Convergence Analysis of a Background Interstage Gain Calibration Technique for Pipelined ADCs*

Dong Wang, John P. Keane, Paul J. Hurst, Bernard C. Levy and Stephen H. Lewis

Department of Electrical and Computer Engineering
University of California, Davis, CA 95616

ABSTRACT

A mathematical framework for the convergence analysis of a pipelined ADC with background gain calibration is presented. The constraints on adaptation step size for mean convergence, for mean-squared convergence, and for signal-to-adaptation-noise ratio are derived. Furthermore, expressions for steady-state tap noise and for signal-to-adaptation-noise ratio are derived. The analysis results are verified with simulations.

1. INTRODUCTION

An analog-to-digital converter (ADC) compares an input sample V_{in} at time n to a reference voltage V_{ref} to generate a digital approximation $i_d[n]$ of the normalized input $i[n] = V_{in}[n]/V_{ref}$. A pipelined ADC is shown in Fig. 1. All signals are normalized to V_{ref}, so the input range is $(-1, +1)$. The analog-to-digital sub-converter (ADSC) generates an M-level digital estimation $D[n]$ of $i[n]$ by comparing it to a set of $M-1$ threshold levels. Using a digital-to-analog sub-converter (DASC), this digital word is then converted to an analog signal that is subtracted from $i[n]$ to form the residue $z[n]$. The residue is the input to a back-end ADC ($\mathrm{ADC_{BE}}$), which generates a digital output $z_d[n]$ that is a digital estimate of the back-end ADC input. The back-end ADC consists of an interstage amplifier with gain $1/m_0$ that generates an output $y[n]$, which is then quantized to produce $y_d[n]$. Assuming an ideal DASC with gain of one, the ADC output is calculated by

$$i_d[n] = D[n] + m_0 y_d[n]. \tag{1}$$

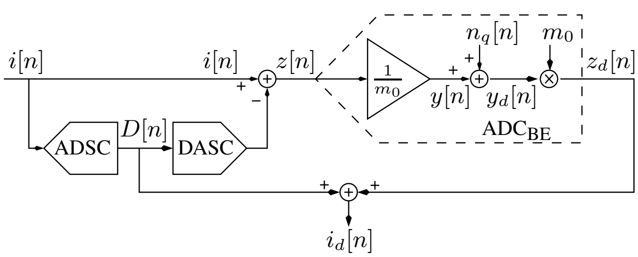

Fig. 1. Traditional pipelined ADC.

To prevent $z[n]$ from exceeding the input range of $\mathrm{ADC_{BE}}$, the magnitude of the residue $z[n]$ must be less than m_0. Even with an optimal spacing of ADSC and DASC levels, this requires that $M \geq 1/m_0$. In practice, due to errors in the stage ADSC and offsets present in the stage, over-range is possible, so a value of $M > 1/m_0$ is often used. This choice allows large comparator offsets without affecting converter linearity and is referred to as redundancy with digital correction [1]. When this redundancy is present, the linearity of the ADC transfer function can be independent of large errors in the ADSC thresholds. The primary remaining sources of linearity error are errors in the interstage amplifier and nonlinearity in the DASC. In a switched-capacitor implementation, DASC nonlinearity can be measured and corrected in the digital domain [2]. Hence, the primary source of ADC nonlinearity is due to errors in the interstage gain $1/m_0$. These errors stem from capacitor mismatch, insufficient op amp gain and charge injection. To address this problem, digital self-calibration is often used, where the value of m_0 in (1) is adjusted to match the actual interstage gain.

A background calibration technique to correct for interstage gain errors was introduced in [3] and extended in [4], [5]. In this method, a randomly modulated calibration signal is added to the DASC output in a pipelined stage, and its effect is measured by correlating the ADC output with the same modulation sequence. If the amplitude of the calibration signal is known, then the magnitude of m_0 can be estimated.

