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PID Controller Synthesis with Shifted Axis Pole Assignment for a Class of MIMO Systems

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***Abstract*— For certain classes of linear, time-invariant, multi-input multi-output plants, a systematic synthesis is developed for stabilization using Proportional+Integral+Derivative (PID) controllers, where the closed-loop poles can be assigned to the left of an axis shifted away from the origin. The real-parts of the closed-loop poles can be smaller than the real-part of the largest transmission-zero of the plant. Plant classes that admit PID controllers with this property include stable and unstable multi-input multi-input plants with transmission-zeros in the left-half complex-plane.**

I. INTRODUCTION

Proportional+Integral+Derivative (PID) controllers are preferred in many control designs since they are simple, have low-order, provide integral-action and hence, achieve asymptotic tracking of step-input references (e.g., [1]). Although the simplicity of PID controllers is desirable due to easy implementation and tuning, the order constraint presents a major restriction that only certain classes of plants can be controlled by using PID controllers. Rigorous PID synthesis methods based on modern control theory are explored recently in e.g., [7], [10], [6]. Sufficient conditions for PID stabilizability of linear, time-invariant (LTI), multi-input multi-output (MIMO) plants were given in [6] and several plant classes that admit PID controllers were identified.

An important criterion for control design is to assign the closed-loop poles sufficiently far from the imaginary-axis of the complex-plane in order for the system to have small time-constants and sufficient damping. Therefore, it is desirable for the closed-loop poles to have real-parts less than $-h$ for a pre-specified positive constant h. This design objective is achievable for certain LTI, MIMO plant classes as identified here. Assignment of the imaginary part of the closed-loop poles is not within the objectives or the scope of this work.

All plant classes that admit PID controllers are necessarily strongly stabilizable, although strong stabilizability is not sufficient for existence of PID controllers [6]. The integral-constant of the PID controller can be non-zero only if the plant has no zeros at the origin. Stable plants are obviously strongly stabilizable and they admit PID controllers. The additional objective of assigning values less than $-h$ to the real-part of the closed-loop poles can be achieved only for certain values of h [3]. The restriction on h is removed for stable plants that have no finite zeros with real-parts larger than the given $-h$; the closed-loop poles can be assigned to the left of this $-h$ for any chosen value of h as shown in [3]. The unstable plant classes investigated here have no finite zeros with real-parts larger than the given $-h$. For these plant classes, Propositions 1, 2 and 3 present systematic PID controller synthesis methods for closed-loop pole assignment to the left of the finite zero with the largest negative real-part.

The main results presented in Section II start with the problem statement and basic definitions. The three plant classes under consideration are studied under three subsections. An illustrative MIMO example is also given in Section II for the case of one plant zero at infinity based on the linearized model of a batch process [8]. The only goals of the design in this example are closed-loop stability with closed-loop poles to the left of a line at $-h$; due to the integral-action in the controller, the steady-state error due to step input references is zero and hence, asymptotic tracking is also achieved. The choice of the free parameters can be optimized with a chosen cost function. Section III gives concluding remarks.

Although we discuss continuous-time systems here, all results also apply to discrete-time systems with appropriate modifications.

Notation:
Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$ denote complex, real, positive real numbers. For $h \in \mathbb{R}_+ \cup \{0\}$, let $\mathcal{U}_h := \{s \in \mathbb{C} \mid \mathcal{R}e(s) \geq -h\} \cup \{\infty\}$. If $h = 0$, $\mathcal{U}_h = \mathcal{U}_0 := \{s \in \mathbb{C} \mid \mathcal{R}e(s) \geq 0\} \cup \{\infty\}$ is the extended closed right-half complex plane. Let $\mathbf{R_p}$ denote real proper rational functions of s. For $h \geq 0$, $\mathbf{S}_h \subset \mathbf{R_p}$ is the subset with no poles in $\mathcal{U}_h$. The set of matrices with entries in $\mathbf{S}_h$ is denoted by $\mathcal{M}(\mathbf{S}_h)$; ${\mathbf{S}_h}^{m\times m}$ is used instead of $\mathcal{M}(\mathbf{S}_h)$ to indicate the matrix size explicitly. A matrix $M \in \mathcal{M}(\mathbf{S}_h)$ is called $\mathbf{S}_h$-stable; $M \in \mathcal{M}(\mathbf{S}_h)$ is called $\mathbf{S}_h$-unimodular iff $M^{-1} \in \mathcal{M}(\mathbf{S}_h)$. The H_∞-norm of $M(s) \in \mathcal{M}(\mathbf{S}_h)$ is $\|M\| := \sup_{s\in\partial\mathcal{U}_h} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}_h$ is the boundary of $\mathcal{U}_h$. We drop (s) in transfer-matrices such as $G(s)$ where this causes no confusion. We use coprime factorizations over $\mathbf{S}_h$; i.e., for $G \in {\mathbf{R_p}}^{m\times m}$, $G = Y^{-1}X$ denotes a left-coprime-factorization (LCF) and $G = N_g D_g^{-1}$ denotes a right-coprime-factorization (RCF) where $X, Y, N_g, D_g \in \mathcal{M}(\mathbf{S}_h)$, $\det Y(\infty) \neq 0$, $\det D_g(\infty) \neq 0$. For MIMO transfer-functions, we refer to transmission-zeros simply as zeros; blocking-zeros are a subset of transmission-zeros. If $G \in {\mathbf{R_p}}^{m\times m}$ is full (normal) rank, then $z_o \in \mathcal{U}_h$ is called a transmission-zero of $G = Y^{-1}X$ if rank$X(z_o) < m$; $z_b \in \mathcal{U}_h$ is called a blocking-zero of $G = Y^{-1}X$ if $X(z_b) = 0$ and equivalenty, $G(z_b) = 0$.

