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Automatica 44 (2008) 1156–1160
www.elsevier.com/locate/automatica

Technical communique

Simultaneous stabilization of MIMO systems with integral action controllers ☆

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Received 31 August 2006; received in revised form 13 July 2007; accepted 6 August 2007
Available online 19 December 2007

Abstract

Simultaneous stabilization with asymptotic tracking of step-input references is explored for linear, time-invariant, multi-input multi-output stable plants. Necessary conditions are presented for existence of simultaneous integral-action controllers and existence of simultaneous PID-controllers. A systematic simultaneous PID-controller synthesis method is proposed under a sufficient condition.

Keywords: Simultaneous tracking and stabilization; Integral action; PID control; Robust asymptotic tracking

1. Introduction

We consider the problem of simultaneously stabilizing a finite class of linear time-invariant (LTI) multi-input multi-output (MIMO) stable plants while achieving asymptotic tracking of step-input references with zero steady-state error. These problems arise in many practical applications; for example, when designing a common controller for multiple operating points of the same system. The simplest controllers that achieve integral-action are proportional + integral + derivative (PID) controllers, which are widely used and preferred for their simplicity. We derive conditions under which general integral-action controllers and particularly PID-controllers exist that achieve simultaneous closed-loop stabilization.

The problem of simultaneously stabilizing stable plants using PID-controllers is equivalent to strong simultaneous stabilization of systems whose unstable poles are at the origin using controllers restricted to order two or less. Even without the order restriction on the controllers, simultaneous stabilization of three or more plants and strong stabilization problems are known to be difficult (Blondel, 1994; Blondel, Gevers, Mortini, & Rupp, 1994; Vidyasagar, 1985). There exists a common controller that simultaneously stabilizes two given plants if and only if a related system satisfies the parity interlacing property (PIP) (Blondel et al., 1994; Vidyasagar, 1985). If the simultaneous stabilization involves more than two unstable plants, PIP is a necessary but not sufficient condition. If this common controller also has to achieve asymptotic tracking of step-input references as considered here, then it is only natural to expect additional conditions to hold even for the case of two plants. Since we consider a finite class of stable plants here, the simultaneous stabilization goal is always achievable. But additional constraints have to be imposed on the DC-gains of these plants in order to achieve the asymptotic tracking requirement using low-order PID-controllers. Rigorous PID design methods exist mostly for single-input single-output (SISO) systems (Aström & Hagglund, 1995; Silva, Datta, & Bhattacharyya, 2002); these methods do not consider simultaneous PID stabilization. General integral action or more restricted PID designs that achieve simultaneous closed-loop stability of MIMO systems have not been explored.

The main results here are: (1) necessary conditions for existence of simultaneous integral-action controllers (Lemma 2) and (2) sufficient conditions and explicit PID synthesis (Proposition 1). The conditions are based on the DC-gains of the plants. For single-output systems, the sufficient conditions

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Mayuresh Kothare under the direction of Editor André Tits.

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doi:10.1016/j.automatica.2007.08.003

coincide with the necessary conditions for the existence of simultaneous PID-controllers. The systematic procedure for simultaneous PID-controller synthesis is then applied to several MIMO examples. The proposed explicit designs allow choice of parameter values. Although the objective here is to achieve simultaneous closed-loop stability with tracking, the flexibility in the choice of the PID parameters offered by the design procedure may be used to satisfy additional performance criteria.