2. GAIN CALIBRATED ADC

The background gain calibration technique presented in [3], [5] is summarized by Fig. 2. A dither $R[n]\Delta D$ is added to both the output of the DASC and to the digital code $D[n]$, where $R[n]$ is a random sequence of ± 1, n is a time index, and ΔD is a constant. This dither can be added using a digital-to-analog converter ($\mathrm{DAC_d}$). In [4], [5], the dither is added in the digital domain to the ADSC output $D[n]$, so that the DASC and the $\mathrm{DAC_d}$ are combined. The dither thus introduced is used for gain calibration.

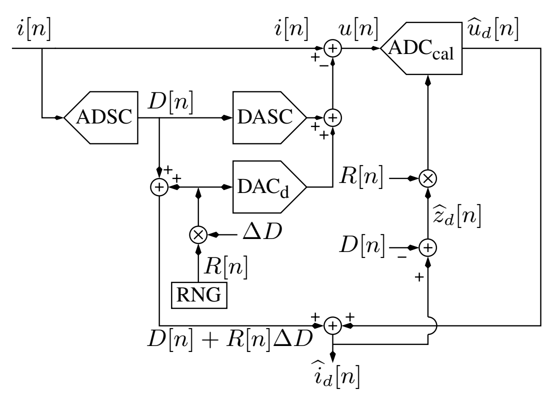

Fig. 2. ADC with gain calibration; $\mathrm{ADC_{cal}}$ is defined in Fig. 4.

Fig. 2 is difficult to analyze since the transfer function from $i[n]$ to $u[n]$ is nonlinear. To simplify the analysis, Fig. 2 is transformed as shown in Fig. 3. The advantage of the system in Fig. 3 is that it consists of a traditional nonlinear part followed by a linear $\mathrm{ADC_{BE}}$. The nonlinear part is not involved in calibration. Now the calibrating ADC ($\mathrm{ADC_{cal}}$) and the added dither are contained within $\mathrm{ADC_{BE}}$.

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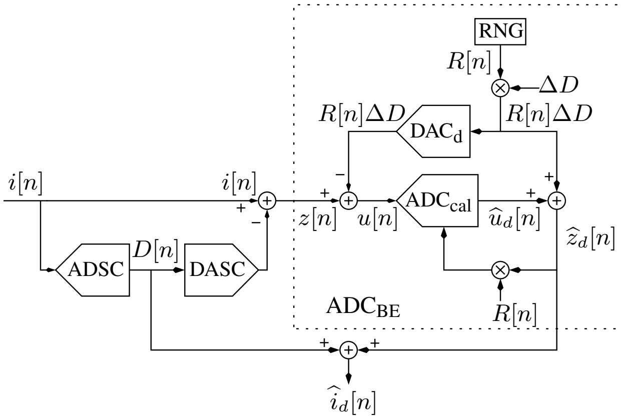

Fig. 3. ADC separated into linear $\mathrm{ADC_{BE}}$ and nonlinear parts.

The analysis in this paper will focus on a linear model of $\mathrm{ADC_{BE}}$ as shown in Fig. 4. Some definitions are given in Table I. When $\widehat{m}[n-1] = m_0$, $z_d[n]$ is defined as $\widehat{z}_d[n]$ in Fig. 4, and

$$z_d[n] \triangleq y_d[n]m_0 + R[n]\Delta D \tag{2}$$

with

$$y_d[n] = \frac{z[n]}{m_0} - \frac{R[n]\Delta D}{m_0} + n_q[n]. \tag{3}$$

Substituting (3) into (2) gives:

$$z_d[n] = z[n] + m_0 n_q[n]. \tag{4}$$

The estimated quantized-residual error ($\widetilde{z}_d[n]$) is defined as

$$\widetilde{z}_d[n] \triangleq z_d[n] - \widehat{z}_d[n]. \tag{5}$$

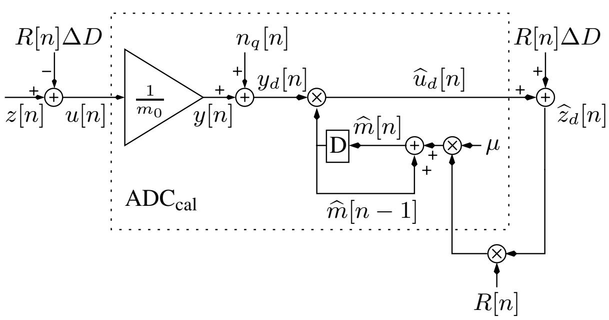

Fig. 4. Model of $\mathrm{ADC_{BE}}$ including $\mathrm{ADC_{cal}}$.

