UNIVERSITY OF CALIFORNIA, DAVIS
Department of Electrical and Computer Engineering

EEC250 Linear Systems and Signals Fall 2009

Lecture 2

Lecture Topics:

a) Elementary row operations, row echelon form

b) Gaussian elimination, solution of linear equations

c) Basis and dimension of a vector space

d) Dimension of the row and column spaces, rank of a matrix

Elementary row operations: To transform an arbitrary $m \times n$ matrix

$$A = \begin{bmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_i^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix}, \tag{1}$$

where $\mathbf{r}_i^T$ denotes the i-th row of A, into a form that reveals its structure, we rely on three types of row operations.

(i) **Row scaling**. Multiplying the i-th row of A by a constant $c \neq 0$ corresponds to multiplying A on the left by the $m \times m$ diagonal matrix

$$M_s = \mathrm{diag}\,\{1, \ldots, 1, c, 1, \ldots, 1\} \tag{2}$$

whose diagonal elements are all equal to 1, except for the ith element which equals c. Then the product

$$M_s A = A_s \triangleq \begin{bmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_{i-1}^T \\ c\mathbf{r}_i^T \\ \mathbf{r}_{i+1}^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix}$$

keeps all the rows od A unchanged, except the i-th row, which is multiplied by c.

(ii) **Row permutation**. Let $\mathbf{e}_k$, $1 \leq k \leq m$ denotes the orthonormal basis of $\mathbb{R}^m$ where vector $\mathbf{e}_k$ has all zero entries, except the k-th entry which equals one. Then for $i < j$, consider the m times m permutation matrix

$$M_p = \begin{bmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_{i-1} & \mathbf{e}_j & \mathbf{e}_{i+1} & \dots & \mathbf{e}_{j-1} & \mathbf{e}_i & \mathbf{e}_{j+1} & \dots & \mathbf{e}_m \end{bmatrix} \tag{3}$$

obtained by exchanging the i-th and j-th columns of the $m \times m$ identity matrix I_m. If A is written row-wise as indicated in (1), by multiplying A on the left by M_p, we obtain the matrix

$$M_pA = A_p = \begin{bmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_{i-1}^T \\ \mathbf{r}_j^T \\ \mathbf{r}_{i+1}^T \\ \vdots \\ \mathbf{r}_{j-1}^T \\ \mathbf{r}_i^T \\ \mathbf{r}_{j+1}^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix}$$

which is obtained by exchanging the i-th and j-th rows of A.

(iii) **Row combination**. Adding to the i-th row of A the j-th multiplied by a constant c corresponds to multiplying A on the left by the $m \times m$ matrix

$$M_c = I_m + c\mathbf{e}_i\mathbf{e}_j^T \ , \tag{4}$$

where $\mathbf{e}_i$ and $\mathbf{e}_j$ denote the i-th and j-th basis vectors of $\mathbb{R}^m$. If A is written row-wise as indicated in (1), multiplying A on the left by M_c gives the matrix

$$M_cA = A_c = \begin{bmatrix} \mathbf{r}_1^T \\ \vdots \\ (\mathbf{r}_i + c\mathbf{r}_j)^T \\ \vdots \\ \mathbf{r}_j^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix}$$

which is obtained by addinf c times row j of A to its i-th row, while keeping all other rows unchanged.

Note that the elementary matrices M_s, M_p and M_c corresponding to the three types of elementary row operations are all invertible, since each row operation is easy to invert.

Specifically, the inverse operation of scaling the i-th row of A by $c \neq 0$ consists in scaling the i-th row of A_s by $1/c$, so

$$M_s^{-1} = \operatorname{diag}\{1, \ldots, 1, c^{-1}, 1, \ldots, 1\}. \tag{5}$$

Similarly, the inverse of exchanging the i-th and j-th rows of A consists of switching them again, so that $M_p^{-1} = M_p$. Finally the inverse of adding the j-th row of A times c to the i-th row consists in subtracting the j-th row of A_c times c from the i-th row of A_c. Thus

$$M_c^{-1} = I_m - c\mathbf{e}_i\mathbf{e}_j^T \,. \tag{6}$$

Row echelon form: A key result is that by elementary row operations, we can bring an arbitrary $m \times n$ matrix A to its row echelon form

$$E = \left[\begin{array}{cccc} 0\ldots0 & 1xx & x\ldots x & x\ldots x \\ 0 & 0 & 1xx & x\ldots x \\ 0 & 0 & 0 & 1x\ldots x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \tag{7}$$

where the x's correspond to arbitrary entries (they can be either zero or nonzero), r is the number of nonzero rows, and the columns corresponding to the first 1 entry of each row are called the pivot columns. The indices $p_1 < p_2 \ldots < p_r$ of the pivot columns are called the pivot indices.