Although we discuss continuous-time systems here, all results apply also to discrete-time systems with appropriate modifications. The following notation is used: $\mathscr{U}$ denotes the extended closed right-half plane, i.e., $\mathscr{U} = \{s \in \mathbb{C} \mid \mathscr{Re}(s) \geqslant 0\} \cup \{\infty\}$; $\mathbb{R}$, $\mathbb{R}_+$ denote real and positive real numbers; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathscr{U}$; $\mathscr{M}(\mathbf{S})$ is the set of matrices (of any size) with entries in $\mathbf{S}$; the notation $\mathbf{S}^{n_y \times n_u}$ is used when the matrix size is emphasized; I_n is the $n \times n$ identity matrix. A square matrix $M \in \mathscr{M}(\mathbf{S})$ is called unimodular iff $M^{-1} \in \mathscr{M}(\mathbf{S})$. The H_∞-norm of $M(s) \in \mathscr{M}(\mathbf{S})$ is denoted by $\|M(s)\|$ (i.e., the norm $\|\cdot\|$ is defined as $\|M\| := \sup_{s \in \partial\mathscr{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathscr{U}$ is the boundary of $\mathscr{U}$). For simplicity, we drop (s) in transfer matrices such as $G(s)$ where this causes no confusion. We use coprime factorizations over $\mathbf{S}$; i.e., for $C \in \mathbf{R}_\mathbf{p}^{n_u \times n_y}$, $C = N_c D_c^{-1}$ denotes a right-coprime-factorization (RCF), where $N_c \in \mathbf{S}^{n_u \times n_y}$, $D_c \in \mathbf{S}^{n_y \times n_y}$, $\det D_c(\infty) \neq 0$.

2. Problem description and preliminaries

Consider the standard LTI, MIMO unity-feedback system $Sys(G_j, C)$ shown in Fig. 1, where $G_j \in \mathbf{S}^{n_y \times n_u}$, $j \in \{1, \ldots, k\}$, and $C \in \mathbf{R}_\mathbf{p}^{n_u \times n_y}$ denote the plant's and the controller's transfer-functions, respectively. It is assumed that the feedback system is well-posed, G_j and C have no unstable hidden-modes, and each plant $G_j \in \mathbf{R}_\mathbf{p}^{n_y \times n_u}$ is full normal rank. The objective is to design a controller C that achieves asymptotic tracking of step-input references with zero steady-state error for a finite class of stable plants G_j simultaneously. Let $C = N_c D_c^{-1}$ be an RCF, where $N_c \in \mathbf{S}^{n_u \times n_y}$, $D_c \in \mathbf{S}^{n_y \times n_y}$, $\det D_c(\infty) \neq 0$. Then C stabilizes $G_j \in \mathscr{M}(\mathbf{S})$ if and only if

$$M_j := D_c + G_j N_c \tag{1}$$

is unimodular, i.e., $M_j^{-1} \in \mathscr{M}(\mathbf{S})$ (Gündeş & Desoer, 1990; Vidyasagar, 1985). Let H_j^{er} denote the (input–error) transfer-function from r to e; let H_j^{yr} denote the (input–output) transfer-function from r to y; then

$$\begin{aligned} H_j^{er} &= (I_{n_y} + G_j C)^{-1} = I_{n_y} - G_j C (I_{n_y} + G_j C)^{-1} \\ &=: I_{n_y} - G_j H_j^{wr} =: I_{n_y} - H_j^{yr}. \end{aligned} \tag{2}$$

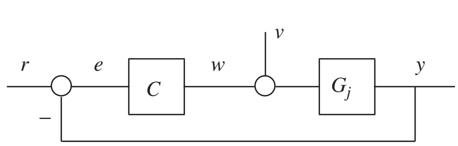

Fig. 1. Unity-feedback system $Sys(G_j, C)$.

Definition 1. (i) The system $Sys(G_j, C)$ is said to be stable iff the closed-loop transfer-function from (r, v) to (y, w) is stable. (ii) The controller C is said to simultaneously stabilize G_j for $j \in \{1, \ldots, k\}$ iff C is proper and the systems $Sys(G_j, C)$ are all stable. (iii) The stable systems $Sys(G_j, C)$ are said to have integral-action iff H_j^{er} has blocking-zeros at $s = 0$, $j \in \{1, \ldots, k\}$. (iv) The controller C is said to be a simultaneously stabilizing integral-action controller iff C stabilizes G_j for $j \in \{1, \ldots, k\}$, and D_c of any RCF $C = N_c D_c^{-1}$ has blocking-zeros at $s = 0$, i.e., $D_c(0) = 0$.