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II. Main results

Consider the LTI, MIMO unity-feedback system $Sys(G,C)$ shown in Fig. 1, where $G \in \mathbf{R_p}^{m\times m}$ and $C \in \mathbf{R_p}^{m\times m}$ are the plant and controller transfer-functions. Assume that $Sys(G,C)$ is well-posed, G and C have no unstable hidden-modes, and $G \in \mathbf{R_p}^{m\times m}$ is full rank.

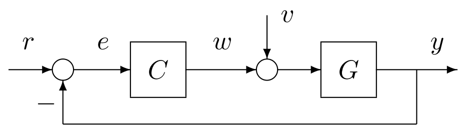

Fig. 1. Unity-Feedback System $Sys(G,C)$.

We consider the realizable form of proper PID controllers given by (1), where K_p, K_i, $K_d \in \mathbb{R}^{m\times m}$ are the proportional, integral, derivative constants, respectively, and $\tau \in \mathbb{R}_+$ (see [4]):

$$C_{pid}(s) = K_p + \frac{1}{s}K_i + \frac{s}{\tau s+1}K_d \ . \tag{1}$$

For implementation, a (typically fast) pole is added to the derivative term so that C_{pid} in (1) is proper. The integral-action in C_{pid} is present when $K_i \neq 0$. Subsets of PID controllers are obtained by setting one or two of the three constants equal to zero: (1) becomes a PI-controller C_{pi} when $K_d = 0$, an ID-controller C_{id} when $K_p = 0$, a PD-controller C_{pd} when $K_i = 0$, a P-controller C_p when $K_d = K_i = 0$, an I-controller C_i when $K_p = K_d = 0$, a D-controller C_d when $K_p = K_i = 0$.

Definition 1: a) $Sys(G,C)$ is said to be $\mathbf{S}_h$-stable if the closed-loop transfer-function from (r,v) to (y,w) is in $\mathcal{M}(\mathbf{S}_h)$. *b)* C is said to $\mathbf{S}_h$-stabilize G if C is proper and $Sys(G,C)$ is $\mathbf{S}_h$-stable. *c)* $G \in \mathbf{R_p}^{m\times m}$ is said to admit a PID controller such that the closed-loop poles of $Sys(G,C)$ are in $\mathbb{C}\setminus\mathcal{U}_h$ if there exists $C = C_{pid}$ as in (1) such that $Sys(G,C_{pid})$ is $\mathbf{S}_h$-stable. We say that G is $\mathbf{S}_h$-stabilizable by a PID controller, and C_{pid} is an $\mathbf{S}_h$-stabilizing PID controller. □

Let $G = Y^{-1}X$ be any LCF of G, $C = N_cD_c^{-1}$ be any RCF of C; for $G \in \mathbf{R_p}^{m\times m}$, $X, Y \in \mathcal{M}(\mathbf{S}_h)$, $\det Y(\infty) \neq 0$, and for $C \in \mathbf{R_p}^{n_u\times n_y}$, $N_c, D_c \in \mathcal{M}(\mathbf{S}_h)$, $\det D_c(\infty) \neq 0$. Then C is a $\mathbf{S}_h$-stabilizing controller for G if and only if

$$M := YD_c + XN_c \tag{2}$$

is $\mathbf{S}_h$-unimodular [11], [5].

The problem addressed here is the following: Suppose that $h \in \mathbb{R}_+$ is a given constant. Is there a PID controller C_{pid} that stabilizes the system $Sys(G,C_{pid})$ with a guaranteed stability margin, i.e., with real-parts of the closed-loop poles of the system $Sys(G,C_{pid})$ less than $-h$? It is clear that this goal is not achievable for some plants. Even when it is achievable, those $h \in \mathbb{R}_+$ for which the closed-loop poles can be placed to the left of a shifted-axis that goes through $-h$ may be restricted.

Let $\hat{s}$ and $\hat{G}$, $\hat{C}_{pid}$ be defined as

$$\hat{s} := s+h, \ \text{equivalently}, \quad s =: \hat{s}-h \ ; \tag{3}$$

$$\hat{G}(\hat{s}) \ := \ G(\hat{s}-h) \ ; \tag{4}$$

$$\begin{aligned}\hat{C}_{pid}(\hat{s}) &:= C_{pid}(\hat{s}-h) \\ &:= K_p + \frac{1}{\hat{s}-h}K_i + \frac{(\hat{s}-h)}{\tau(\hat{s}-h)+1}K_d \, .\end{aligned} \tag{5}$$

Then $C_{pid}(s)$ $\mathbf{S}_h$-stabilizes $G(s)$ if and only if $\hat{C}_{pid}(\hat{s})$ $\mathbf{S}_0$-stabilizes $\hat{G}(\hat{s})$. For any $\rho \in \mathbb{R}_+$, an RCF of $\hat{C}_{pid}(\hat{s})$ is given by

$$\hat{C}_{pid} = \left(\ \frac{(\hat{s}-h)}{\hat{s}+\rho}\hat{C}_{pid} \ \right) \left(\frac{(\hat{s}-h)}{\hat{s}+\rho}I \ \right)^{-1} \ . \tag{6}$$

We consider plant classes that admit PID controllers and identify values of $h \in \mathbb{R}$ such that the closed-loop poles lie to the left of $-h$. A necessary condition for existence of PID controllers with nonzero integral-constant K_i is that the plant $G(s)$ has no zeros (transmission-zeros or blocking-zeros) at $s = 0$ [6]. Therefore, all plants under consideration are assumed to be free of zeros at the origin (of the s-plane).

