Topics:

a) Properties of the matrix exponential

b) Eigenvalue/eigenvector expressions for $e^{At}$

Matrix exponential: Let $A$ be an $n \times n$ real matrix. Then the matrix exponential $\Phi(t) = \exp(At)$ is defined as the solution of the differential equation

$$\Phi(t) = A\Phi(t)$$

for $t \geq 0$, with initial condition $\Phi(0) = I_n$. If $F(s)$ denotes the Laplace transform of $\Phi(t)$, by Laplace transforming equation (1), we obtain

$$sF(s) - \Phi(0) = AF(s),$$

so that

$$(sI - A)F(s) = \Phi(0) = I_n.$$  

This implies $F(s) = (sI - A)^{-1}$, so that $\exp(At)$ can also be obtained through the Laplace transform relation

$$\exp(At) \leftrightarrow (sI - A)^{-1}.$$  

Property 1: $\exp(At)$ admits the power series expansion

$$\exp(At) = I_n + At + \frac{(At)^2}{2!} + \ldots + \frac{(At)^k}{k!} + \ldots,$$

for $t \geq 0$, which can be viewed as a matrix version of the power series expansion

$$\exp(at) = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}$$

of the scalar exponential function. To verify (3), denote the power series on the right hand side of (3) by $\Phi(t)$ and assume that it converges and can be differentiated term by term. Differentiating each term gives

$$\frac{d}{dt} \Phi(t) = A \left( I_n + At + \ldots + \frac{(At)^{k-1}}{(k-1)!} + \ldots \right) = A\Phi(t)$$

for $t \geq 0$, with $\Phi(0) = I_n$, so that $\Phi(t)$ obeys the differential equation (1) defining the matrix exponential.
Another way of deriving (3) relies on the observation that for \( s \) sufficiently large, \((sI - A)^{-1}\) admits the power series expansion
\[
(sI - A)^{-1} = \frac{1}{s}(I - \frac{A}{s})^{-1} = \frac{1}{s}[I + \frac{A}{s} + (\frac{A}{s})^2 + \ldots + (\frac{A}{s})^k + \ldots].
\] (4)

Then, using the fact that
\[
\frac{t^k}{k!} \overset{LT}{\longleftrightarrow} \frac{1}{s^{k+1}},
\]
and taking the inverse Laplace transform of (4) gives (3).

**Examples:** (i) Consider the \( r \times r \) nilpotent matrix
\[
N = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]
which has ones on its first superdiagonal, and zeros everywhere else. It has property that for \( \ell < r \) the matrix
\[
N^\ell = \begin{bmatrix}
0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 1 & 0 & \ldots & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0
\end{bmatrix}
\]
has ones on its \( \ell \)-th superdiagonal, and zeros everywhere else, and \( N^\ell = 0 \) for \( \ell \geq r \). Then in the power series expansion (3) for \( \exp(Nt) \), only the first \( r \) terms are nonzero, so that
\[
\exp(Nt) = I_r + Nt + \ldots + \frac{(Nt)^{r-1}}{(r-1)!} + \frac{t^r}{(r-1)!}.
\]

(ii) Let
\[
A = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}.
\]
Then $A^{2p} = (-1)^p I_2$ and $A^{2p+1} = (-1)^p A$, where $I_2$ denotes the $2 \times 2$ identity matrix, so that
\[
\exp(At) = \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} \cdots \right) I_2 + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \right) A
\]
\[
= \cos t I_2 + \sin t A = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.
\]

**Property 2:** The matrix exponential has the *transition property*
\[
\exp(A(t_1 + t_2)) = \exp(At_1) \exp(At_2)
\]
for $t_1, t_2 \geq 0$. To see how this property arises, consider solving the differential equation (1) for $0 \leq t \leq t_1 + t_2$. We can solve the equation in one step over the whole interval, in which case the solution at $t = t_1 + t_2$ is $\exp(A(t_1 + t_2))$. Alternatively, we can first solve the equation over $[0, t_2]$ and then over $[t_2, t_1 + t_2]$. In this case the solution at $t = t_2$ is $\exp(At_2)$, and the solution for $t_2 \leq t \leq t_1 + t_2$ is obtained by solving
\[
\frac{d}{dt} \Phi(t) = A\Phi(t)
\]
over $[t_2, t_1 + t_2]$ with initial condition $\Phi(t_2) = \exp(At_2)$. But the system is LTI, so that the solution at $t = t' + t_2$ with $t' \geq 0$ is given by
\[
\Phi(t) = \exp(A t') \Phi(t_2).
\]
Setting $t = t_1 + t_2$ and $t' = t_1$ in this identity gives (5).

