UNIVERSITY OF CALIFORNIA, DAVIS
Department of Electrical and Computer Engineering

EEC 250 | Linear Systems and Signals | Fall 2009

Lecture 1

Lecture Topics:

a) Vector space, subspace

b) Matrices: sum, product, invertible matrix

c) Column and row spaces, right and left nullspaces

Vector space: A vector space is constituted of a pair (V, S) where V is a set of vectors and S a field of scalars. Throughout this course S will be either $\mathbb{R}$ (the field of real numbers) or $\mathbb{C}$ (the field of complex numbers). The pair (V, S) is endowed with two operations: the addition of two vectors, and the multiplication of a vector by a scalar. In the following a vector $\mathbf{x} \in V$ will be denoted with in bold font and a scalar $a \in S$ in regular font.

Vector addition: For $\mathbf{x}$, $\mathbf{y} \in V$, the addition $\mathbf{x} + \mathbf{y} \in V$ must satisfy the following axioms.

(i) Associativity:
$$\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$$
for all $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in V$.

(ii) Commutativity:
$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}\,.$$

(iii) There exists a zero vector $\mathbf{0}$ such that
$$\mathbf{x} + \mathbf{0} = \mathbf{x}\,.$$

(iv) For every vector $\mathbf{x} \in V$, there exists an additive inverse $-\mathbf{x} \in V$ such that $\mathbf{x} - \mathbf{x} = \mathbf{0}$.

Scalar multiplication: For $a \in S$ and $\mathbf{x} \in V$, the multiplication $a\mathbf{x} \in V$ must satisfy the following axioms.

(i) Associativity: $a(b\mathbf{x}) = (ab)\mathbf{x}$ for all $a, b \in S$ and $\mathbf{x} \in V$.

(ii) Distributivity of the scalar multiplication with respect to the vector addition:
$$a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + b\mathbf{y}$$
for all $a \in S$ and $\mathbf{x}$, $\mathbf{y} \in V$.

(iii) Distributivity of the multiplication with respect to the scalar addition:

$$(a+b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$$

for all $a, b \in S$ and $\mathbf{x} \in V$.

(iv) There exists a unit scalar $1 \in S$ such that $1.\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.

Example 1: $V = \mathbb{R}^n$, $S = \mathbb{R}$. In this case, an arbitrary vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$

is an n-tuple of real entries x_i with $1 \leq i \leq n$. Then if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} \quad , \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_i \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$$

and $a \in \mathbb{R}$, we have

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_i + y_i \\ \vdots \\ x_n + y_n \end{bmatrix} \quad \text{and} \quad a\mathbf{x} = \begin{bmatrix} ax_1 \\ \vdots \\ ax_i \\ \vdots \\ ax_n \end{bmatrix} .$$

The zero-vector and additive inverse $-\mathbf{x}$ are given respectively by

$$\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad -\mathbf{x} = \begin{bmatrix} -x_1 \\ \vdots \\ -x_i \\ \vdots \\ -x_n \end{bmatrix} .$$

When $n = 3$, $\mathbb{R}^3$ is the set of real vectors in 3 dimensions, and the addition $\mathbf{x} + \mathbf{y}$ and scalar multiplication $a\mathbf{x}$ can be represented geometrically as shown in Fig. 1 below. Note that the vector $a\mathbf{x}$ is colinear with $\mathbf{x}$.

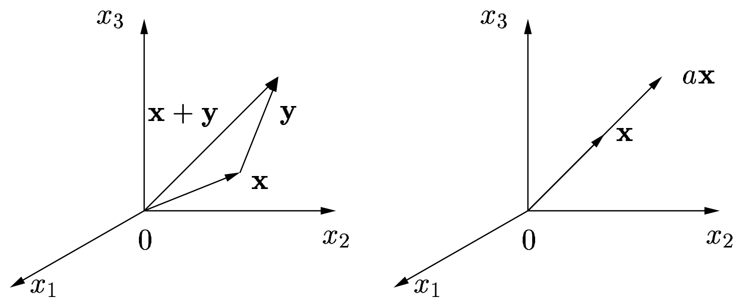

Figure 1: Vector addition and multiplication of a vector by a scalar in $\mathbb{R}^3$.

Also, an arbitrary vector of $\mathbb{R}^n$ can be represented as

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i \tag{1}$$

where for $1 \leq i \leq n$, the entries of basis vector are all zero, except for its i-th entry which equals one. The basis $\{\mathbf{e}_i,\ 1 \leq i \leq n\}$ is orthonormal since

$$\mathbf{e}_i^T \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j\,. \end{cases}$$

Example 2: Let $S = \mathbb{R}$, and let V be the space of piecewise continuous real functions over interval $[0, T]$. A function $f(t)$, $0 \leq t \leq T$ belonging to this space is shown in Fig. 2.