Let $\mathcal{G}_{ph}$ be the set of $\mathbf{S}_h$-stable $m\times m$ plants that have no (transmission or blocking) zeros at $s = 0$; i.e., for a given $h \in \mathbb{R}_+ \cup \{0\}$, let $\mathcal{G}_{ph} \subset \mathbf{S}_h{}^{m\times m}$ be defined as

$$\mathcal{G}_{ph} := \{ \ G(s) \in \mathbf{S}_h{}^{m\times m} \ | \ \det G(0) \neq 0\}. \tag{7}$$

For $G(s) \in \mathcal{G}_{ph}$, with $\hat{G}(\hat{s}) \ := \ G(\hat{s}-h)$, $\det G(0) \neq 0$ is equivalent to $\det \hat{G}(h) \neq 0$. The plants $G \in \mathcal{G}_{ph}$ may have transmission-zeros or blocking-zeros anywhere in $\mathbb{C}$ other than $s = 0$. There exist $\mathbf{S}_h$-stabilizing PID controllers for this classes of plants for certain values of $h \in \mathbb{R}_+$ [3].

In this paper we focus on plants that may have poles in the region $\mathcal{U}_h$. We consider the following three plant classes, which have no restrictions in the poles anywhere in the complex-plane $\mathbb{C}$ and no restrictions on the zeros in the region $\mathbb{C}\setminus\mathcal{U}_h$:

1) The first class of plants $\mathcal{G}_{zh}$ is the set of $m\times m$ plants that have no (transmission or blocking) zeros in $\mathcal{U}_h$; i.e., for a given $h \in \mathbb{R}_+ \cup \{0\}$, let $\mathcal{G}_{zh} \subset \mathbf{R_p}^{m\times m}$ be defined as

$$\mathcal{G}_{zh} := \{ \ G(s) \in \mathbf{R_p}^{m\times m} \ | \ G^{-1}(s) \in \mathbf{S}_h{}^{m\times m} \ \}. \tag{8}$$

In the single-input single-output (SISO) case, this class represents plants without zeros in $\mathcal{U}_h$ that have relative degree zero. Some plants in the set $\mathcal{G}_{zh}$ are not $\mathbf{S}_h$-stable; therefore, these plants either have poles in $\mathcal{U}_0$, or they are $\mathbf{S}_0$-stable but some poles have negative real-parts larger than the specified $-h$. Obviously, the plants in $\mathcal{G}_{zh}$ satisfy the necessary condition for existence of PID controllers with nonzero integral-constant K_i since the fact that they have no zeros in $\mathcal{U}_h$ implies that they have no zeros at $s = 0$.

2) The second class of plants $\mathcal{G}_{1\infty}$ is the set of $m\times m$ strictly-proper plants that have no (transmission or blocking) zeros in $\mathcal{U}_h$ except at infinity with multiplicity one, as

defined below: For a given $h \in \mathbb{R}_+ \cup \{0\}$, let $\mathcal{G}_{1\infty} \subset \mathbf{R_p}^{m\times m}$ be defined as

$$\mathcal{G}_{1\infty} := \{G(s) \in \mathbf{R_p}^{m\times m} \mid \frac{1}{s+a} G^{-1}(s) \in \mathbf{S}_h{}^{m\times m}, a > h\}. \quad (9)$$

In the SISO case, this class represents plants without zeros in $\mathcal{U}_h$ that have relative degree one. Some plants in the set $\mathcal{G}_{1\infty}$ are not $\mathbf{S}_h$-stable; these plants either have poles in $\mathcal{U}_0$, or they are $\mathbf{S}_0$-stable but some poles have negative real-parts larger than the specified $-h$. Obviously, the plants in $\mathcal{G}_{1\infty}$ satisfy the necessary condition for existence of PID controllers with nonzero integral-constant K_i since the fact that they have no zeros in $\mathcal{U}_h$ (other than at infinity) implies that they have no zeros at $s = 0$.

3) The third class of plants $\mathcal{G}_{2\infty}$ is the set of $m \times m$ strictly-proper plants that have no (transmission or blocking) zeros in $\mathcal{U}_h$ except at infinity with multiplicity two, as defined below: For a given $h \in \mathbb{R}_+ \cup \{0\}$, let $\mathcal{G}_{2\infty} \subset \mathbf{R_p}^{m\times m}$ be defined as

$$\mathcal{G}_{2\infty} := \{G(s) \in \mathbf{R_p}^{m\times m} \mid \frac{1}{(s+a)^2} G^{-1}(s) \in \mathbf{S}_h{}^{m\times m} \text{ for any } a > h\}. \quad (10)$$

In the SISO case, this class represents plants without zeros in $\mathcal{U}_h$ that have relative degree two. Some plants in the set $\mathcal{G}_{2\infty}$ are not $\mathbf{S}_h$-stable; these plants either have poles in $\mathcal{U}_0$, or they are $\mathbf{S}_0$-stable but some poles have negative real-parts larger than the specified $-h$. Obviously, the plants in $\mathcal{G}_{2\infty}$ satisfy the necessary condition for existence of PID controllers with nonzero integral-constant K_i since the fact that they have no zeros in $\mathcal{U}_h$ (other than at infinity) implies that they have no zeros at $s = 0$.

The set $\mathcal{G}_{ph} \cap \mathcal{G}_{1\infty}$ and the set $\mathcal{G}_{ph} \cap \mathcal{G}_{2\infty}$ correspond to $\mathbf{S}_h$-stable plants with no poles in $\mathcal{U}_h$, and no zeros in $\mathcal{U}_h$ other than (one or two, respectively) zeros at infinity.

A. *Plants with no zeros in $\mathcal{U}_h$*

Consider the class $\mathcal{G}_{zh}$ of $m \times m$ plants with no (transmission or blocking) zeros in $\mathcal{U}_h$ as described in (8). The plants $G \in \mathcal{G}_{zh}$ may not be $\mathbf{S}_h$-stable but $G^{-1} \in \mathcal{M}(\mathbf{S}_h)$; an LCF of $G(s)$ is

$$G = Y^{-1}X = (G^{-1})^{-1} I \ . \quad (11)$$

The plants in $\mathcal{G}_{zh}$ are strongly stabilizable, and they admit $\mathbf{S}_0$-stabilizing PID controllers [6]. Proposition 1 shows that these plants also admit $\mathbf{S}_h$-stabilizing PID controllers for any pre-specified $h \in \mathbb{R}_+$, and proposes a systematic PID controller synthesis.