Suppose that $Sys(G_j, C)$ is stable and that step input references are applied to the system. Then the steady-state error $e(t)$ due to step inputs at $r(t)$ goes to zero as $t \to \infty$ if and only if $H_j^{er}(0) = 0$. Therefore, by Definition 1, the stable system $Sys(G_j, C)$ achieves asymptotic tracking of constant reference inputs with zero steady-state error if and only if it has integral-action. Write H_j^{er} as $H_j^{er} = (I_{n_y} + G_j C)^{-1} = D_c M_j^{-1}$. Then by Definition 1, $Sys(G_j, C)$ has integral-action if and only if $C = N_c D_c^{-1}$ is an integral-action controller since M_j unimodular means $H_j^{er}(0) = (D_c M_j^{-1})(0) = 0$ if and only if $D_c(0) = 0$. The controller's denominator matrix having blocking-zeros at $s = 0$ is equivalent to the well-known internal model principle, i.e., the controller duplicates the dynamic structure of the exogenous signals that the regulator has to process (Francis & Wonham, 1975).

The simplest integral-action controllers are in PID form. We consider the following (realizable) form of proper PID-controllers, where $K_P, K_I, K_D \in \mathbb{R}^{n_u \times n_y}$ are called the proportional constant, the integral constant, and the derivative constant, respectively (Goodwin, Graebe, & Salgado, 2001):

$$C_{\text{pid}} = K_P + \frac{K_I}{s} + \frac{K_D s}{\tau s + 1}. \tag{3}$$

Due to implementation issues of the derivative action, a pole is typically added to the derivative term (with $\tau > 0$) so that the transfer-function C_{pid} in (3) is proper. The only $\mathscr{U}$-pole of the PID-controller in (3) is at zero. The constants K_P, K_D, K_I may be negative; in the scalar case, this would imply that the zeros of C_{pid} may be in the unstable region $\mathscr{U}$. The integral-action in the PID-controller is present when the integral constant K_I is nonzero. Subsets of the PID-controller in (3) are: proportional + integral (PI) $C_{\text{pi}} = K_P + (K_I/s)$ (when $K_D = 0$); proportional + derivative (PD) $C_{\text{id}} = K_P + (K_D s/\tau s + 1)$ (when $K_I = 0$); integral + derivative (ID) $C_{\text{id}} = (K_I/s) + (K_D s/\tau s + 1)$ (when $K_P = 0$); integral (I) $C_i = (K_I/s)$ (when $K_P = K_D = 0$); derivative (D) $C_d = (K_D s/\tau s + 1)$ (when $K_P = K_I = 0$); proportional (P) $C_p = K_P$ (when $K_I = K_D = 0$).

3. Main results

We first derive necessary conditions for existence of simultaneously stabilizing integral-action controllers, and particularly PID-controllers. Then we propose explicit PID-controller design under a sufficient condition, which turns out to be necessary for plants with a single-output. Lemma 1 states the basic necessary condition on G_j for existence of integral-action controllers:

Lemma 1 (*Necessary condition for integral-action*). *Let* $G_j \in \mathbf{S}^{n_y \times n_u}$. *If the stable system* $Sys(G_j, C)$ *has integral-action, then* $\operatorname{rank} G_j(0) = n_y \leqslant n_u$, *i.e.*, G_j *has no transmission-zeros at* $s = 0$.

Although there exist integral-action controllers such that the closed-loop system is stable for each individual plant G_j that has no transmission-zeros at $s=0$, existence of integral-action controllers simultaneously stabilizing all G_j for $j \in \{1, \ldots, k\}$ requires additional necessary conditions as stated in Lemma 2. If the integral-action controllers are further restricted to be PID, these necessary conditions are imposed except when all plants in the class are minimum-phase and non-strictly proper:

Lemma 2 (*Necessary conditions for simultaneous integral-action*). *Let* $G_j \in \mathbf{S}^{n_y \times n_u}$, $j \in \{1, \ldots, k\}$. *Let* $\operatorname{rank} G_j(0) = n_y \leqslant n_u$. *Let* $G_j(0)^I \in \mathbb{R}^{n_u \times n_y}$ *denote any arbitrary right-inverse of* $G_j(0)$.

