of the set $\mathcal{D}_2$, then agents will not update their opinions because either they are too closed-minded to the opinions of the other agents or they do not have sufficient incentives to interact. On the other hand, as depicted in Fig. 2, if the threshold $\tau \gg \sup(\mathcal{D}_2)$, agents will interact if their squared opinion distance d_{ij} lies in the open interval $(\sup(\mathcal{D}_2), \tau)$. Hence, it can be deduced that the system will form one or multiple opinion clusters. Within each cluster, the squared opinion distances are upper bounded by $\sup(\mathcal{D}_2)$. And the squared opinion distances between the clusters are lower bounded by the threshold τ.

III. Numerical Results

The purpose of this section is to numerically validate the analytical results presented earlier. Since the choice of the underlying communication network $\mathcal{G}$ is arbitrary and the analytical results hold for any connected network, we start by generating $\mathcal{G}$ using a random geometric graph (RGG), i.e., $\mathcal{G} = \mathcal{G}(n, r)$, consisting of $n = 50$ social agents randomly distributed social agents over an unit disk with a radius of communication $r = 0.8$. Each initial opinion profile $\mathbf{x}_i[0]$ for $\forall i \in \mathcal{V}$ is uniformly distributed in the opinion space $\mathcal{X}$ with $q = 3$ possible decision states. Without loss of generality, the 2-norm is used to measure the opinion distance between agents, i.e., $d(\mathbf{x}_i, \mathbf{x}_j) = \|\mathbf{x}_i - \mathbf{x}_j\|_A^2$ with $A = I_q$. We define the trust function to be $\mu(d) = 0.5 - 0.4d^2$ for $\forall d < \tau$, which satisfies our assumptions on μ.

Case Study (i): Consider the subgraph $\mathcal{G}_{\text{eff}}[k] = (\mathcal{V}, \mathcal{E}_{\text{eff}}[k])$ of $\mathcal{G}$ at each time k, where $\mathcal{E}_{\text{eff}}[k]$ contains all the edges $(i,j) \in \mathcal{E}$ whose corresponding distances $d_{ij}[k]$ are such that $d_{ij}[k] < \tau$ if $\mathcal{D}_1 = \emptyset$ and $d_{ij} < \inf(\mathcal{D}_1)$ otherwise. Fig. 3 shows the (normalized) algebraic connectivity of the graph $\mathcal{G}_{\text{eff}}[k]$ for k sufficiently large, and for various values of τ and $\inf(\mathcal{D}_1)$, the plot is averaged over 400 realizations (of topology, initial opinion profile, and interactions). Each realization starts with an uniformly distributed initial opinion profile. Observe that when $\inf(\mathcal{D}_1)$ is small (i.e., γ_1/α is large), the agents are less likely to reach a consensus for any value of τ. In contrast, when $\inf(\mathcal{D}_1)$ is large, i.e., $\inf(\mathcal{D}_1) > 0.64$ approximately, the society tends to form a convergent opinion almost surely for large values of τ (approximately above 0.64).

Case Study (ii): Fig. 4 shows the final outcome of the interactions (top panel) and the squared distance distribution (bottom panel) with $\tau = 0.09$ and $\sup(\mathcal{D}_2) = 0.0158$ (i.e., $\gamma_2/\alpha = 4.0016$.) Observe from the top panel that three opinion clusters are formed. Within each clusters, the squared opinion distances are upper bounded by $\sup(\mathcal{D}_2) = 0.0158$, as shown in the bottom panel of Fig. 4. Also, the squared distances between clusters are at least 0.18, which is much

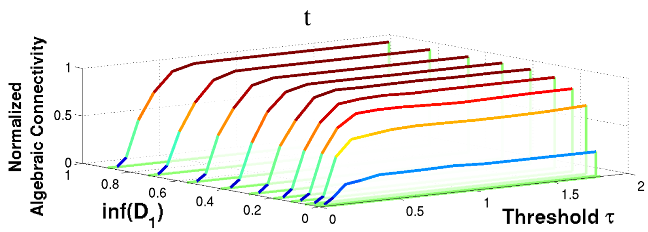

Fig. 3: Phase Transitions for different values of $\inf(\mathcal{D}_1)$

larger than the threshold $\tau = 0.09$. On the other hand, Fig. 5 shows the final outcome of the interactions (top panel) and the squared distance distribution (bottom panel) when $\tau = 0.64$ and $\sup(\mathcal{D}_2) = 0.0158$. In this case, agents form a single opinion cluster as shown in the top panel. Also, the squared distances within this cluster is upper bounded by 0.012, which is less than $\sup(\mathcal{D}_2)$.