Thus, for an arbitrary $m \times n$ matrix A, we can find an $m \times n$ matrix M, where $M = \prod_{k=1}^{N} M_k$ is the product of elementary matrices, such that

$$MA = E \,. \tag{8}$$

Example: Let

$$A = \left[\begin{array}{rrrrr} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & -2 \end{array} \right] .$$

To compute simultaneously the echelon form E and transformation matrix M, it is convenient to operate on the expanded 3×6 block matrix

$$A_e = \left[\begin{array}{cc} A & I_3 \end{array} \right] ,$$

where I_3 denotes the idenity matrix of size 3, instead of A only. This expansion will make it easier to keep track of the transformation matrix M. Then by adding the 1st row of A_e

to the 2nd row, and subtracting the 1st row from the 3rd row, we obtain

$$\begin{aligned} A_e &= \left[\begin{array}{rrrrr:rrr} 1 & 2 & 0 & 2 & 1 & 1 & 0 & 0 \\ -1 & -2 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & -3 & -7 & -2 & 0 & 0 & 1 \end{array}\right] \\ &\longrightarrow \left[\begin{array}{rrrrr:rrr} 1 & 2 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & -3 & -9 & -3 & -1 & 0 & 1 \end{array}\right]. \end{aligned}$$

Next, adding 3 times the 2nd row to the the 3rd row gives

$$\left[\begin{array}{rrrrr:rrr} 1 & 2 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 \end{array}\right] = \left[\begin{array}{cc} E & M \end{array}\right].$$

The above example required only the use of row combinations. However, in general it ia necessary to employ row exchange and row scaling operations. For example

$$\begin{aligned} A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 2 & 8 & 8 \end{bmatrix} &\longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 2 & 4 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

where in the first line we substract two times the first row from the second and third row, and in the second line the third row is scaled by $1/2$ and exchanged with the second row.

Reduced row echelon form: For some problems, it is convenient to work with the reduced row echelon form E_r, which is obtained from E by using the 1's in the pivot columns to eliminate the nonzero entries located in the same column.

For the example we have just considered, subtracting 3 times the second row of E from the first row gives

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = E_r .$$

When A is a square invertible matrix, $E_r = I$ and from $MA = E_r = I$, we see that $M = A^{-1}$.

Example: Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 9 \\ 2 & 8 & 8 \end{bmatrix}.$$

Then

$$\left[\begin{array}{ccc:ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 6 & 9 & 0 & 1 & 0 \\ 2 & 8 & 8 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 5 & -2 & 1 & 0 \\ 0 & 2 & 4 & -2 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc:ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1/2 \\ 0 & 0 & 1 & -2/5 & 1/5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:ccc} 1 & 3 & 0 & 9/5 & -2/5 & 0 \\ 0 & 1 & 0 & -1/5 & -2/5 & 1/2 \\ 0 & 0 & 1 & -2/5 & 1/5 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 12/5 & 4/5 & -3/2 \\ 0 & 1 & 0 & -1/5 & -2/5 & 1/2 \\ 0 & 0 & 1 & -2/5 & 1/5 & 0 \end{array}\right] = \left[\begin{array}{c:c} E_r = I_3 & A^{-1} \end{array}\right].$$

An important fact is that elementary row operations do not change the solutions of $A\mathbf{x} = \mathbf{b}$, and do not change the row space of A.

Lemma 1: Let A be an $m \times n$ matrix and M be an invertible $m \times m$ matrix. Then

a) $\{\mathbf{x} \in \mathbb{R}^n \, : \, A\mathbf{x} = \mathbf{b}\} = \{\mathbf{x} \in \mathbb{R}^n \, : \, MA\mathbf{x} = M\mathbf{b}\}$

b) row space of MA = row space of A.