**Property 3:** In general if $A$ and $B$ are two arbitrary matrices
\[
\exp((A + B)t) \neq \exp(At) \exp(Bt).
\]
To see this, let
\[
A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
\]
Both $A$ and $B$ are nilpotent with
\[
\exp(At) = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \exp(Bt) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix},
\]
so
\[
\exp(At) \exp(Bt) = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} 1 - t^2 & -t \\ t & 1 \end{bmatrix}.
\]
On the other hand
\[
A + B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\]
and it was shown earlier that
\[
\exp((A + B)t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.
\]
which is clearly different from \( \exp(At) \exp(Bt) \).

However if \( A \) and \( B \) commute, i.e., \( AB = BA \), we have

\[
\exp((A + B)t) = \exp(At) \exp(Bt) .
\] (7)

To prove this, note that when \( A \) and \( B \) commute

\[
(A + B)^k = A^k + \binom{k}{1} A^{k-1} B + \ldots + \binom{k}{1} A B^{k-1} + B^k .
\]

Taking this identity into account, and multiplying power series term by term gives

\[
\exp(At) \exp(Bt) = \left[ I + At + \frac{(At)^2}{2!} + \ldots \right] \left[ I + Bt + \frac{(Bt)^2}{2!} + \ldots \right] = I + (A + B)t + \frac{(A + B)t^2}{2!} + \ldots = \exp((A + B)t) .
\]

**Examples:** (i) Consider the \( r \times r \) Jordan block

\[
J = \begin{bmatrix}
\lambda & 1 & & 0 \\
& \lambda & \ddots & \\
& & \ddots & \ddots \\
0 & & & \lambda \\
\end{bmatrix} = \lambda I_r + N
\]

where \( \lambda I_r \) and \( N \) commute. We have

\[
e^{\lambda t} = \left( 1 + \lambda t + \frac{(\lambda t)^2}{2!} + \ldots \right) I_r = e^{\lambda} I_r
\]

and

\[
\exp(Jt) = \exp((\lambda I_r + N)t) = \exp(\lambda I_r) \exp(Nt)
\]

\[
= \exp(\lambda t) \begin{bmatrix} 1 & t & t^r-1/(r-1)! \end{bmatrix} .
\]

(ii) Let

\[
A = \begin{bmatrix} \sigma & -\omega \\
\omega & \sigma \end{bmatrix} = \sigma_2 + \omega Q
\]

with

\[
Q \triangleq \begin{bmatrix} 0 & -1 \\
1 & 0 \end{bmatrix} ,
\]
where $I_2$ and $Q$ commute. Then
$$
\exp(At) = \exp(\sigma t) \exp(\omega Qt) = \exp(\sigma t) \begin{bmatrix}
\cos \omega t & -\sin \omega t \\
\sin \omega t & \cos \omega t
\end{bmatrix}.
$$

Property 4: $\exp(At)$ is an invertible matrix for all $t \geq 0$ and
$$
\left( \exp(At) \right)^{-1} = \exp((-A)t). \tag{8}
$$
To see this, note that $A$ and $-A$ commute and $A + (-A) = 0$, so that
$$
e^{0t} = I_n = \exp(At) \exp((-A)t).
$$

Eigenvalue/eigenvector expressions: The matrix exponential $\exp(At)$ can also be expressed in terms of the eigenvalues and eigenvectors (or generalized eigenvectors) of $A$.

Case 1: $A$ is diagonalizable: In this case $A$ admits $n$ independent eigenvectors $p_i$ corresponding to eigenvalues $\lambda_i$, i.e.
$$
Ap_i = \lambda_i p_i \tag{9}
$$
for $1 \leq i \leq n$. The relations (9) can be combined as a single matrix equation
$$
AP = PA
$$
with
$$
P = \begin{bmatrix} p_1 & \cdots & p_i & \cdots & p_n \end{bmatrix} \quad \text{and} \quad A = \text{diag} \{ \lambda_i, 1 \leq i \leq n \},
$$
so that $A = PA^{-1}$. The matrix
$$
P^{-1} = Q = \begin{bmatrix}
q_1^T \\
\vdots \\
q_i^T \\
\vdots \\
q_n^T
\end{bmatrix}
$$
yields the left eigenvectors of $A$ since $P^{-1}A = P^{-1}A$, or equivalently
$$
q_i^T A = \lambda_i q_i^T,
$$
for $1 \leq i \leq n$. Then
$$
A^2 = (PAP^{-1})(PAP^{-1}) = PA^2P^{-1}
$$
and $A^k = P\Lambda^kP^{-1}$, so that
$$
\exp(At) = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = P \left[ \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} \right] P^{-1}
= P \exp(\Lambda t) P^{-1}. \tag{10}
$$
The main advantage of expression (10) is that, since $A$ is diagonal, its exponential matrix is easy to compute and is also diagonal, i.e.