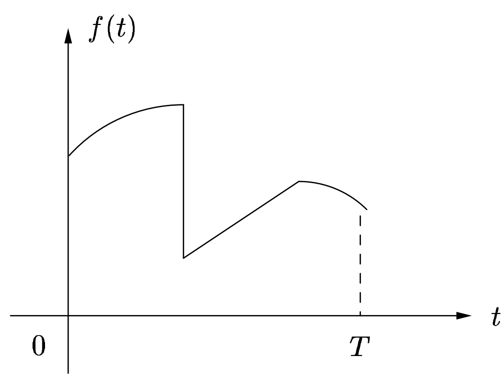

Figure 2: Piecewise continuous function over interval $[0, T]$.

The sum of two such functions is given by

$$(f + g)(t) = f(t) + g(t)\ ,$$

for $0 \leq t \leq T$, which is also piecewise continuous. The scalar multiplication $af(t)$ corresponds to scaling the function $f(t)$ by a.

The inner product of two functions is given by

$$\langle f, g \rangle = \int_0^T f(t)\, g(t)\, dt.$$

Then, if we consider

$$c_k(t) = \sqrt{\frac{2}{T}} \cos(k\omega_0 t) \quad , \quad s_k(t) = \sqrt{\frac{2}{T}} \sin(k\omega_0 t) \; ,$$

with k integer, where $\omega_0 = 2\pi/T$ is the fundamental frequency associated to the interval $[0, T]$, the functions $\{c_k(t),\, k \geq 0\}$ and $\{s_k(t),\, k \geq 1\}$ are orthonormal since

$$\begin{aligned} \langle c_k,\, c_\ell \rangle &= \langle s_k,\, s_\ell \rangle = \delta_{k\ell} \\ \langle c_k,\, s_\ell \rangle &= 0 \end{aligned}$$

for all k, ℓ. Furthermore, an arbitrary piecewise continuous function $f(t)$ over $[0, T]$ can be expressed in Fourier series as

$$f(t) = \sum_{k=0}^{\infty} a_k c_k(t) + \sum_{k=1}^{\infty} b_k s_k(t) \tag{2}$$

where the Fourier coefficients $\{a_k,\, k \geq 0\}$ and $\{b_k,\, k \geq 1\}$ are given by

$$a_k = \int_0^T f(t) c_k(t) dt \quad , \quad b_k = \int_0^T f(t) s_k(t) dt \; .$$

Subspace: If V is a vector space and W is a set of vectors from V, W is a subspace of V if it is closed under the operation of vector addition and scalar multiplication, i.e., if $\mathbf{x}$ and $\mathbf{y}$ are arbitrary vectors of W and a, $b \in S$, we have $a\mathbf{x} + b\mathbf{y} \in W$.

Example 1: Let $V = \mathbb{R}^3$, and let W be the set of vectors belonging to the x_1-x_2 plane, i.e.,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} .$$

This space is depicted in Fig. 3 below.

W is a subspace of $\mathbb{R}^3$ since for

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \quad , \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} \in W$$

we have

$$a\mathbf{x} + b\mathbf{y} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ 0 \end{bmatrix} ,$$

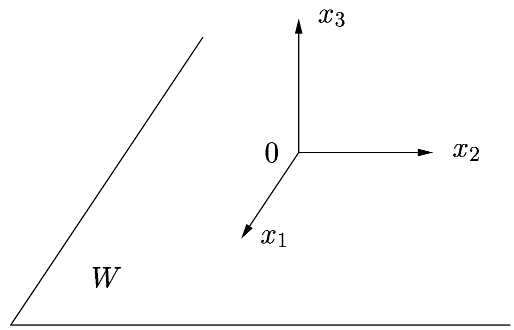

Figure 3: Space of vectors belonging to the x_1-x_2 plane.

where the vector $a\mathbf{x} + b\mathbf{y}$ belongs to the x_1-x_2 plane, since its last entry is zero.

Example 2: Let $V = \mathbb{R}^2$, and let W be the set of vectors $\mathbf{x}$ colinear with the two axes, i.e.,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

with either x_1 or x_2 zero. W is shown in Fig. 4 below. Then W is not a subspace since for

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad , \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in W$$

the vector

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin W .$$

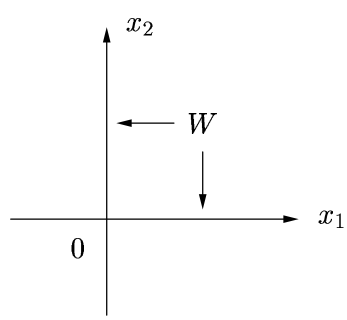

Figure 4: Set W of vectors colinear with the two axes.