Proof: a) If $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$, then $MA\mathbf{x} = M\mathbf{b}$; if $\mathbf{x}$ satisfies the latter equation, premultiplying by M^{-1} gives $A\mathbf{x} = \mathbf{b}$.
[b) If $\mathbf{c}$ is in the row space of A, then $\mathbf{y}^T A = \mathbf{c}^T$ for some $\mathbf{y} \in \mathbb{R}^m$. Choosing $\mathbf{z}^T = \mathbf{y}^T M^{-1}$, we have $\mathbf{z}^T MA = \mathbf{c}^T$, and c is in the row space of MA. Conversely if $\mathbf{c}^T = \mathbf{z}^T MA$ for some $\mathbf{z} \in \mathbb{R}^m$, then $\mathbf{c}^T = \mathbf{y}^T A$ for $\mathbf{y}^T = \mathbf{z}^T M$. □

Gaussian elimination, solution of linear equations: Lemma 1 provides the basis for solving linear equations of the form $A\mathbf{x} = \mathbf{b}$ by the method of Gaussian elimination. This method requires 3 steps:

1) Use elementary row operations to transform $A\mathbf{x} = \mathbf{b}$ to the form $E_r\mathbf{x} = \mathbf{b}_r$ where $MA = E_r$ is the reduced row echelon form and $\mathbf{b}_r = M\mathbf{b}$. The computation of $\mathbf{b}_r$ can be achieved by operating on
$$[\, A \,\vdots\, \mathbf{b} \,]$$
instead of A only.

2) Determine whether $E_r\mathbf{x} = \mathbf{b}_r$ has (i) no solution, (ii) a unique solution, (iii) an infinite number of solutions. This can be done as follows. Let r be the number of nonzero

rows of E_r. Then, if the last $m-r$ entries of $\mathbf{b}_r$ are zero, i.e.

$$\mathbf{b}_r = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{9}$$

where entries β_i, $1 \leq i \leq r$ are arbitrary, $E_r\mathbf{x} = \mathbf{b}_r$ will admit a solution. Otherwise it has no solution. Assume now that $E_r\mathbf{x} = \mathbf{b}_r$ admits a solution. If $r = n$, i.e., if all columns of E_r are pivot columns, the solution is unique. On the other hand, if $r < n$ (there exists nonpivot columns), there is an infinite number of solutions.

3) The entries $x_{p_1}, x_{p_2} \ldots x_{p_r}$ of $\mathbf{x}$ corresponding to the pivot columns are called *basic variables*. The remaining entries are called *nonbasic*. Then if $E_r\mathbf{x} = \mathbf{b}_r$ admits a solution, the set of all solutions can be obtained by using $E_r\mathbf{x} = \mathbf{b}_r$ to express the basic variables in terms of both $\mathbf{b}_r$ and the nonbasic variables, which act as free coefficients parametrizing the set of all solutions.

Example: Consider the two equations

$$A\mathbf{x} = \mathbf{b}_1 \quad \text{and} \quad A\mathbf{x} = \mathbf{b}_2 \,,$$

with

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & -2 \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

By applying elementary operations, we find

$$\begin{aligned}
\left[\begin{array}{c|cc} A & b_1 & b_1 \end{array}\right] &= \left[\begin{array}{ccccc|cc} 1 & 2 & 0 & 2 & 1 & 1 & 1 \\ -1 & -2 & 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & -3 & -7 & -2 & -2 & 1 \end{array}\right] \\
\longrightarrow \left[\begin{array}{ccccc|cc} 1 & 2 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & -3 & -9 & -3 & -3 & 0 \end{array}\right] &\longrightarrow \left[\begin{array}{ccccc|cc} 1 & 2 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{array}\right] \\
&= \left[\begin{array}{c|cc} E_r & \mathbf{b}_{r1} & \mathbf{b}_{r2} \end{array}\right].
\end{aligned}$$

If we consider the equation

$$E_r\mathbf{x} = \mathbf{b}_{r1} \,, \tag{10}$$

with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix},$$

by observing that the third entry of $\mathbf{b}_{r1}$ is zero, coinciding with the zero third row of E_r, we conclude that (10) admits a solution. Since the pivot columns of E_r are the first and third columns, x_1 and x_3 are the basic variables, and the nonbasic variables are x_2, x_4 and x_5. The set of solutions of (10) is given by

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix}.$$

Since x_2, x_4 and x_5 are free, there exists an infinite number of solutions.

On the other hand, equation

$$E_r\mathbf{x} = \mathbf{b}_{r2}$$

has no solution, since the third entry of $\mathbf{b}_{r2}$ is nonzero.

Basis: A set of vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ forms a basis of a vector space V if:

(i) the vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ are linearly independent;

(ii) they span the space V, i.e. any vector $\mathbf{v} \in V$ can be expressed as

$$\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{e}_i$$

with $a_i \in \mathbb{R}$ for $1 \leq i \leq n$.