TABLE I
DEFINITION OF TERMS.

Term	Notation
1/(Amplifier gain)	m_0
Estimate of m	$\widehat{m}[n]$
Error in estimate of m	$\widetilde{m}[n]$
Residual	$z[n]$
Quantized residual	$z_d[n]$
Estimate of quantized residual	$\widehat{z}_d[n]$
Error in estimate of quantized residual	$\widetilde{z}_d[n]$
Residual for calibration	$u[n]$
Estimate of quantized residual for calibration	$\widehat{u}_d[n]$
Dither amplitude	ΔD
Random sequence of $\{+1, -1\}$	$R[n]$

This analysis assumes the binary sequence $R[n]$ is white and stationary with zero mean. The residual sequence $z[n]$ is also assumed to be white and stationary. In addition, $z[j]$ is assumed to be independent of $R[k]$ for all possible combinations of j, k.

The analysis is divided into three steps. Section 3 starts with the mean convergence analysis. Mean-squared convergence is analyzed in Section 4. Finally, steady-state performance is analyzed in Section 5.

3. MEAN CONVERGENCE

A constraint on adaptation step-size (μ) for mean convergence is derived in this section. Start with the update equation as defined in [5] and shown in Fig. 4:

$$\widehat{m}[n] \triangleq \widehat{m}[n-1] + \mu R[n]\widehat{z}_d[n]. \tag{6}$$

Subtracting m_0 from both sides and noting that $\widetilde{m}[n] \triangleq m_0 - \widehat{m}[n]$ gives:

$$\widetilde{m}[n] = \widetilde{m}[n-1] - \mu R[n]\widehat{z}_d[n]. \tag{7}$$

Substituting $\widehat{z}_d[n]$ from (5) into (7) gives:

$$\widetilde{m}[n] = \widetilde{m}[n-1] - \mu R[n](z_d[n] - \widetilde{z}_d[n]). \tag{8}$$

From Fig. 4 and (5), $\widetilde{z}_d[n]$ can be shown to be:

$$\widetilde{z}_d[n] = \widetilde{m}[n-1]y_d[n]. \tag{9}$$

Substituting $\widetilde{z}_d[n]$ from (9) into (8) and simplifying results in:

$$\widetilde{m}[n] = \widetilde{m}[n-1]A[n] - \mu R[n]z_d[n], \tag{10}$$

where

$$A[n] \triangleq 1 + \mu R[n]y_d[n]. \tag{11}$$

Taking the expectation of (10) and noting that $A[n]$ is independent of $\widetilde{m}[n-1]$ gives:

$$E\{\widetilde{m}[n]\} = E\{\widetilde{m}[n-1]\}E\{A[n]\} - \mu E\{R[n]z_d[n]\}. \tag{12}$$

Expanding (12) using (11), (3), and (4), then further noting that $R[n]$ is independent of $z[n]$ and has zero mean, results in:

$$E\{\widetilde{m}[n]\} = E\{\widetilde{m}[n-1]\}\left(1 - \mu\frac{\Delta D}{m_0}\right) + \mu E\{R[n]n_q[n]\}\Big(E\{\widetilde{m}[n-1]\} - m_0\Big). \tag{13}$$