Example 3: Let $V = \mathbb{R}^2$, and let W be the set of vectors $\mathbf{x}$ in the first quadrant, i.e.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $x_1 \geq 0$, $x_2 \geq 0$. This set is shown in Fig. 5 below. W is not a subspace since

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in W \text{ but } -\mathbf{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin W\,.$$

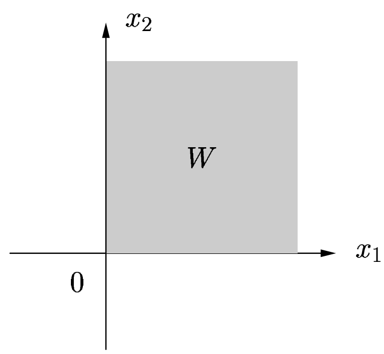

Figure 5: First quadrant of $\mathbb{R}^2$.

Linear dependence/independence: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is said to be linearly dependent if we can find a set of scalars a_i, $1 \leq i \leq k$, which not all zero, such that

$$\sum_{i=1}^{k} a_i \mathbf{v}_i = \mathbf{0}\,. \tag{3}$$

If no such scalars exist, i.e., if

$$\sum_{i=1}^{k} a_i \mathbf{v}_i = \mathbf{0} \Rightarrow a_1 = a_2 \ldots = a_k = 0\,,$$

the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are linearly independent.

Example 1: The vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

are linearly dependent since

$$\mathbf{v}_1 + \mathbf{v}_3 - 2\mathbf{v}_2 = \mathbf{0}\,.$$

Example 2: The vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

are linearly independent since

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \begin{bmatrix} a_1 + 2a_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has

$$a_2 = a_1 = 0$$

as its unique solution.

Spanning set: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ spans a vector space V if for an arbitrary vector $\mathbf{x}$ of V we can find some scalars a_i, $1 \le i \le k$ such that

$$\mathbf{x} = \sum_{i=1}^{k} a_i \mathbf{v}_i \ , \tag{4}$$

i.e., $\mathbf{x}$ is a linear combination of the vectors $\mathbf{v}_i$.

Example:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

form a spanning set of $\mathbb{R}^2$, but it is not a minimum spanning set since $\{\mathbf{v}_1, \mathbf{v}_2\}$ or $\{\mathbf{v}_1, \mathbf{v}_3\}$ are also spanning

Matrix: A matrix A is an m by n array of real or complex numbers a_{ij}:

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\ \vdots & & & & \vdots \\ a_{i1} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & & & & \vdots \\ a_{m1} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix} ,$$

where a_{ij} represents the element of row i and column j with $1 \le i \le m$ and $1 \le j \le n$. The special case of $n = 1$ is a column vector, and $n = 1$ is a row vector. Note that A can be written column-wise as

$$A = [\ \mathbf{c}_1 \ \ \ldots \ \ \mathbf{c}_j \ \ \ldots \ \ \mathbf{c}_n \ ]$$

where

$$\mathbf{c}_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix}$$

denotes the j-th column of A. Similarly, A can be written row-wise as

$$A = \begin{bmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_i^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix}$$

where

$$\mathbf{r}_i^T = \begin{bmatrix} a_{i1} & \dots & a_{ij} & \dots & a_{in} \end{bmatrix}$$

denotes the i-th row of A.

If both A and B are $m \times n$ matrices, their **sum** $C = A + B$ is an m by n matrix with elements $c_{ij} = a_{ij} + b_{ij}$. It is easy to check that the sum is associative and commutative, i.e.

$$(A + B) + C = A + (B + C) \qquad A + B = B + A\,.$$

If A and B are $m \times q$ and $q \times n$ matrices, respectively, the **product** $C = AB$ is an $m \times n$ matrix with elements

$$c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}\,, \tag{5}$$

i.e., the elements of the i-th row of A are multiplied term by term with those of the j-th column of B and the terms summed.