Example: Let $\{\mathbf{e}_i,\ 1 \leq i \leq n\}$ be the family of vectors of $\mathbb{R}^n$ such that all entries of $\mathbf{e}_i$ are zero, except its i-th entry wich equals one. These vectors form a basis of $\mathbb{R}^n$ since

$$\sum_{i=1}^{n} a_i \mathbf{e}_i = \begin{bmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix} = \mathbf{0}$$

implies $a_i = 0$ for all i, so the vectors $\mathbf{e}_i$ are linearly independent. Furthermore an arbitrary vector $\mathbf{x}$ of $\mathbb{R}^n$ can be expressed as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i \mathbf{e}_i\ ,$$

so that $\{\mathbf{e}_i,\ 1 \leq i \leq n\}$ spans $\mathbb{R}^n$.

Lemma 2: The nonzero rows of the echelon form E (or reduced row echelon form E_r) of A form a basis of the row space of A.

Proof: We have $E = MA$ with M invertible since it is the product of invertible elementary matrices. According to part b) of Lemma 1, the row space of E = row space of A, so that the nonzero rows of E span the row space of A. Let $\mathbf{s}_i^T$, $1 \leq i \leq r$ be the nonzero rows of E, and let $\sum_{i=1}^{r} a_i \mathbf{s}_i^T = \mathbf{0}^T$ be a zero linear combination of these rows. This can be expressed in in matrix form as

$$\mathbf{a}^T \tilde{E} = \mathbf{0}^T \ . \tag{11}$$

where

$$\tilde{E} = \begin{bmatrix} 0\ldots0 & 1xx & x\ldots x & x\ldots x & x\ldots x \\ 0 & 0 & 1xx & x\ldots x & x\ldots x \\ 0 & 0 & 0 & 1x\ldots x & x\ldots x \\ 0 & 0 & 0 & 0 & 1x\ldots x \end{bmatrix}$$

is the matrix formed by retaining the nonzero rows of E. By matching both sides of equation (11) from left to right we find

$$a_1 = 0,\ a_2 = 0,\ \ldots,\ a_r = 0$$

so that the nonzero rows $\{\mathbf{s}_i^T,\ 1 \leq i \leq r\}$ of E are linearly independent. Thus, they form a basis of the row space of A. $\square$

Lemma 3: If A is $m \times n$ with $m < n$, the homogeneous system $A\mathbf{x} = \mathbf{0}$ admits a solution $\mathbf{x} \neq \mathbf{0}$.
Proof: $A\mathbf{x} = \mathbf{0}$ is equivalent to $E_r\mathbf{x} = \mathbf{0}$. Since $m < n$, some of the entries x_i of $\mathbf{x}$ are nonbasic variables. Selecting a nonzero value for at least one of the nonbasic variables, and solving for the basic variables, we obtain a nonzero solution of $E_r\mathbf{x} = \mathbf{0}$, and therefore of $A\mathbf{x} = \mathbf{0}$. $\square$

An important result is that all bases of a vector space V admit the same number of basis vectors, which is called the *dimension* of V.

Theorem 1: If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ are two bases of V, then $n = m$.

Proof: Assume that $m < n$, and let

$$V = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} \mathbf{w}_1 & \ldots & \mathbf{w}_m \end{bmatrix}$$

be matrices with columns $\{\mathbf{v}_j,\ 1 \leq j \leq qn\}$ and $\{\mathbf{w}_i,\ 1 \leq i \leq m\}$, respectively. Since $\{\mathbf{w}_i,\ 1 \leq i \leq m\}$ is a basis, each vector $\mathbf{v}_j$ can be expressed as a linear combination

$$\mathbf{v}_j = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i \ .$$

This can be written in matrix form as

$$V = WA \, ,$$

where A is the $m \times n$ matrix with entries a_{ij} for $1 \leq i \leq m$, $1 \leq j \leq n$. Since $m < n$, according to Lemma 3, we can can find $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$ so that $V\mathbf{x} = \mathbf{0}$. This means that the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly dependent, so they cannot form a basis of the vector space V, a contradiction. Thus $m = n$. $\square$

Example: Let V be the subspace of $\mathbb{R}^3$ formed by the vectors perpendicular to

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

This subspace is shown in Fig. 1 below.

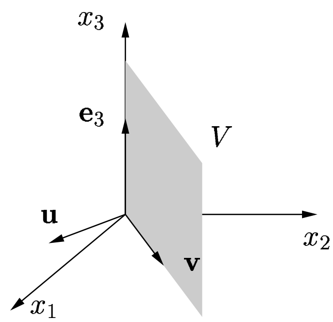

Figure 1: Suspace V of vectors orthogonal to $\mathbf{u}$.