It is generally true [5] that if the input signal varies enough to provide sufficient dither, then

$$E\{R[n]n_q[n]\} \approx 0. \tag{14}$$

When (14) is true, (13) reduces to:

$$E\{\widetilde{m}[n]\} = E\{\widetilde{m}[n-1]\}B, \tag{15}$$

where

$$B \triangleq 1 - \mu\frac{\Delta D}{m_0}. \tag{16}$$

Eq. (15) can be re-written in closed form as:

$$E\{\widetilde{m}[n]\} = E\{\widetilde{m}[0]\}B^n. \tag{17}$$

From (17), it can be seen that the sequence $E\{\widetilde{m}[n]\}$ converges if $-1 < B < 1$. This constrains μ to be in the range

$$\mathbf{0 < \mu < 2m_0/\Delta D}. \tag{18}$$

Thus when (18) is satisfied, $E\{\widehat{m}[n]\}$ converges to m_0.

It is interesting to note that $E\{\widehat{m}[n]\}$ can converge to m_0 in one time step when $\mu = \mu_{opt} \triangleq m_0/\Delta D$ [6, p. 180]. In practice, the optimum μ (μ_{opt}) cannot be used due to other limitations. A large μ can cause excessive error power in $\widehat{m}[n]$ ($E\{\widetilde{m}^2[n]\}$) and excessive noise power in $\widehat{z}_d[n]$ ($E\{\widetilde{z}_d^2[n]\}$). In some cases, it can cause $E\{\widetilde{m}^2[n]\}$ and $E\{\widetilde{z}_d^2[n]\}$ to diverge as shown in the next two sections.

4. MEAN-SQUARED CONVERGENCE

The mean convergence criterion as shown in (18) is a necessary but not a sufficient condition for the safe operation of the adaptive calibration in (6). The criterion for mean-squared convergence ($\lim_{n\to\infty} E\{\widetilde{m}^2[n]\} < \infty$) is also relevant in addition to the criterion for mean convergence ($\lim_{n\to\infty} E\{\widetilde{m}[n]\} \to 0$).

The criterion for mean-squared convergence is derived below. The steps in the derivation are: finding a relation between $\widetilde{m}^2[n]$ and $\widetilde{m}^2[n-1]$, taking expectation of $\widetilde{m}^2[n]$, relating $E\{\widetilde{m}^2[n]\}$ to $E\{\widetilde{m}^2[n-1]\}$, deriving an expression for $E\{\widetilde{m}^2[n]\}$ assuming mean convergence, and finding the convergence criterion for the sequence $E\{\widetilde{m}^2[n]\}$.

To find the relation between $\widetilde{m}^2[n]$ and $\widetilde{m}^2[n-1]$, start by squaring both sides of (10) to get:

$$\widetilde{m}^2[n] = \widetilde{m}^2[n-1]A^2[n] + \mu^2 {z_d}^2[n] - 2\mu R[n]\widetilde{m}[n-1]A[n]z_d[n]. \quad (19)$$

Taking the expectation of (19) and noting that $\widetilde{m}[n-1]$ is independent of $A[n]$, and $\widetilde{m}[n-1]$ is independent of $R[n]A[n]z_d[n]$ gives:

$$E\{\widetilde{m}^2[n]\} = E\{\widetilde{m}^2[n-1]\}E\{A^2[n]\} + \mu^2 E\{{z_d}^2[n]\} - 2\mu E\{\widetilde{m}[n-1]\}E\{R[n]A[n]z_d[n]\}. \quad (20)$$

Assuming mean convergence,

$$E\{\widetilde{m}^2[n]\} \overset{mean\ conv.}{\to} E\{\widetilde{m}^2[n-1]\}C + D, \quad (21)$$

where

$$C \triangleq E\{A^2[n]\}, \quad (22)$$

and

$$D \triangleq \mu^2 E\{{z_d}^2[n]\}. \quad (23)$$

The expression in (21) can be re-written as:

$$E\{\widetilde{m}^2[n]\} \overset{mean\ conv.}{\to} E\{\widetilde{m}^2[0]\}C^n + D\sum_{j=0}^{j=n-1} C^j. \quad (24)$$