Proposition 1: (*PID for plants with no $\mathcal{U}_h$-zeros*): Let $G \in \mathcal{G}_{zh}$. Then there exists an $\mathbf{S}_h$-stabilizing PID controller C_{pid}. Furthermore, C_{pid} can be designed as follows: Choose any nonsingular $\hat{K}_p \in \mathbb{R}^{m\times m}$. Choose any $K_d \in \mathbb{R}^{m\times m}$, and $\tau \in \mathbb{R}_+$ satisfying $\tau < 1/h$. Choose any $\alpha \in \mathbb{R}_+$ satisfying $\alpha > 2h$. Define $\Phi(\hat{s})$ as

$$\Phi(\hat{s}) := \hat{K}_p^{-1} \ [\ \hat{G}^{-1}(\hat{s}) + \frac{(\hat{s}-h)}{\tau(\hat{s}-h)+1} K_d \ ] \ . \quad (12)$$

Let $K_p = \beta \hat{K}_p$, $K_i = \alpha\,\beta \hat{K}_p$, where $\beta \in \mathbb{R}_+$ satisfies

$$\beta > \|\ \Phi(\hat{s})\ \| \ . \quad (13)$$

Then an $\mathbf{S}_h$-stabilizing PID controller C_{pid} is given by

$$C_{pid} = \beta \hat{K}_p + \frac{\alpha\,\beta}{s} \hat{K}_p + \frac{s}{\tau s + 1} K_d \ . \quad (14)$$

For $K_d = 0$, (14) is a PI-controller. $\square$

Proof of Proposition 1: Substitute $\hat{s} = s + h$ as in (3)-(5). Then an LCF of $\hat{G}(\hat{s})$ is $\hat{G}(\hat{s}) = \hat{Y}^{-1}\hat{X} := (\ \hat{G}^{-1}(\hat{s})\ )^{-1} I$. Write the controller $C_{pid}(s)$ given in (14) as

$$\begin{aligned} C_{pid}(s) &= (\frac{s}{s+\alpha} C_{pid})(\frac{s\,I}{s+\alpha})^{-1} \\ &= (\beta \hat{K}_p + \frac{s}{(s+\alpha)} \frac{s}{(\tau\, s+1)} K_d)(\frac{s\,I}{s+\alpha})^{-1}. \quad (15) \end{aligned}$$

Substitute $\hat{s} = s + h$ into (15) to obtain an RCF of $\hat{C}_{pid}(\hat{s})$ as in (6), with $\rho = \alpha - h$. Then

$$\begin{aligned} \hat{C}_{pid}(\hat{s}) \ = \ (\ &\beta \hat{K}_p \\ &+ \frac{(\hat{s}-h)}{(\hat{s}-h+\alpha)} \frac{(\hat{s}-h)}{(\tau(\hat{s}-h)+1)} K_d)(\frac{(\hat{s}-h)}{\hat{s}-h+\alpha} I)^{-1}, \quad (16) \end{aligned}$$

where $(1 - \tau h) \in \mathbb{R}_+$ and $(\alpha - h) \in \mathbb{R}_+$ by assumption. By (2), $\hat{C}_{pid}(\hat{s})$ in (16) stabilizes $\hat{G}(\hat{s})$ if and only if $M_\beta(\hat{s})$ is $\mathbf{S}_0$-unimodular:

$$\begin{aligned} M_\beta(\hat{s}) &= \hat{Y}(\hat{s})\, \frac{(\hat{s}-h)}{\hat{s}-h+\alpha} I + \hat{X}(\hat{s})\, \frac{(\hat{s}-h)}{\hat{s}-h+\alpha} \hat{C}_{pid}(\hat{s}) \\ &= \hat{G}^{-1}(\hat{s})\, \frac{(\hat{s}-h)}{\hat{s}-h+\alpha} I + \ \frac{(\hat{s}-h)}{\hat{s}-h+\alpha} \hat{C}_{pid}(\hat{s}) \\ &= \beta\, \hat{K}_p + \frac{(\hat{s}-h)}{(\hat{s}-h+\alpha)} [\ \hat{G}^{-1}(\hat{s}) + \frac{(\hat{s}-h)}{\tau(\hat{s}-h)+1} K_d\ ] \\ &= \beta \hat{K}_p (I + \frac{1}{\beta} \hat{K}_p^{-1} \frac{(\hat{s}-h)}{(\hat{s}-h+\alpha)} [\hat{G}^{-1}(\hat{s}) + \frac{(\hat{s}-h)}{\tau(\hat{s}-h)+1} K_d]) \\ &= \beta\, \hat{K}_p\ (\ I + \frac{1}{\beta}\ \Phi(\hat{s})\ \frac{(\hat{s}-h)}{(\hat{s}-h+\alpha)}\ )\ , \quad (17) \end{aligned}$$

where $\hat{K}_p$ is unimodular and $G^{-1}(s) \in \mathcal{M}(\mathbf{S}_h)$ by assumption. If $\alpha > 2h$ as assumed, then $\beta > \|\Phi(\hat{s})\|$ implies

$$\begin{aligned} \|\frac{1}{\beta}\ \Phi(\hat{s})\ \frac{(\hat{s}-h)}{(\hat{s}-h+\alpha)}\| \ &\le\ \frac{1}{\beta}\ \|\ \Phi(\hat{s})\ \|\, \|\frac{(\hat{s}-h)}{(\hat{s}-h+\alpha)}\| \\ &= \frac{1}{\beta}\ \|\ \Phi(\hat{s})\ \| < 1; \quad (18) \end{aligned}$$

hence, $M_\beta(\hat{s})$ in (17) is $\mathbf{S}_0$-unimodular. Therefore, $\hat{C}_{pid}(\hat{s})$ an $\mathbf{S}_0$-stabilizing controller for $\hat{G}(\hat{s})$; hence, C_{pid} is an $\mathbf{S}_h$-stabilizing controller for G. $\square$