Then a basis of V is given by

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and V has dimension 2. Geometrically, V is the plane perpendicular to the x_1-x_2 plane which bisects the first quadrant.

Because of Theorem 1, the dimension of a vector space is defined unambiguously as the number of vectors in a basis. Let us now define the *rank* of a matrix as the dimension of its row space. A key result is as follows.

Theorem 2: The rank of a matrix A is equal to the dimension of its column space, i.e.

$$\begin{aligned} \text{rank}\,(A) &\triangleq \text{dim row space } \mathcal{R}(A^T) &(12)\\ &= \text{dim column space } \mathcal{R}(A)\,. &(13) \end{aligned}$$

Proof: We show that

$$\begin{aligned} \text{dim row space of } A &= \text{dim row space of } E_r = r &(14)\\ = \text{dim column space of } E_r &= \text{dim column space of } A\,. &(15) \end{aligned}$$

Note in this respect that although elementary row operations do not change the row space, i.e., row space of A = row space of $E_r = MA$, they change the column space, i.e., in general we have column space of $A \neq$ column space of E_r. However, as is shown below, the dimension of the column space is not affected by elementary row operations. To prove (14), note that according to Lemma 2, the dimension of the row space of $A =$ the number of nonzero rows of $E_r = r$.

Then, if we consider the reduced row echelon form

$$E_r = \begin{bmatrix} 0\ldots 0 & 1xx & 0x\ldots x & 0x\ldots x \\ 0 & 0 & 1x\ldots x & 0x\ldots x \\ 0 & 0 & 0 & 1x\ldots x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we see that the pivot columns

$$\mathbf{f}_{p_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_1 \ , \ \mathbf{f}_{p_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_2 \ , \ \mathbf{f}_{p_r} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_r$$

are clearly linearly independent, since they are the first r vectors of the orthonormal basis of $\mathbb{R}^m$. Furthermore, all the nonpivot columns of E_r can be expressed by inspection as linear combinations of the pivot columns $\{\mathbf{f}_{p_1}, \ldots, \mathbf{f}_{p_r}\}$. Thus, the pivot columns $\{\mathbf{f}_{p_1}, \ldots, \mathbf{f}_{p_r}\}$ form a basis of the column space of E_r, so

$$r = \text{ dim column space of } E_r \, .$$

Next, we observe that $A\mathbf{x} = \mathbf{0}$ and $E_r\mathbf{x} = \mathbf{0}$ are equivalent since they can be obtained from each other by premultiplication by M or M^{-1}, respectively. This means that the set of dependence relations between the columns of E_r and those of A are the same. Thus, if $\{\mathbf{c}_{p_1} \ldots \mathbf{c}_{p_r}\}$ are the columns of A corresponding to the pivot columns of E_r, they are linearly independent (the pivot columns of E_r are independent) and all the other columns of A can be expressed as linear combinations of $\{\mathbf{c}_{p_1}, \ldots, \mathbf{c}_{p_r}\}$. This means that $\{\mathbf{c}_{p_1}, \ldots, \mathbf{c}_{p_r}\}$ is a basis of the column space of A, so that

$$r = \text{ dim column space of } A \, , \tag{16}$$

which proves Theorem 2. □

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} .$$

Using elementary row operations, we find

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E_r$$

A basis of the row space of A is given by

$$\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \} .$$

Also, since the pivots of E_r are located in the 1st and 2nd columns, a basis of the column space is

$$\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \} ,$$

so that dim row space of $A =$ dim column space of $A = \text{rank}(A) = 2$. Note also that the column space of A is different from the column space of E_r, which is spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \} .$$

Yet they have the same dimension.

A consequence of the previous result is as follows.

Theorem 3: An $n \times n$ matrix A is invertible if and only if rank $(A) = n$.

Proof: If A is invertible, there exists a matrix B such that $BA = AB = I_n$. From these identities we can immediately conclude that the columns and rows of A span

$$\text{column space of } A = \text{ row space of } A = \mathbb{R}^n \ ,$$

so that $\text{rank}(A) = n$. Conversely if A is $n \times n$ with $\text{rank}(A) = n$, its reduced row echelon form E_r will have no zero row, i.e., each row contains a pivot. Since it is square, all its columns will be pivot columns, so that $E_r = MA = I_n$, which shows that A is invertible.□