From (24), it can be seen that the sequence $E\{\widetilde{m}^2[n]\}$ converges if

$$-1 < C < 1. \quad (25)$$

Next, an expanded expression for C is derived to find the constraint on μ for mean-squared convergence. Start by squaring $A[n]$ as defined in (11) to get

$$A^2[n] = 1 + 2\mu R[n]y_d[n] + \mu^2 {y_d}^2[n]. \quad (26)$$

Expanding the right hand side of (26) using (3), taking expectation, and using (14) gives:

$$C \triangleq E\{A^2[n]\} = 1 - 2\mu\frac{\Delta D}{m_0} + \mu^2 E\{{y_d}^2[n]\}. \quad (27)$$

With (25) and (27), it can be shown that the constraint on μ for mean-squared convergence is:

$$\mu < \frac{\mathbf{2\Delta D/m_0}}{\mathbf{E\{{y_d}^2[n]\}}}. \quad (28)$$

When this inequality is satisfied, the mean-squared error in $\widehat{m}[n]$ does not diverge. This result does not imply that $E\{\widetilde{m}^2[n]\}$ converges to zero. In Section 5, the steady-state value for $E\{\widetilde{m}^2[n]\}$ is found.

5. STEADY-STATE PERFORMANCE

In the previous two sections, expressions were derived for mean and mean-squared convergence. In Section 5-A an expression for steady-state adaptation noise in $\widehat{m}[n]$ is derived assuming mean-squared convergence. Furthermore, steady-state signal-to-adaption-noise ratio ($\mathrm{SNR_{ss}}$) and its limitation on μ are derived in Section 5-B.

A. Mean-squared error

Assuming mean-squared convergence, steady-state mean-squared error can be found by taking the limit of (24) as time n tends to infinity to give:

$$\lim_{n\to\infty} E\{\widetilde{m}^2[n]\} = \frac{D}{1-C}. \quad (29)$$

Plugging C from (27) and D from (23) into (29) gives the steady-state mean-squared error in $\widehat{m}[n]$:

$$\lim_{\mathbf{n}\to\infty} \mathbf{E\{\widetilde{m}^2[n]\}} = \frac{\mu\mathbf{E\{{z_d}^2[n]\}}}{\mathbf{2\Delta D/m_0} - \mu\mathbf{E\{{y_d}^2[n]\}}}. \quad (30)$$

For small μ, (30) can be approximated by

$$\lim_{n\to\infty} E\{\widetilde{m}^2[n]\} = \frac{\mu E\{{z_d}^2[n]\}}{2\Delta D/m_0}, \quad (31)$$

and $\widetilde{m}[n]$ has a Gaussian distribution [7, pp. 103–107].

Section 5-B will show that μ is limited by the SNR requirement on $\mathrm{ADC_{BE}}$. In practice this is often the most severe limitation on μ.

B. signal-to-adaptation-noise ratio

The steady-state signal-to-adaptation-noise ratio at the output of $\mathrm{ADC_{BE}}$ is derived below. First, steady-state signal power is defined as:

$$Q_{S_{ss}} \triangleq \lim_{n\to\infty} E\{{z_d}^2[n]\}. \quad (32)$$

Next, adaptation noise is defined here as $\widetilde{z}_d[n]$. Finally, steady-state adaptation noise power is defined as:

$$Q_{N_{ss}} \triangleq \lim_{n\to\infty} E\{\widetilde{z}_d^2[n]\}. \quad (33)$$

Plugging $\widetilde{z}_d[n]$ from (9) into (33) and noting that $\widetilde{m}[n-1]$ is independent of $y_d[n]$ gives:

$$Q_{N_{ss}} = \left(\lim_{n\to\infty} E\{\widetilde{m}^2[n-1]\}\right)E\{{y_d}^2[n]\}. \quad (34)$$

Plugging (30) into (34) gives:

$$Q_{N_{ss}} = \left(\frac{\mu E\{{z_d}^2[n-1]\}}{2\Delta D/m_0 - \mu E\{{y_d}^2[n-1]\}}\right)E\{{y_d}^2[n]\}. \quad (35)$$