$$e^{At} = \text{diag} \{ \exp(\lambda_i t), 1 \leq i \leq n \}.$$  

The identity (10) expresses $\exp(At)$ completely in terms of the eigenvalues and right and left eigenvectors of $A$. To see this, note that

$$\exp(At) = P \exp(\Lambda t) Q$$

$$= \begin{bmatrix} p_1 & \ldots & p_i & \ldots & p_n \end{bmatrix} \text{diag} \{ \exp(\lambda_i t), 1 \leq i \leq n \}$$

$$= \begin{bmatrix} q_1^T \\ \vdots \\ q_i^T \\ \vdots \\ q_n^T \end{bmatrix}$$

can be rewritten as

$$e^{At} = \sum_{i=1}^{n} p_i q_i^T \exp(\lambda_i t).$$  (11)

**Example:** Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ 6 & 0 & -6 \end{bmatrix}.$$  

Then

$$sI - A = \begin{bmatrix} s-1 & 0 & 2 \\ 0 & s+1 & 0 \\ -6 & 0 & s+6 \end{bmatrix}$$

and

$$a(s) = \det(sI - A) = (s+1) \left[ (s-1)(s+6) + 12 \right] = (s+1)(s+2)(s+3).$$

The eigenvector $p_1$ corresponding to $\lambda_1 = -1$ is obtained by solving

$$(\lambda_1 I - A)p_1 = \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ -6 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$  

This gives $x_1 = x_3 = 0$ with $x_2$ free, and since the scaling of $p_1$ is arbitrary, we select

$$p_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$  

Similarly the eigenvector $p_2$ corresponding to $\lambda_2 = -2$ is obtained by solving

$$(\lambda_2 I - A)p_2 = \begin{bmatrix} -3 & 0 & 2 \\ 0 & -1 & 0 \\ -6 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$p_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$
which gives \( x_2 = 0 \) and \( 3x_1 = 2x_3 \), so that we can select

\[
p_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}.
\]

Finally, to find eigenvector \( p_3 \) corresponding to \( \lambda_3 = -3 \), we solve

\[
(\lambda_3I - A)p_3 = \begin{bmatrix} -4 & 0 & 2 \\ 0 & -2 & 0 \\ -6 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

This gives \( x_2 = 0 \) and \( x_3 = 2x_1 \), so that we can choose

\[
p_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.
\]

Inverting the matrix

\[
P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix}
\]

gives

\[
Q = P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \\ -3 & 0 & 2 \end{bmatrix},
\]

so that

\[
\exp(At) = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} \exp(-t) & 0 & 0 \\ 0 & \exp(-2t) & 0 \\ 0 & 0 & \exp(-3t) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \\ -3 & 0 & 2 \end{bmatrix}
\]

\[
= \begin{bmatrix} 4\exp(-2t) - 3\exp(-3t) & 0 & 2[-\exp(-2t) + \exp(-3t)] \\ 0 & \exp(-t) & 0 \\ 6[\exp(-2t) - \exp(-3t)] & 0 & -3\exp(-2t) + 4\exp(-3t) \end{bmatrix}.
\]

In the expression (11) for \( \exp(At) \), although \( \exp(At) \) is real, the eigenvalues \( \lambda_i \) and eigenvectors \( p_i \) and \( q_i \) may be complex.

**Example:** Let

\[
A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.
\]

Then

\[
sI - A = \begin{bmatrix} s + 2 & 0 & 1 \\ 0 & s + 2 & 0 \\ -2 & 0 & s \end{bmatrix},
\]

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and
\[ a(s) = \det sI - A = (s^2 + 2s + 2)(s + 2) = (s + 1 - j)(s + 1 + j)(s + 2)i. \]

Thus \( A \) has two complex conjugate eigenvalues \( \lambda_\pm = -1 \pm j \) and a real eigenvalue \( \lambda_3 = -2 \). The eigenvector \( p_+ \) corresponding to \( \lambda_+ \) is obtained by solving

\[ (\lambda_+ I - A)p_+ = \begin{bmatrix} 1 + j & 0 & 1 \\ 0 & 1 + j & 0 \\ -2 & 0 & -1 + j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \]

which gives \( x_2 = 0, \ x_3 = -(1 + j)x_1 \), so that we can select

\[ p_+ = \begin{bmatrix} 1 \\ 0 \\ -(1 + j) \end{bmatrix}. \]

