Sampling and reconstruction of bandlimited CT signals

Consider a CT bandlimited signal $x(t)$ with bandwidth $B$

so that

$$X(j\omega) = 0 \text{ for } |\omega| > B \quad (1)$$

e.g.

![Diagram showing the frequency response $X(j\omega)$ with a triangle indicating the frequency range from $-B$ to $B$.]

where

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t) \, dt = \text{CT FT of } x(t).$$

A very important result which underpins all of DSP products, consumer tools such as CDs, DVDs, digital TV, digital communications, etc... is that $x(t)$ can be reconstructed exactly from its samples $x(nT_s)$ provided

$$T_s \leq \frac{T_N}{B}$$

Nyquist sampling period.
or equivalently provided

\[ \omega_s = \frac{2\pi}{T_s} = \text{sampling frequency} \rightarrow \frac{2\pi}{T_N} = 2B = \omega_N \quad (2) \]

This important result is known as the **Nyquist sampling theorem**. One formula for reconstructing \( x(t) \) from its samples is given by

\[ x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \left( \frac{\pi}{T_s} \frac{(t-nT_s)}{T_s} \right)}{\pi \frac{(t-nT_s)}{T_s}} \quad (3) \]
with

\[ k(t) = \frac{\sin \left( \frac{\pi t}{T_s} \right)}{\pi t/T_s} = \text{interpolating function} \]

but we shall see later that when \( T_s < T_N \)

say \( T_s = 0.9T_N \) or \( 0.8T_N \), other choices of

interpolating functions are possible.

Note that the interpolation function \( k(t) \)

is such that

\[ k(0) = 1 \]

\[ k(\pi T_s) = \frac{\sin \left( \pi \frac{T_s}{T_s} \right)}{\pi \frac{T_s}{T_s}} = 0 \quad \text{for } n \neq 0 \]

so clearly by setting \( t = kT_s \) in (3) we find \( x(kT_s) = x(kT_s) \).
i.e. the interpolated function evaluated at the sampling times $kT_s$ coincides with the corresponding DT samples.

To analyze the sampling and interpolation process, it is convenient to model CT sampling as follows.

\[ x(\tilde{t}) \uparrow \mathcal{B} \downarrow \rightarrow x(t) \]

\[ s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \text{train of impulses located at } 0, \pm T_s, \pm 2T_s \]

\[ = \text{the sampling function} \]

Then sampled values of

\[ z(k) = s(t) \circ x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s) \]

where we have used

\[ x(kT_s) \delta(t - kT_s) = x(kT_s) \delta(t - kT_s) \]
So the information contained in $z(t)$ is equivalent to the information contained in the samples $x(nT_s)$. Let us now evaluate the CT FT of $z(t)$.

**Evaluation ($\mathcal{F}$)**

**First evaluation (obvious)**

\[
\mathcal{F}[z(t)] = Z(j\omega) = \sum_{n=-\infty}^{\infty} x(nT_s) \mathcal{F}[\delta(t-nT_s)]
\]

\[
= \sum_{n=-\infty}^{\infty} x(nT_s) \exp(-j\omega nT_s) \quad (8)
\]

(recall $\delta(t) \leftrightarrow 1$

$\delta(t-t_0) \leftrightarrow \exp(-j\omega t_0)$ here $t_0 = nT_s$)

If

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(nT_s) \exp(-j\omega n) \quad (9)
\]

\[
= \mathcal{F}\text{ FT of samples } x(nT_s) \quad \text{DT frequency}
\]

\[
Z(j\omega) = X(e^{j\omega T_s}) \quad \text{i.e. } \omega_0 = \omega T_s \quad \text{CT frequency}
\]

\[
(10)
\]
so $Z(j\omega)$ is just a way of representing the DT FT $X(e^{j\omega t})$ of the samples $x(nT)$. This justifies the introduction of the strange sampled-impulsive sampled signal $z(t)$.