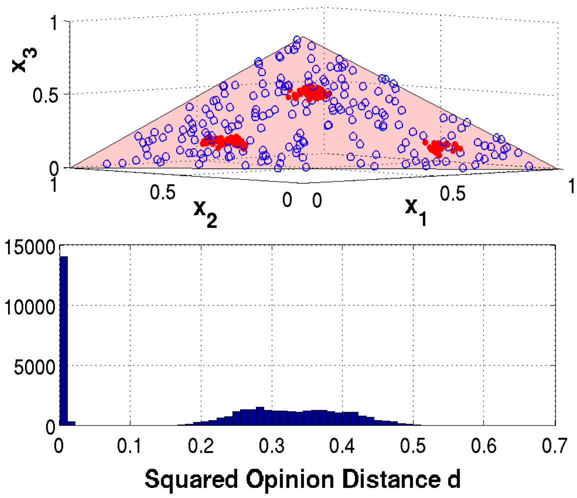

Fig. 4: Opinion formation when $\tau = 0.09$ and $\sup(\mathcal{D}_2) = 0.0158$. The blue circles are initial opinions, and red dots represent the final opinion profile.

IV. CONCLUSIONS

In this paper, we proposed a strategic communication scheme. We showed how the opinion formation processes are affected by the individual incentives behind interactions. In particular, we explored in detail two specific utility functions that lead to two different asymptotic opinion patterns.

V. APPENDIX

Proof of Eqn. (8): Let $f_k(d|A)$ be the average distribution of $d_{ij}[k]$ conditioned on event $A = \{(i,j) \text{ interacts}\}$, i.e.,

$$f_k(d \mid A) = \frac{2}{n(n-1)} \sum_{p=1}^{|\mathcal{V}|} \sum_{l>p}^{|\mathcal{V}|} f_{d_{pl}[k]}(d \mid A)$$

where $f_{d_{pl}[k]}(d \mid A)$ is the conditional distribution of the squared distance between the pair (p,l). If event A happens, for sufficiently large n, the value of $D[k]$, say $D^{ij}[k]$ should

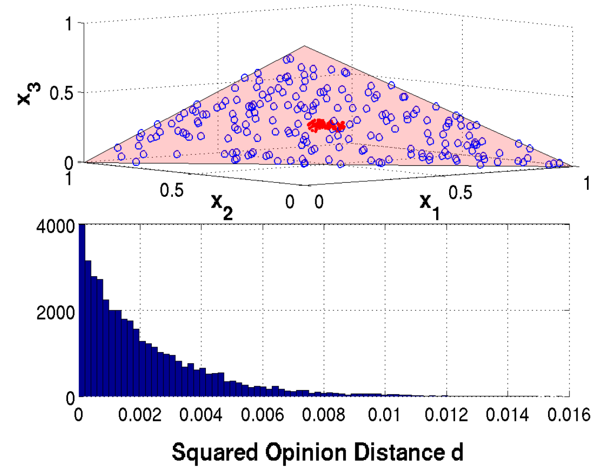

Fig. 5: Opinion formation when $\tau = 0.64$ and $\sup(\mathcal{D}_2) = 0.0158$.

be such that $\int u f_k(u|A) du = \mathbb{E}\{d[k] \mid A\} \approx \frac{D^{ij}[k]}{n(n-1)/2}$. Hence, the conditional expectation of the squared distance with respect to the average distribution can be approximated by the sample mean:

$$\begin{aligned}\mathbb{E}\{d[k+1]\} &= \sum\nolimits_{(i,j)\in\mathcal{E}} \overline{P}_{ij}\, \mathbb{E}\left\{d[k+1] \mid \mid (i,j) \text{ interacts}\right\} \\ &\approx \sum\nolimits_{(i,j)\in\mathcal{E}} \overline{P}_{ij} \frac{D^{ij}[k+1]}{n(n-1)/2}.\end{aligned}$$

where $\overline{P}_{ij} = p_i P_{ij} + p_j P_{ji} = \frac{1}{n}(P_{ij} + P_{ji})$. It then follows from the relation in (5) that $\mathbb{E}\{d[k+1]\} - \mathbb{E}\{d[k]\} = -\frac{4}{n+1}\sum_{(i,j)\in\mathcal{E}} \overline{P}_{ij}\rho(d_{ij}[k])d_{ij}[k]$. Using Euler's approximation and setting $h = \frac{4}{n+1}$, the following *ordinary differential equation* (ODE) can be derived $\dot{d}(t) = -\sum_{(i,j)\in\mathcal{E}} \overline{P}_{ij}\rho(d_{ij}(t))d_{ij}(t)$.

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