B. *Strictly-proper plants with no other zeros in $\mathcal{U}_h$*

Consider the class $\mathcal{G}_{1\infty}$ of $m \times m$ strictly-proper plants that have no other (transmission or blocking) zeros in $\mathcal{U}_h$ as described in (9). The plants $G \in \mathcal{G}_{1\infty}$ are not all $\mathbf{S}_h$-stable but $\frac{1}{s+a} G^{-1} \in \mathcal{M}(\mathbf{S}_h)$ for any $a > h$. An LCF of $G(s)$ is

$$G = Y^{-1}X = (\frac{1}{s+a} G^{-1})^{-1} (\ \frac{1}{s+a} I\ )\ ; \quad (19)$$

in (19), $G(\infty) = 0$, and $Y(\infty)^{-1} = (s+a)G(s)|_{s\to\infty} = s\,G(s)|_{s\to\infty}$. The plants in $\mathcal{G}_{1\infty}$ are strongly stabilizable, and they admit $\mathbf{S}_0$-stabilizing PID controllers [6]. Proposition 2 shows that these plants also admit $\mathbf{S}_h$-stabilizing PID controllers for any pre-specified $h \in \mathbb{R}_+$, and proposes a systematic PID controller synthesis procedure.

Proposition 2: (*PID for plants with one zero at infinity*): Let $G \in \mathcal{G}_{1\infty}$. Then there exists an $\mathbf{S}_h$-stabilizing PID controller, and C_{pid} can be designed as follows: Let $Y(\infty)^{-1} := s\,G(s)|_{s\to\infty}$. Choose any $K_d \in \mathbb{R}^{m\times m}$, and $\tau \in \mathbb{R}_+$ satisfying $\tau < 1/h$. Choose any $\alpha \in \mathbb{R}_+$ satisfying $\alpha > h$. Define $\Psi(\hat{s})$ as

$$\Psi(\hat{s}) := [\hat{G}^{-1}(\hat{s}) + \frac{(\hat{s}-h)}{\tau(\hat{s}-h)+1}K_d]\frac{(\hat{s}-h)}{(\hat{s}-h+\alpha)}Y(\infty)^{-1} - (\hat{s}-h)I. \quad (20)$$

Let $K_p = \delta Y(\infty)$, $K_i = \alpha\,\delta Y(\infty)$, where $\delta \in \mathbb{R}_+$ satisfies

$$\delta > \|\,\Psi(\hat{s})\,\| + h\,. \quad (21)$$

Then an $\mathbf{S}_h$-stabilizing PID controller C_{pid} is given by

$$C_{pid} = \delta\,Y(\infty) + \frac{\alpha\,\delta}{s}Y(\infty) + \frac{s}{\tau s+1}K_d\,. \quad (22)$$

For $K_d = 0$, (22) is a PI-controller. □

Proof of Proposition 2: Substitute $\hat{s} = s + h$ as in (3)-(5). Then an LCF of $\hat{G}(\hat{s})$ is $\hat{G}(\hat{s}) = \hat{Y}^{-1}\hat{X} := (\frac{1}{\hat{s}-h+a}\hat{G}^{-1}(\hat{s}))^{-1}(\frac{1}{\hat{s}-h+a}I)$. Write the controller $C_{pid}(s)$ given in (22) as

$$C_{pid}(s) = (\frac{s}{s+\delta}C_{pid})(\frac{s\,I}{s+\delta})^{-1} = (\frac{(s+\alpha)}{s+\delta}\delta Y(\infty) + \frac{s}{(s+\delta)}\frac{s}{(\tau\,s+1)}K_d)(\frac{s\,I}{s+\delta})^{-1}. \quad (23)$$

Substitute $\hat{s} = s + h$ into (23) to obtain an RCF of $\hat{C}_{pid}(\hat{s})$ as in (6), with $\rho = \delta - h$. Then

$$\hat{C}_{pid}(\hat{s}) = (\frac{(\hat{s}-h+\alpha)}{\hat{s}-h+\delta}\delta\,Y(\infty) + \frac{(\hat{s}-h)}{(\hat{s}-h+\delta)}\frac{(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}K_d)(\frac{(\hat{s}-h)}{\hat{s}-h+\delta}I)^{-1}, \quad (24)$$

where $(1-\tau h) \in \mathbb{R}_+$ and $(\delta - h) \in \mathbb{R}_+$ by assumption. By (2), $\hat{C}_{pid}(\hat{s})$ in (24) stabilizes $\hat{G}(\hat{s})$ if and only if $M_\delta(\hat{s})$ is $\mathbf{S}_0$-unimodular:

$$\begin{aligned}
M_\delta(\hat{s}) &= \hat{Y}(\hat{s})\,\frac{(\hat{s}-h)}{(\hat{s}-h+\delta)}I + \hat{X}(\hat{s})\,\frac{(\hat{s}-h)}{(\hat{s}-h+\delta)}\hat{C}_{pid}(\hat{s}) \\
&= \hat{Y}(\hat{s})\,\frac{(\hat{s}-h)}{(\hat{s}-h+\delta)}I + \frac{1}{(\hat{s}-h+a)}I\,\frac{(\hat{s}-h)}{(\hat{s}-h+\delta)}\hat{C}_{pid}(\hat{s}) \\
&= [\frac{\delta}{\hat{s}-h+\delta}I + \frac{(\hat{s}-h)}{(\hat{s}-h+\delta)(\hat{s}-h+\alpha)}[\hat{Y}(\hat{s})(\hat{s}-h+a) \\
&\quad \frac{(\hat{s}-h)}{(\tau(\hat{s}-h)+1)}K_d]Y(\infty)^{-1}]\frac{(\hat{s}-h+\alpha)}{(\hat{s}-h+a)}Y(\infty) \\
&= [I + \frac{(\hat{s}-h)}{(\hat{s}-h+\delta)}[(\hat{Y}(\hat{s})Y(\infty)^{-1}\frac{(\hat{s}-h+a)}{(\hat{s}-h+\alpha)} - I) + \\
&\quad \frac{(\hat{s}-h)}{(\tau(\hat{s}-h)+1)(\hat{s}-h+\alpha)}K_d Y(\infty)^{-1}]]\frac{(\hat{s}-h+\alpha)}{(\hat{s}-h+a)}Y(\infty) \\
&= [I + \frac{1}{(\hat{s}-h+\delta)}\Psi(\hat{s})]\frac{(\hat{s}-h+\alpha)}{(\hat{s}-h+a)}Y(\infty). \quad (25)
\end{aligned}$$

Then $M_\delta(\hat{s})$ in (25) is $\mathbf{S}_0$-unimodular for $\delta \in \mathbb{R}_+$ satisfying (21) since $\delta > \|\Psi(\hat{s})\| + h$ implies

$$\begin{aligned}
\|\frac{1}{(\hat{s}-h+\delta)}\;\Psi(\hat{s})\,\| \;\le\; \|\;\frac{1}{(\hat{s}-h+\delta)}\;\|\,\|\,\Psi(\hat{s})\,\| \\
= \;\frac{1}{\delta-h}\;\|\,\Psi(\hat{s})\,\| < \;1.
\end{aligned}$$

Therefore, $\hat{C}_{pid}(\hat{s})$ an $\mathbf{S}_0$-stabilizing controller for $\hat{G}(\hat{s})$; hence, C_{pid} is an $\mathbf{S}_h$-stabilizing controller for G. □

In Example 1, we consider the two-input two-output linearized process model of an unstable batch reactor (also considered in e.g., [8], [9]):

Example 1: Consider the linearized model of an unstable batch reactor as $\dot{x} = Ax + Bu$, $y = Cx + Du$, where

$$A = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix},\; C = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix},\; D = 0\,.$$

The transfer-function G is given by:

$$G = \frac{1}{d}\begin{bmatrix} g_{11} + h_{11} & g_{12} + h_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

where $d = s^4 + 11.6680s^3 + 15.7538s^2 - 88.2911s + 5.5406$, $g_{11} = 0.0008s^2 + 29.2256s + 233.6673$, $g_{12} = -(21.1254s^2 + 111.0942s + 26.2766)$, $g_{21} = 5.6790s^3 + 42.6665s^2 - 68.8304s - 106.8024$, $g_{22} = 9.4304s + 15.1503$, $h_{11} = 0.0008s + 29.7745$, $h_{12} = -3.1460s^3 - 11.5490s^2 + 21.2688s - 5.5279$. The poles of G are at $\{1.9910, 0.0635, -5.0566, -8.6659\}$. The finite transmission-zeros are at $\{-1.1916, -5.0362, -5.5098\}$ and G has a blocking-zero at infinity. Therefore, we can design PID controllers such that the closed-loop poles have real-parts less than $-h$ for any $h < 1.1916$ following the procedure in Proposition 2. Suppose $h = 1$. Since the transmission zero -5.0598 and the pole -5.0566 are close to each other, $G^{-1}(s)$ can be approximated by a one order lower transfer matrix. By substituting $s = \hat{s} - h$, $\hat{G}^{-1}(\hat{s}) = G^{-1}(\hat{s} - h)$ becomes

$$\hat{G}^{-1}(\hat{s}) = \frac{1}{\Delta}\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} ,$$

where $G_{11} = 9.4304\hat{s} + 5.7199$, $G_{12} = 3.1460\hat{s}^3 + 23.2364\hat{s}^2 + 33.9146\hat{s} - 28.4925$, $G_{21} = -5.6790\hat{s}^3 - 25.6295\hat{s}^2 + 137.1264\hat{s} + 0.9845$, $G_{22} = 0.0008\hat{s}^2 + 29.2248\hat{s} + 234.2162$, $\Delta = 17.8661\hat{s}^2 + 75.5340\hat{s} + 13.8160$.

For simplicity, we choose $K_d = 0$ and design a PI-controller; the term K_d can be varied depending on other design specifications that are to be satisfied. Choosing $\alpha = 11 > h$ satisfying $\alpha > h$, with $Y(\infty)^{-1} = \begin{bmatrix} 0 & -3.146 \\ 5.679 & 0 \end{bmatrix}$, we compute $\| \Psi(\hat{s}) \| = 9.9385$. Therefore, (21) is satisfied for $\delta > 10.9385$. We choose $\delta = 12$ and obtain the PI-controller from (22) as

$$C_{pi} = \delta\, Y(\infty) + \frac{\alpha\delta}{s} Y(\infty) = (1 + \frac{11}{s}) \begin{bmatrix} 0 & 2.1130 \\ -3.8144 & 0 \end{bmatrix}.$$

The corresponding closed-loop poles with this controller are at $\{-8.0772 \pm j7.9389, -5.5164 \pm j7.8895, -7.2698, -5.0565, -1.2107\}$, and all have negative real-parts less than $-h = -1$. Due to the integral-action in the designed controller, the constant reference inputs applied at r are tracked asymptotically at the output y with zero steady-state error. Other values can be chosen for the various parameters if additional performance specifications are to be satisfied in addition to closed-loop stability with sufficient damping and asymptotic tracking of constant reference inputs. □

Example 2: The systematic design procedure in Proposition 2 has some free parameters that may be possible to choose in order to fulfill additional performance criteria. In this example, we will show a preliminary result regarding using these free parameters along the same line as in [2]. Consider the same system as in Example 1. In Fig. 2, the step response of the closed loop system is shown by the dotted line.