(a) *Suppose that all* $G_i - G_j$ *for* $i, j \in \{1, \ldots, k\}$ *have at least one common blocking-zero in* $\mathbb{R} \cap \mathscr{U}$ *(including infinity). Under this condition, if there exist simultaneously stabilizing integral-action controllers, then*

$$\det[G_j(0) G_i(0)^I] > 0 \quad \text{for all } i, j \in \{1, \ldots, k\}. \tag{4}$$

(b) *Suppose that each* G_j *has at least one blocking-zero in* $\mathbb{R} \cap \mathscr{U}$ *(including infinity) for* $j \in \{1, \ldots, k\}$. *Under this condition, if there exist simultaneously stabilizing PID-controllers, then* (4) *holds.*

In Lemma 2(a), if all plants in the class have the same blocking-zero $z_0 \in \mathscr{U}$ on the extended positive real axis, then $G_i(z_0) - G_j(z_0) = 0$, and (4) becomes a necessary condition for existence of a common integral-action controller. For example, if all plants in the class are strictly proper, then they all have a blocking-zero at $\infty \in \mathscr{U}$. The plants G_j may have additional individual $\mathscr{U}$-zeros other than this common zero at $s = z_0$. In Lemma 2(b), if each plant in the class has some blocking-zero on the extended positive real axis (not necessarily all at the same location), then (4) becomes a necessary condition for existence of a common PID-controller.

Proposition 1 presents a method for designing PID-controllers that simultaneously stabilize $\{G_1, \ldots, G_k\}$. A sufficient condition for existence of such controllers is that the eigenvalues of $G_j(0)G_1(0)^I$ are positive real for all plants G_j in the class, where an arbitrary member G_1 is called the nominal plant.

Proposition 1 (*Simultaneous PID-controller synthesis*). *Let* $G_j \in \mathbf{S}^{n_y \times n_u}$, $j \in \{1, \ldots, k\}$. *Let* $\operatorname{rank} G_j(0) = n_y \leqslant n_u$. *Designate an arbitrary plant* G_1 *as the nominal plant. Let* $G_1(0)^I \in \mathbb{R}^{n_u \times n_y}$ *denote any arbitrary right-inverse of* $G_1(0)$. *Suppose that, for* $j \in \{2, \ldots, k\}$, *all eigenvalues of* $G_j(0)G_1(0)^I$ *are real and positive. Then simultaneously stabilizing PID-controllers exist. Furthermore, PID-controllers stabilizing* $\{G_1, \ldots, G_k\}$ *can be designed as follows: Choose any* $\hat{K}_P, \hat{K}_D \in \mathbb{R}^{n_u \times n_y}$, $\tau > 0$. *Let* $K_P = \beta \hat{K}_P, K_D = \beta \hat{K}_D$, $K_I = \beta G_1(0)^I$ *for any positive* $\beta \in \mathbb{R}_+$ *satisfying*

$$\beta < \min_{j \in \{1, \ldots, k\}} \left\| G_j \left(\hat{K}_P + \frac{\hat{K}_D s}{\tau s + 1} \right) + \frac{[G_j(s) - G_j(0)] G_1(0)^I}{s} \right\|^{-1}. \tag{5}$$

Then a PID-controller that simultaneously stabilizes all G_j *for* $j \in \{1, \ldots, k\}$ *is given by*

$$C_{\text{pid}} = \beta \hat{K}_P + \frac{\beta G_1(0)^I}{s} + \frac{\beta \hat{K}_D s}{\tau s + 1}. \tag{6}$$

PI, ID, I-controllers are obtained by choosing $\hat{K}_D = 0$, $\hat{K}_P = 0$, *and* $\hat{K}_D = \hat{K}_P = 0$, *respectively.*

The sufficient condition of positive real eigenvalues for $G_j(0)G_1(0)^I$ becomes a necessary condition for plant classes that have single-output ($n_y = 1$ and $n_u \geqslant 1$, i.e., $G_j(0)G_1(0)^I \in \mathbb{R}$) when each plant has at least one blocking-zero on the extended positive real axis (these zeros may all be different). Transmission-zeros and blocking-zeros are the same for these plants.

Corollary 1 (*Necessary and sufficient existence conditions for simultaneous PID-controllers*). *Let* $G_j \in \mathbf{S}^{1 \times n_u}$, $j \in \{1, \ldots, k\}$. *Let* $G_j(0) \neq 0$.