Steady-state signal-to-adaptation-noise ratio is defined as:

$$SNR_{ss} \triangleq \frac{Q_{S_{ss}}}{Q_{N_{ss}}}. \quad (36)$$

Substituting (32) for $Q_{S_{ss}}$ and (35) for $Q_{N_{ss}}$ into (36) gives:

$$SNR_{ss} = \frac{E\{{z_d}^2[n]\}}{E\{{z_d}^2[n-1]\}}\frac{2\Delta D/m_0 - \mu E\{{y_d}^2[n-1]\}}{\mu E\{{y_d}^2[n]\}}. \quad (37)$$

Noting that both ${z_d}^2[n]$ and ${y_d}^2[n]$ are stationary gives:

$$SNR_{ss} = \frac{2\Delta D/m_0 - \mu E\{{y_d}^2[n]\}}{\mu E\{{y_d}^2[n]\}}. \quad (38)$$

Eq. (38) further simplifies to:

$$\mathbf{SNR_{ss}} = \frac{\mathbf{2}}{\mu}\frac{\mathbf{\Delta D/m_0}}{\mathbf{E\{{y_d}^2[n]\}}} - \mathbf{1}. \quad (39)$$

Eq. (39) shows that $\mathrm{SNR_{ss}}$ is inversely proportional to μ. Thus, constraining $\mathrm{SNR_{ss}}$ by a lower bound ($\mathrm{SNR_{min}}$) would imply an upper bound for μ (μ_{max}). This can be shown by re-arranging (39) to get:

$$\mu < \mu_{\mathbf{max}} = \mathbf{2}\frac{\mathbf{\Delta D/m_0}}{\mathbf{E\{y_d{}^2[n]\}}}\frac{\mathbf{1}}{\mathbf{SNR_{min}+1}}. \quad (40)$$

Eq. (40) allows the designer to pick μ and ΔD to satisfy a specific SNR requirement for $\mathrm{ADC_{BE}}$. In the next section, a practical example is simulated to verify the analysis.

6. Simulation results

A back-end ADC ($\mathrm{ADC_{BE}}$) with parameters listed in Table II was simulated. $R[n]$ is a sequence of independent and identically distributed (iid) random $\pm$1s with zero mean ($E\{R[n]\} = 0$). $z[n]$ is a sequence of iid random real numbers uniformly distributed within the interval $\left[\frac{-1}{16}, \frac{1}{16}\right)$. Furthermore, $z[j]$ is independent of $R[k]$ for all possible combinations of j, k. The constraints on μ for this particular setup are calculated to be: $\mu < 4$ for mean convergence, $\mu < 3$ for mean-squared convergence, and $\mu < 1.1 \times 10^{-5}$ for 9-bit SNR.

TABLE II
PARAMETERS FOR SIMULATED $\mathrm{ADC_{BE}}$.

Parameter	Value	Parameter	Value
Stages	9	bits	9
Slicers per Stage	2	ΔD	$\frac{1}{16}$
$1/m_0$ for first stage	8	$1/m_0$ for last 8 stages	2
Range for $z[n]$	$\left[\frac{-1}{16}, \frac{1}{16}\right)$	Range for $y[n]$	$[-1, 1)$

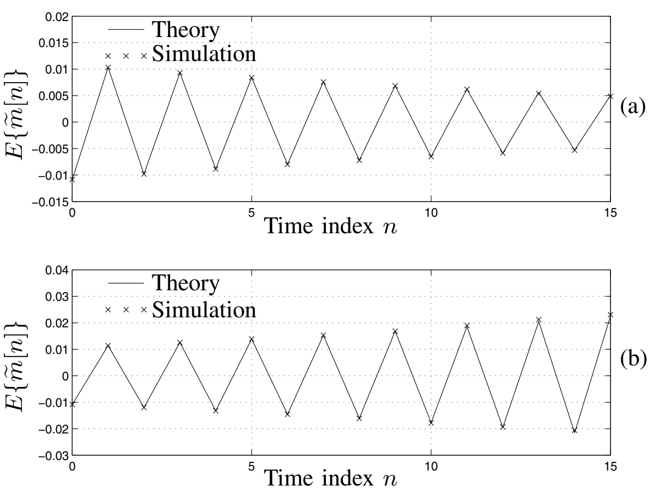

Fig. 5. $E\{\widehat{m}[n]\}$ versus time index for a) $\mu = 3.9$, b) $\mu = 4.1$.