Suppose that we want to get less overshoot with a slower rising time. We can choose a model system $T_m(s)$ such that $T_{m12}(s) = T_{m21}(s) = 0$ and $T_{m11}(s) = T_{m22}(s)$ equals the same prototype second order model plant, with $\zeta = 0.7$ and $\omega_n = 10.56$; i.e., $T_m = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega s + \omega_n^2}$. The step response for $T_m(s)$ is shown in Fig. 2 as the dashed lines.

To make a comparison, we use the same $h = 1$ as in Example 1. To maintain the PID controller structure, we choose a small $\tau = 0.05$. Denote the step response of the model plant T_m as $s_m(t)$. The goal is to make the actual closed-loop step response as close as possible to $s_m(t)$. That is, we consider the cost function

$$error = \frac{1}{2.5}\sum_{i=1}^{2}\sum_{j=1}^{2}\int_0^{2.5} (s_{oij}(t) - s_{mij}(t))^2 dt, \quad (26)$$

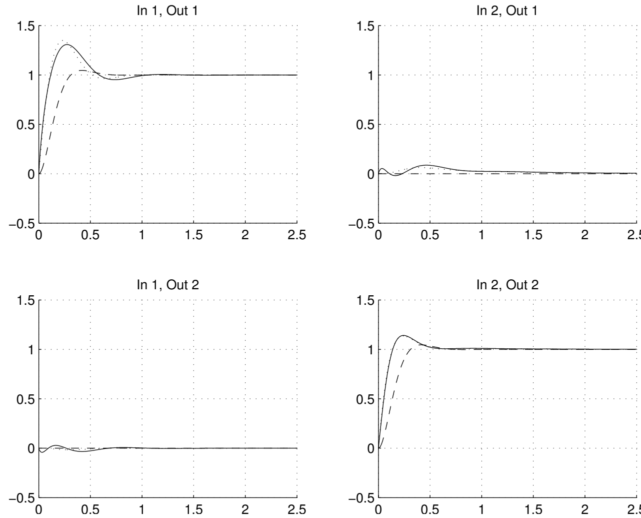

Fig. 2. Step responses for three transfer matrices in Example 2

where $s_o(t)$ denotes the step response for a choice of (α, δ, K_d). The goal is to minimize $error$ by choosing the best (α, δ, K_d), subject the simple linear constraint ($\alpha > h$ and the complex nonlinear constraint (21). The MATLAB function "fmincon" is used to solve the problem by using the controller in Example 1 as the initial point. The optimal controller has the following coefficients in (22), and shows the derivative part does help to improve the performance:

$$\alpha = 10.3368, \quad \delta = 11.6285, \quad K_d = \begin{bmatrix} -0.0280 & -0.0169 \\ -0.0848 & -0.0597 \end{bmatrix}.$$

The step response of the optimal design corresponds to the given model plant is shown in Fig. 2 by the solid lines. We can see that it is closer to the given model plant than the original design. □

C. Strictly-proper plants with two zeros at infinity

Consider the class $\mathcal{G}_{2\infty}$ of $m \times m$ strictly-proper plants that have no other (transmission or blocking) zeros in $\mathcal{U}_h$ as described in (10). The plants $G \in \mathcal{G}_{2\infty}$ are not all $\mathbf{S}_h$-stable but $\frac{1}{(s+a)^2} G^{-1} \in \mathcal{M}(\mathbf{S}_h)$ for any $a > h$. An LCF of $G(s)$ is

$$G = Y^{-1}X = (\frac{1}{(s+a)^2} G^{-1})^{-1} (\frac{1}{(s+a)^2} I) \; ; \quad (27)$$

in (27), $G(\infty) = 0$, and $Y(\infty)^{-1} = (s+a)^2 G(s)|_{s\to\infty} = s^2 G(s)|_{s\to\infty}$. The plants in $\mathcal{G}_{2\infty}$ are strongly stabilizable, and they admit $\mathbf{S}_0$-stabilizing PID controllers [6]. Proposition 3 shows that these plants also admit $\mathbf{S}_h$-stabilizing PID controllers for any pre-specified $h \in \mathbb{R}_+$, and proposes a systematic PID controller synthesis procedure.

Proposition 3: (*PID for plants with two zeros at infinity*): Let $G \in \mathcal{G}_{2\infty}$. Then there exists an $\mathbf{S}_h$-stabilizing PID controller, and C_{pid} can be designed as follows: Let $Y(\infty)^{-1} := s^2 G(s)|_{s\to\infty}$. Choose any $\alpha \in \mathbb{R}_+$ satisfying $\alpha > h$. Define $\Gamma(\hat{s})$ as

$$\Gamma(\hat{s}) := \frac{(\hat{s} - h)}{(\hat{s} - h + \alpha)^2} \hat{G}^{-1}(\hat{s}) Y(\infty)^{-1} - \hat{s}\, I \; . \quad (28)$$

Let $\mu \in \mathbb{R}_+$ satisfy

$$\mu > 2 \parallel \Gamma(\hat{s}) \parallel + h \ . \tag{29}$$

Then an $\mathbf{S}_h$-stabilizing PID controller C_{pid} is given by

$$C_{pid} = \frac{(\mu - h)^2\,(s+\alpha)^2}{s\,(s+2\mu-h)}\,Y(\infty) \ . \tag{30}$$