By observing that \( \lambda_- = \lambda_+^* \) and \( A \) is real, and taking the complex conjugate of the eigenvector equation for \( \lambda_+ \) we find that

\[ p_- = p_+^* = \begin{bmatrix} 1 \\ 0 \\ -(1 - j) \end{bmatrix} \]

satisfies

\[ (\lambda_- I - A)p_- = 0, \]

so it is the eigenvector corresponding to eigenvalue \( \lambda_- \). Finally, the eigenvector corresponding to \( \lambda_3 = -2 \) is obtained by solving

\[ (\lambda_3 I - A)p_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \]

which yields \( x_1 = x_3 = 0 \), so that we can select

\[ p_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \]

Then if
\[ P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -(1 + j) & -(1 - j) & 0 \end{bmatrix} \]

we find
\[ Q = P^{-1} = \begin{bmatrix} (1 + j)/2 & 0 & j/2 \\ (1 - j)/2 & 0 & -j/2 \\ 0 & 1 & 0 \end{bmatrix} \]

and
\[ \exp(At) = P \exp(At)Q \]
\[ \exp(A) = \begin{bmatrix} \exp(-t + j\tau) & 0 & 0 \\ 0 & \exp(-t - j\tau) & 0 \\ 0 & 0 & \exp(-2\tau) \end{bmatrix}. \]

**Real form of the eigenvalue/eigenvector expansion of** \( \exp(At) \): To obtain expressions involving only real quantities, one can proceed as follows. First observe that since \( A \) is a real matrix, if \( (\lambda_i, p_i) \) is an eigenvalue/eigenvector pair for \( A \), so is \( (\lambda_i^*, p_i^*) \). Assume now that \( A \) has \( r \) pairs of complex conjugate eigenvalues and \( q = n - 2r \) real eigenvalues. Then order the eigenvalues of \( A \) as \( \{\lambda_1, \lambda_1^*, \lambda_2, \lambda_2^*, \ldots, \lambda_r, \lambda_r^*, \lambda_{2r+1}, \ldots, \lambda_n\} \). In the eigenvalue/eigenvector relation \( A p_k = \lambda_k p_k \), the eigenvalue \( \lambda_k \) and eigenvector \( p_k \) can be decomposed into their real and imaginary parts as
\[ \lambda_k = a_k + j b_k \]
\[ p_k = p_k^R + j p_k^I. \]

This implies that
\[ A p_k^R = a_k p_k^R - b_k p_k^I \]
\[ A p_k^I = b_k p_k^R + a_k p_k^I. \]

Then, if we consider the matrix
\[ M = \begin{bmatrix} p_1^R p_1^I & \cdots & p_k^R p_k^I & \cdots & p_r^R p_r^I & p_{2r+1} & \cdots & p_n \end{bmatrix}, \]

\( M \) is real and
\[ AM = MD \]

with
\[ D = \begin{bmatrix} D_1 \\ \vdots \\ D_k \\ \cdots \\ D_r \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \lambda_{2r+1} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \lambda_n \end{bmatrix}, \]

where
\[ D_k \triangleq \begin{bmatrix} a_k & b_k \\ -b_k & a_k \end{bmatrix}. \]

If \( N = M^{-1} \), this implies
\[ A = MDN \]

(12)

where \( D \) is block diagonal. It is constituted of \( r \) \( 2 \times 2 \) blocks corresponding to the \( r \) pairs \( (\lambda_k, \lambda_k^*) \) of complex conjugate eigenvalues, as well as \( n - 2r \) \( 1 \times 1 \) blocks corresponding to the \( n - 2r \) real eigenvalues \( \lambda_{2r+1}, \ldots, \lambda_n \) of \( A \).
The expression (12) for $A$ implies

$$\exp(At) = M \exp(Dt)N,$$

(13)

with

$$\exp(Dt) = \begin{bmatrix}
\exp(D_1 t) & & \\
& \ddots & \\
& & \exp(D_k t)
\end{bmatrix}
\begin{bmatrix}
0 & & \\
& \ddots & \\
& & \exp(\lambda_{2r+1} t)
\end{bmatrix},$$

and

$$\exp(D_k t) = \exp(a_k t)
\begin{bmatrix}
\cos(b_k t) & \sin(b_k t) \\
-\sin(b_k t) & \cos(b_k t)
\end{bmatrix},$$

where the expression (13) involves only real matrices.