(a) *Suppose that all* $G_i - G_j$ *for* $i, j \in \{1, \ldots, k\}$ *have at least one common blocking-zero in* $\mathbb{R} \cap \mathscr{U}$ *(including infinity). Under this condition, there exist simultaneously stabilizing integral-action controllers if and only if* $G_j(0)G_i(0)^I > 0$, *for all* $i, j \in \{1, \ldots, k\}$.

(b) *Suppose that each* G_j *has at least one blocking-zero in* $\mathbb{R} \cap \mathscr{U}$ *(including infinity) for* $j \in \{1, \ldots, k\}$. *Under this condition, there exist simultaneously stabilizing PID-controllers if and only if* $G_j(0)G_1(0)^I > 0$, *for all* $j \in \{2, \ldots, k\}$.

We apply the systematic design procedure of Proposition 1 to several MIMO plant classes with no transmission-zeros at $s = 0$. The plants in Example 1 have varying number of transmission-zeros on the extended right-half-plane and only one has a blocking-zero, which means that the necessary conditions of Lemma 2 do not apply. Since the plants in Example 2 all share a common blocking-zero, the necessary conditions of Lemma 2 apply. These plants all have exactly one positive transmission-zero. The class considered in Example 3 includes plants without any transmission-zeros in $\mathscr{U}$ as well as those with different numbers of transmission-zeros in $\mathscr{U} \cap \mathbb{R}$.

Example 1. Consider the class of plants $\{G_1, G_2, G_3\}$:

$$G_1 = \begin{bmatrix} \dfrac{-1}{s+1} & \dfrac{-4}{s+2} \\ 0 & \dfrac{(s-2)(s-5)}{(s+1)(s+2)} \end{bmatrix},$$

$$G_2 = \begin{bmatrix} \dfrac{-5(s^2+9)}{(s+3)^2} & \dfrac{s+12}{s+3} \\ \dfrac{-(s^2+9)}{(s+3)^2} & \dfrac{9}{(s+3)} \end{bmatrix},$$

$$G_3 = \begin{bmatrix} \dfrac{s-2}{s+5} & \dfrac{s-2}{(s+1)(s+5)} \\ \dfrac{-(s-2)}{s+3} & \dfrac{-10(s-2)}{(s+1)(s+3)} \end{bmatrix}.$$

The plant G_1 has transmission-zeros at $s=2, 5, \infty \in \mathscr{U}$, but no blocking-zeros; G_2 has transmission-zeros at $s=0\pm j3$, $s=33 \in \mathscr{U}$, but no blocking-zeros; G_3 has a transmission-zero at infinity and a blocking-zero at $s = 2 \in \mathscr{U}$. The eigenvalues of $G_2(0)G_1(0)^{-1}$ and $G_3(0)G_1(0)^{-1}$ are all positive, and hence, the class satisfies the sufficient conditions for existence of simultaneously stabilizing PID-controllers given in Proposition 1. Following the synthesis method, we choose $\hat{K}_P = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.1 \end{bmatrix}$, $\hat{K}_D = 0$. We take $\beta = 0.39 < \min\{0.5713, 0.3919, 0.6364\}$ satisfying (5). The corresponding simultaneous PI-controller $C_{\text{pi}} = \beta\hat{K}_P + (\beta G_1(0)^{-1}/s)$, which is one among many that can be designed using the procedure of Proposition 1, is

$$C_{\text{pi}} = \begin{bmatrix} \dfrac{-0.195(s+2)}{s} & \dfrac{-0.156}{s} \\ 0 & \dfrac{0.039(s+2)}{s} \end{bmatrix}.$$

Example 2. Consider the quadruple tank process (Johansson, 2000), which consists of four interconnected water tanks. The objective is to control the level in two lower tanks with two pumps. The linearized system dynamics has a transmission-zero, which can be moved between the positive and negative real axis by changing a valve. With two input voltages to the tanks and two output voltages from level measurements (the parameters γ_{ij}, c_{ij} and the time-constants T_{ij} as in Johansson, 2000), the linearized plant model G_j at the jth operating point is given by

$$G_j = \begin{bmatrix} \dfrac{\gamma_{1j}\alpha_{1j}}{1+sT_{1j}} & \dfrac{(1-\gamma_{2j})\alpha_{1j}}{(1+sT_{1j})(1+sT_{3j})} \\ \dfrac{(1-\gamma_{1j})\alpha_{2j}}{(1+sT_{2j})(1+sT_{4j})} & \dfrac{\gamma_{2j}\alpha_{2j}}{1+sT_{2j}} \end{bmatrix}.$$