Let us consider now a second, more interesting, evaluation of $Z(j\omega)$. Note that since

$$z(t) = s(t) * x(t)$$

we have

$$Z(j\omega) = \frac{1}{2\pi} S(j\omega) * X(j\omega)$$

where $S(j\omega) = \text{FT of sampling function } s(t)$

$$= \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \quad \text{with } \omega_s = \frac{2\pi}{T_s}$$

Lathi Table 4.1
To develop an intuition about (13) note that if a function is impulsive in one domain, it is periodic in the other domain, e.g.

- Time Fourier

\[ \delta(t-kT) \leftrightarrow \exp(-j\omega t) \text{ periodic} \]

- Impulsive

\[ \exp(-j\omega t) \leftrightarrow \frac{2\pi}{\omega} \delta(w-\omega) \text{ periodic} \]

Since \( s(t) \) is impulsive and periodic

\[ s(j\omega) \] is impulsive and periodic

Same form as \( s(t) \) but amplitude \( \omega_s \) instead of 1

Density between \( \omega_s \) and \( \omega_s \)

To evaluate (12) note that

\[ \mathcal{Z}(j\omega) = \frac{\omega_s}{2\pi} \sum_{k=\infty}^{\infty} \delta(w-k\omega_s) * X(j\omega) \]

\[ \mathcal{Z} = 1 \sum_{\infty}^{\infty} \frac{X(j\omega-k\omega_s)}{T_s} \]
This last expression is the key to understanding sampling.

We consider 2 cases:

**Case 1**

\[
\omega_s > 2B \quad \text{(no aliasing)}
\]

\[
\frac{1}{Ts} X(j(w + \omega_s)) \rightarrow \sum_{n=-\infty}^{\infty} X(jw) \frac{1}{Ts} \delta(w - nTs - \omega_s)
\]

Because \( \omega_s - B > B \), the copies \( \frac{1}{Ts} X(j(w + \omega_s)) \)

of the DTFT of signal \( x(t) \) do not overlap. We can therefore recover \( X(jw) \) by passing \( z(t) \) through a low-pass filter \( H(jw) \) such that

\[
H(jw) = \begin{cases} 
\frac{1}{Ts} & |w| \leq B \\
0 & |w| > \omega_s - B \end{cases}
\]
Then \( \hat{x}(\omega) = H(\omega)Z(\omega) = X(\omega) \)

\[
\hat{x}(t) = x(t) \quad (16)
\]

reconstructed
signal

so the signal \( x(t) \) is reconstructed exactly from
its samples \( x(nT_s) \).

Note that any filter satisfying (15) will work.
One filter that does the job is the ideal lowpass filter.

\[ H(j\omega) = \frac{\sin(\pi \frac{\omega}{\omega_s})}{\pi \frac{\omega}{\omega_s}} \]

\[ h(t) = \frac{\omega_s}{2\pi} \frac{\sin(\omega_s t)}{2} = \frac{\sin(\pi \frac{t}{\tau_s})}{\pi \frac{t}{\tau_s}} \] \hspace{1cm} (17)

which is the filter (4).

In this case

\[ \sum \left[ h(k) \right] \rightarrow x(k) \]

\[ x(k) = h(k) \ast \overline{z}(k) = h(k) \ast \sum_{n=-\infty}^{\infty} z(n\tau_s) \delta(k-n\tau_s) \]

\[ = \sum_{n=-\infty}^{\infty} z(n\tau_s) h(k-n\tau_s) \] \hspace{1cm} (18)

which is interpolation formula (3).
The advantage of the ideal lowpass filter is that it has no transition band, so it can be used even if \( \omega_s = 2B + \epsilon \) is an arbitrary small number.

Its disadvantage is that

\[
h(t) = \frac{\sin(\pi t/T_s)}{\pi t/T_s}
\]

decays slowly like \( 1/t \), so in the sum

\[
\sum_{k=-\infty}^{\infty} x(nT_s) \sin\left(\frac{\pi (t-kT_s)}{T_s}\right)
\]

more terms need to be retained to evaluate \( x(t) \) at some time \( t \).
A more practical choice would be to use the
raised cosine pulse filter

\[ h(w) = \begin{cases} 
  T_s & 0 \leq |w| \leq \frac{(1 - \alpha) \omega_s}{2} \\
  T_s \left( 1 - \sin \left( \frac{T_s}{2\alpha} (1 - \alpha) \omega_s \right) \right) & \frac{(1 + \alpha) \omega_s}{2} \leq |w| \leq \frac{(1 - \alpha) \omega_s}{2} \\
  0 & |w| > \frac{(1 + \alpha) \omega_s}{2} \end{cases} \]  