Fig. 5 shows the theoretical and simulated expected value of the error in $\widehat{m}[n]$ for two cases: with $\mu = 3.9$ (where it converges) and with $\mu = 4.1$ (where it diverges). Fig. 6 shows the theoretical and simulated expected value of the mean-squared error in $\widehat{m}[n]$ for two cases: with $\mu = 2.9$ (where convergence is reached) and with $\mu = 3.1$ (where divergence occurs). Both Fig. 5 and Fig. 6 show good agreement between theory and simulation.

Fig. 7 plots the theoretical and simulated signal-to-adaptation-noise ratio (SNR) for $\mathrm{ADC_{BE}}$ with $\mu = 10^{-6}$. Furthermore, SNR including both quantization noise and adaptation noise ($\mathrm{SNR_{total}}$) is also plotted. This plot shows that when adaptation noise is high, $\mathrm{SNR_{total}}$ is dominated by adaptation noise; when adaptation noise is low, $\mathrm{SNR_{total}}$ is dominated by quantization noise.

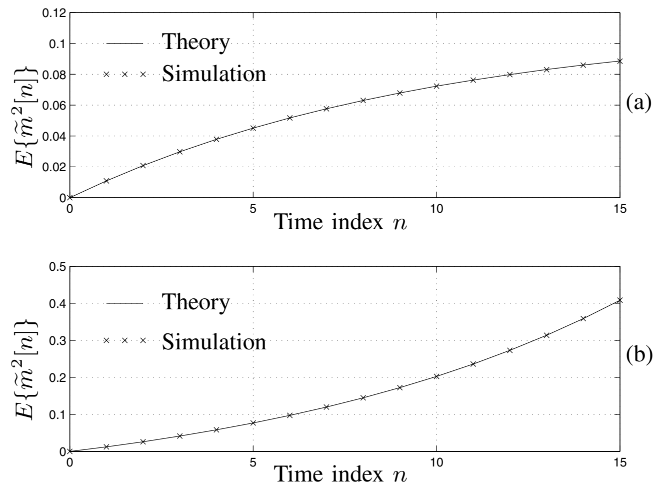

Fig. 6. Tap-noise power versus time index for a) $\mu = 2.9$, b) $\mu = 3.1$.

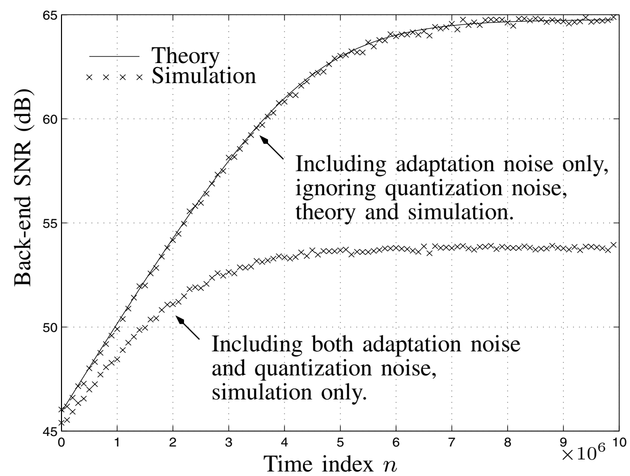

Fig. 7. Back-end SNR versus time index for $\mu = 10^{-6}$.

7. Conclusion

The gain-calibrated ADC as described in [5] was analyzed. Three constraints on μ were derived: a mean convergence constraint, a mean-squared convergence constraint, and a constraint due to SNR limitations. All three were verified with simulations. For most practical applications, μ is constrained by SNR as shown in (40).

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