$\square$

Proof of Proposition 3: Substitute $\hat{s} = s + h$ as in (3)-(5). Then an LCF of $\hat{G}(\hat{s})$ is $\hat{G}(\hat{s}) = \hat{Y}^{-1}\hat{X} := (\frac{1}{(\hat{s}-h+a)^2}\hat{G}^{-1}(\hat{s}))^{-1}(\frac{1}{(\hat{s}-h+a)^2}I)$. Write the controller $C_{pid}(s)$ given in (30) as

$$\begin{aligned} C_{pid}(s) &= (\frac{s(s+2\mu-h)}{(s+\mu)^2}C_{pid})(\frac{s(s+2\mu-h)}{(s+\mu)^2}I)^{-1} \\ &= ((\mu-h)^2\frac{(s+\alpha)^2}{(s+\mu)^2}Y(\infty))(\frac{s(s+2\mu-h)}{(s+\mu)^2}I)^{-1}. \end{aligned} \tag{31}$$

Substitute $\hat{s} = s + h$ into (31) to obtain an RCF of $\hat{C}_{pid}(\hat{s})$ as

$$\begin{aligned} &\hat{C}_{pid}(\hat{s}) = \\ &(\frac{(\mu-h)^2(\hat{s}-h+\alpha)^2}{(\hat{s}-h+\mu)^2}Y(\infty))(\frac{(\hat{s}-h)(\hat{s}+2\mu-2h)}{(\hat{s}-h+\mu)^2}I)^{-1}, \end{aligned} \tag{32}$$

where $(\mu - h) \in \mathbb{R}_+$ by (29). By (2), $\hat{C}_{pid}(\hat{s})$ in (32) stabilizes $\hat{G}(\hat{s})$ if and only if $M_\mu(\hat{s})$ is $\mathbf{S}_0$-unimodular:

$$\begin{aligned} M_\mu(\hat{s}) &= (\hat{Y}(\hat{s}) + \hat{X}(\hat{s})\hat{C}_{pid}(\hat{s}))\frac{(\hat{s}-h)(\hat{s}+2\mu-2h)}{(\hat{s}-h+\mu)^2} \\ &= \hat{Y}(\hat{s})\,\frac{(\hat{s}-h)(\hat{s}+2\mu-2h)}{(\hat{s}-h+\mu)^2}I \\ &\quad + \frac{1}{(\hat{s}-h+a)^2}I\frac{(\mu-h)^2(\hat{s}-h+\alpha)^2}{(\hat{s}-h+\mu)^2}Y(\infty) \\ &= \frac{(\hat{s}-h+\alpha)^2}{(\hat{s}-h+a)^2}[\frac{(\mu-h)^2}{(\hat{s}-h+\mu)^2}I + \\ &\frac{(\hat{s}+2(\mu-h))}{(\hat{s}-h+\mu)^2}\hat{Y}(\hat{s})\frac{(\hat{s}-h+a)^2}{(\hat{s}-h+\alpha)^2}(\hat{s}-h)Y(\infty)^{-1}]Y(\infty) \\ &= \frac{(\hat{s}-h+\alpha)^2}{(\hat{s}-h+a)^2}[I + \frac{(\hat{s}+2(\mu-h))}{(\hat{s}-h+\mu)^2}\ \Gamma(\hat{s})\ ]Y(\infty). \end{aligned} \tag{33}$$

Then $M_\mu(\hat{s})$ in (33) is $\mathbf{S}_0$-unimodular for $\mu \in \mathbb{R}_+$ satisfying (29) since $\mu > 2\,\|\Gamma(\hat{s})\| + h$ implies

$$\begin{aligned} \|\frac{(\hat{s}+2(\mu-h))}{(\hat{s}-h+\mu)^2}\ \Gamma(\hat{s})\ \| \ &\leq \ \|\ \frac{(\hat{s}+2(\mu-h))}{(\hat{s}-h+\mu)^2}\|\,\|\,\Gamma(\hat{s})\,\| \\ &= \ \frac{2}{\mu-h}\ \|\,\Gamma(\hat{s})\,\| < \ 1. \end{aligned}$$

Therefore, $\hat{C}_{pid}(\hat{s})$ is an $\mathbf{S}_0$-stabilizing controller for $\hat{G}(\hat{s})$; hence, C_{pid} is an $\mathbf{S}_h$-stabilizing controller for G. $\square$

Example 3: Consider the MIMO system

$$G = \begin{bmatrix} \frac{2(s+3)}{(s+2)(s-4)(s-8)} & \frac{1}{(s+5)(s+20)} \\ \frac{(s+5)}{10(s+3)(s+6)(s+7)} & \frac{(s+4)}{(s+9)(s^2-6s+12)} \end{bmatrix} , \tag{34}$$

which has no finite zeros with real-parts larger than -2.3064. Thus we can choose $h = 1$. By choosing $\alpha = 2$, we can compute $\parallel \Gamma(\hat{s}) \parallel = 79.1849$ from (28). Choose $\mu = 162$ to satisfy the inequality in (29). The maximum of the real-parts of the closed-loop poles is less than -1.1319. Thus the requirement is fulfilled. $\square$

III. Conclusions

Systematic PID controller designs were proposed for LTI, MIMO plants, where closed-loop poles are placed in the left-half complex-plane to the left of the plant zero with the largest negative real-part. The plants under consideration may be stable or unstable; there are no restrictions on the plant poles and no restrictions on the zeros in the region of stability with real-parts less than $-h$. However, the zeros in the unstable region are restricted. We showed that for plants that have no zeros in the unstable region and either one or two zeros at infinity (as described in (19) and (27), respectively) it is possible to design PID controllers such that the closed-loop poles have negative real-part less than any prescribed $-h$. The synthesis method focuses on the objective of shifting the real-part of the closed-loop poles away from the origin, which ensures stability margins; assignment of the imaginary-part is not within the design objectives and scope.

Due to using PID controllers with non-zero integral-terms, the closed-loop systems are guaranteed to achieve asymptotic tracking of constant reference inputs with zero steady-state error. The proposed synthesis method allows freedom in the choice of parameters, which may be used to satisfy additional performance specifications.

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