**Example:** It was found earlier that the matrix

$$A = \begin{bmatrix}
-2 & 0 & -1 \\
0 & -2 & 0 \\
2 & 0 & 0
\end{bmatrix}$$

has eigenvalues $\lambda_{\pm} = -1 \pm j$ and $\lambda_3 = -2$. Its eigenvectors are

$$p_{\pm} = \begin{bmatrix} 1 \\ 0 \\ -(1 \pm j) \end{bmatrix} \text{ and } p_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The vector $p_+$ can be decomposed as $p_+ = p_R + j p_I$ with

$$p_R = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } p_I = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Let

$$M = \begin{bmatrix} p_R & p_I & p_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$N = M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$
Then
\[
\exp(At) = M \exp(Dt)N
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
-1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
\exp(-t) \cos t & \exp(-t) \sin t & 0 \\
-\exp(-t) \sin t & \exp(-t) \cos t & 0 \\
0 & 0 & \exp(-2t)
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
-1 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}.
\]

**Case 2:** \( A \) is not diagonalizable. In this case \( A \) has a Jordan block of size 2 or larger, or equivalently it has a repeated eigenvalue which admits fewer independent eigenvectors than its multiplicity. In this case, we can find an invertible matrix \( T \) constituted of the eigenvectors and generalized eigenvectors of \( A \) such that
\[
A = TJJ^{-1}
\]
with
\[
J = \begin{bmatrix}
J_1 & & \\
& \ddots & \\
& & J_i
\end{bmatrix}
\]
\[
J_i = \begin{bmatrix}
\lambda_i & 1 & 0 \\
& \ddots & \ddots \\
& & \lambda_i & 1 \\
0 & & & \lambda_i
\end{bmatrix} = n_i \times n_i \text{ Jordan block ,}
\]
where \( \sum_{i=1}^{\ell} n_i = n \). The matrix exponential \( \exp(At) \) is then given by
\[
\exp(At) = T \exp(Jt)T^{-1},
\]
with
\[
\exp(Jt) = \begin{bmatrix}
\exp(J_1 t) & 0 \\
& \ddots & 0 \\
& & \exp(J_\ell t)
\end{bmatrix}
\]
and
\[
\exp(J_i t) = \exp(\lambda_i t)
\]
\[
\begin{bmatrix}
1 & t & \frac{t^{n_i-1}}{(n_i-1)!} \\
& \ddots & \ddots \\
& & 1
\end{bmatrix}.
\]
**Example:** Let

\[ A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

Then

\[ sI_4 - A = \begin{bmatrix} s - 1 & -2 & 0 & -1 \\ 0 & s - 1 & 0 & 0 \\ 0 & 1 & s - 1 & 0 \\ 0 & 0 & 0 & s - 1 \end{bmatrix}, \]

and using Laplace’s expansion of \( a(s) = \text{det} \ sI - A \) with respect to the first column yields \( a(s) = (s - 1)^4 \). The eigenvectors \( \mathbf{p} \) of \( A \) corresponding to \( \lambda_1 = 1 \) satisfy

\[(\lambda_1 I - A)\mathbf{p} = \begin{bmatrix} 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \]

This gives \( x_2 = x_4 = 0 \), so that \( A \) has the two eigenvectors

\[ \mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \]

Since the number of eigenvectors (two) is less than the multiplicity (four) of \( \lambda_1, A \) is not diagonalizable. Observing that \( (\lambda_1 I - A)^2 = 0 \), we can conclude that \( A \) will have 2 Jordan blocks of size \( 2 \times 2 \) corresponding to \( \lambda_1 = 1 \). The generalized eigenvector \( \mathbf{g}_1 \) corresponding to \( \mathbf{p}_1 \) is given by

\[(A - \lambda_1 I)\mathbf{g}_1 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \]

which gives \( x_2 = 0 \) and \( x_4 = 1 \), so that we can select

\[ \mathbf{g}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \]

Similarly the generalized eigenvector \( \mathbf{g}_2 \) corresponding to \( \mathbf{p}_2 \) satisfies

\[(A - \lambda_1 I)\mathbf{g}_2 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \]
We obtain $x_2 = -1$ and $x_4 = 2$, so that we can select
\[
g_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}.
\]

Then if
\[
T = \begin{bmatrix} p_1 & g_1 & p_2 & g_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}
\]
\[
T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}
\]
\[
J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]
we have $A = TJT^{-1}$ and
\[
\exp(At) = T \exp(Jt) T^{-1}
\]
with
\[
\exp(Jt) = \exp(t) = \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

Note that the Jordan decomposition $A = TJT^{-1}$ may be complex. There exists a real form of this decomposition, but it will not be needed in the remainder of this course.