Example: Let

$$A = \begin{bmatrix} 1 & 3 & 7 \\ -1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$AB = C = \begin{bmatrix} 4 & 9 \\ -1 & 3 \end{bmatrix},$$

where the entries of C are obtained by using (5). For example

$$c_{11} = 1.1 + 3.1 + 7.0 = 4\,.$$

Note that in order to be able to compute the product of A and B, the number of columns of A must be equal to the number of rows of B. The product is associative: $(AB)C = A(BC)$, but not necessarily commutative, i.e., in general $AB \neq BA$. To see this, consider

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so that $AB \neq BA$. Also, the sum and product are distributive, i.e., $(A+B)C = AC + BC$ and $C(A+B) = CA + CB$.
A square matrix is an $m \times n$ matrix with $m = n$. The identity matrix I_n is an $n \times n$ matrix with ones on the diagonal and zeros elsewhere, i.e.,

$$a_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise} . \end{cases}$$

The identity has the property $AI_n = I_n A = A$ for any $n \times n$ matrix A.

An $n \times n$ matrix A is invertible if there exists an $n \times n$ matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$. We shall see later that if A is invertible then A^{-1} is unique; A^{-1} is called the inverse of A.

Theorem: If A is invertible, then $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for each choice of $\mathbf{b}$, and $\mathbf{y}^T A = \mathbf{c}^T$ has a unique solution $\mathbf{y}$ for each $\mathbf{c}$.

Proof: If $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$, pre-multiplying this equation by A^{-1} yields $A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$, or equivalently

$$I_n \mathbf{x} = \mathbf{x} = A^{-1}\mathbf{b} .$$

Since $A^{-1}\mathbf{b}$ is uniquely specified, the solution is unique. Similarly post-multiplying $\mathbf{y}^T A = \mathbf{c}^T$ by A^{-1} yields $\mathbf{y}^T = \mathbf{c}^T A^{-1}$. $\square$

Matrix transpose: The transpose of an arbitrary $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix $A^T = (b_{ij})$ with $b_{ij} = a_{ji}$. Under transposition the rows of A become the columns of A^T, and vice-versa.

Example: If

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -7 & 3 & 2 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & -7 \\ 0 & 3 \\ 1 & 2 \end{bmatrix} .$$

To an arbitrary $m \times n$ matrix A, we can associate four fundamental spaces.

The **column space** $\mathcal{R}(A)$ is the subspace of $\mathbb{R}^m$ spanned by the columns $\mathbf{c}_j$ of A, i.e., it is the set of vectors $\mathbf{b} \in \mathbb{R}^m$ that can be expressed as $\mathbf{b} = \sum_{j=1}^n x_j \mathbf{c}_j$ for some real numbers x_j, $1 \leq j \leq n$. In matrix form this gives $\mathbf{b} = A\mathbf{x}$ with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_j \\ x_n \end{bmatrix} ,$$

so

$$\mathcal{R}(A) = \{\mathbf{b} \in \mathbb{R}^m \ : \ \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\} .$$

The **row space** $R(A^T)$ is the space spanned by the rows $\mathbf{r}_i$ of A, or equivalently by the columns of A^T. It is the set of vectors $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{c}^T = \sum_{i=1}^m y_j \underline{r}_i$, or equivalently

$\mathbf{c}^T = \mathbf{y}^T A$ for some $\mathbf{y} \in \mathbb{R}^m$. Thus

$$\begin{aligned} \mathcal{R}(A^T) &= \{\mathbf{c} \in \mathbb{R}^n \,:\, \mathbf{c}^T = \mathbf{y}^T A \text{ for some } \mathbf{y} \in \mathbb{R}^m\} \\ &= \{\mathbf{c} \in \mathbb{R}^n \,:\, \mathbf{c} = A^T \mathbf{y} \text{ for some } \mathbf{y} \in \mathbb{R}^m\}\,. \end{aligned}$$

The **right null space** $N(A)$ is the set of solutions $\mathbf{x}$ of the homogeneous equation $A\mathbf{x} = \mathbf{0}$, or equivalently the set of dependence relations existing between the columns of A, i.e.

$$\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}\,.$$

The **left null space** $\mathcal{N}(A^T)$ is the set of solutions $\mathbf{y} \in \mathbb{R}^m$ of $\mathbf{y}^T A = \mathbf{0}^T$, i.e.,

$$\mathcal{N}(A^T) = \{\mathbf{y} \in \mathbb{R}^m \,:\, \mathbf{y}^T A = \mathbf{0}^T\} = \{\mathbf{y} \in \mathbb{R}^m : A^T \mathbf{y} = \mathbf{0}\}\,.$$

Example: If

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

the right null space $\mathcal{N}(A)$ is obtained by solving

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives

$$x_1 = x_3 = -x_2\,,$$

so that $\mathcal{N}(A)$ is the space spanned by

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$