(19)

with impulse response

\[ h(t) = \frac{\sin \left( \pi \frac{t}{T} \right)}{\pi \frac{t}{T}} \frac{\cos \left( \alpha \pi \frac{t}{T} \right)}{1 - (2\alpha \frac{t}{T})^2} \]

(20)

an ideal LP filter

impulse response
Disadvantage

It requires

\[
\frac{(1-\alpha)\omega_s}{2} > B
\]

or equivalently

\[
\omega_s > (1-\alpha)\omega_s > 2B \quad \alpha = \text{excess ratio}
\]

\[\text{oversampling}\]

Advantage

\( h(t) \) decays like \( 1/t^3 \) fewer terms need to be used in interpolation formula (18)

Case 2: aliasing \( \omega_s < 2B \)

\[
A \quad \frac{2(j\omega)}{Ts} \quad \frac{1}{Ts} \quad X(j\omega)
\]

\[
\frac{A}{Ts} \quad 1 \quad X(j(\omega-\omega_s))
\]

\[
\text{The carriers } 1 \quad X(j(\omega-\omega_s)) \text{ overlap and it is not possible to recover } X(j\omega) / x(t)
\]

\[
\text{from } Z(j\omega) / z(t)
\]
Example 1

\[ x(t) = \cos(1000 \pi t) - 2 \cos(3000 \pi t) \]

\[ X(j\omega) = \pi \left[ \delta(\omega - 1000\pi) + \delta(\omega + 1000\pi) \right] - 2 \left[ \delta(\omega - 3000\pi) + \delta(\omega + 3000\pi) \right] \]

Signal bandwidth = largest frequency

\[ B = 3000\pi \]

so

\[ \omega_N = 3000\pi \]

= Nyquist frequency
Choice: $w_s = 5000\pi < w_n$ does not satisfy the Nyquist sampling criterion

$$T_s - \frac{2\pi}{w_n} = \frac{1}{2500} = 0.4 \text{ m sec}$$

$Z(jw) \frac{1}{T_s} \times \frac{1}{j(w - w_s)}$

Note that the frequency at $2000\pi$ is due to $\frac{1}{T_s} \times \frac{1}{j(w - w_s)}$

$-2000\pi$ is due to $\frac{1}{T_s} \times \frac{1}{j(w + w_s)}$

Suppose now we use the ideal lowpass filter

$\Delta H(jw)$

$T_s = \frac{1}{2500}$

$-2500\pi \leq \omega \leq 2500\pi$
Then

\[ \hat{X}(j\omega) = H(j\omega)Z(j\omega) \]

\[ = \pi \left( \delta(\omega-1000\pi) + \delta(\omega+1000\pi) \right) \]

\[ - 2 \left( \delta(\omega-2000\pi) + \delta(\omega+2000\pi) \right) \]

\[ \Rightarrow \text{reconstructed correctly} \]

\[ \hat{x}(t) = \cos(1000\pi t) - 2 \cos(2000\pi t) \]

cosine at the wrong frequency!

Choice 2: \( \omega_S = 8000\pi > \omega_N \) satisfies the Nyquist criterion

\[ T_S = \frac{2\pi}{\omega_S} = \frac{1}{4000} = 0.25 \text{ msec} \]

\[ \frac{1}{T_S} \times (j\omega) \quad \frac{1}{\pi} \times (j\omega) \quad \frac{1}{\pi} \times (j\omega - \omega_0) \]
and if we use the ideal low pass filter

\[ H(\omega) = \begin{cases} 1 & |\omega| < \omega_s \\ 0 & \text{otherwise} \end{cases} \]

\[ ts = \frac{1}{4000} \]

we get

\[ x(t) = x(t) \cdot \cos(1000\pi t) - 2\cos(3000\pi t) \]

exact reconstruction