The plant models for each operating point all have a common blocking-zero at infinity. Therefore, by Lemma 2(a), simultaneously stabilizing integral-action controllers (and in particular PID-controllers) exist only if $\det[G_j(0)G_i(0)^{-1}] = \det G_j(0) \det G_i(0)^{-1} > 0$ for all i, j, i.e., $\det G_j(0) = \gamma_{1j} + \gamma_{2j} - 1$, has the same sign as $\det G_i(0) = \gamma_{1i} + \gamma_{2i} - 1$. The plants G_j have no transmission-zeros at $s = 0$ if and only if $\gamma_{1j} + \gamma_{2j} \neq 1$. In addition to the blocking-zero at infinity, G_j has a positive transmission-zero if $\gamma_{1j} + \gamma_{2j} < 1$, which shifts to the negative real-axis if $\gamma_{1j} + \gamma_{2j} > 1$. Therefore, the necessary condition of Lemma 2(a) concludes that simultaneously stabilizing integral-action controllers exist only if (i) either all plants to be simultaneously stabilized have this transmission-zero on the positive real-axis (so that the sign of $\gamma_{1j} + \gamma_{2j} - 1$ is negative for all plants G_j), (ii) or all plants to be simultaneously stabilized have this transmission-zero on the negative real-axis (so that the sign of $\gamma_{1j} + \gamma_{2j} - 1$ is positive for all plants G_j). If $\gamma_{1j} + \gamma_{2j} - 1 < 0$ for G_j, whereas $\gamma_{1i} + \gamma_{2i} - 1 > 0$ for G_i, then the necessary condition of Lemma 2(a) is violated and hence, G_j and G_i cannot be simultaneously stabilized using PID-controllers.

Consider the case $\alpha_{1j} = \alpha_{2j} = 1$, $T_{ij} = 1$, $\gamma_{1j} = \gamma_{2j} = \gamma_j$ as in Aström, Johansson, and Wang (2002). Let all plants have $\gamma_j < \frac{1}{2}$, so that one of the two transmission-zeros is positive for each G_j. Assign $\gamma_1 = \min_j \gamma_j$, i.e., G_1 corresponds to the plant with the smallest γ. Under these conditions, $G_j(0)G_1(0)^{-1}$ is symmetric, positive-definite and hence, the sufficient conditions for existence of simultaneously stabilizing PID-controllers given in Proposition 1 are satisfied for any number of plants G_j representing different operating points. Let $\gamma_1 = \frac{1}{5}$, $\gamma_2 = \frac{1}{4}$, $\gamma_3 = \frac{1}{3}$ to design a controller that simultaneously stabilizes G_1 (with transmission-zeros at $s = 3, -5$), G_2 (with transmission-zeros at $s = 2, -4$), and G_3 (with transmission-zeros at $s = 1, -3$). For example, choose $\hat{K}_P = \begin{bmatrix} -0.1 & 2 \\ 0.5 & -0.1 \end{bmatrix}$, $\hat{K}_D = 0$. We take $\beta = 0.54 < \min\{0.5438, 0.5834, 0.6612\}$ satisfying (5). With $G_1(0)^{-1} = \begin{bmatrix} -1/3 & 4/3 \\ 4/3 & -1/3 \end{bmatrix}$, the corresponding simultaneous PI-controller $C_{\text{pi}} = \beta\hat{K}_P + (\beta G_1(0)^{-1}/s)$ is

$$C_{\text{pi}} = \begin{bmatrix} \dfrac{-(0.054s+0.18)}{s} & \dfrac{1.08s+0.72}{s} \\ \dfrac{0.27s+0.72}{s} & \dfrac{-(0.054s+0.18)}{s} \end{bmatrix}.$$

Example 3. Consider the plants G_j, $j \in \{1, 2, 3, 4\}$:

$$G_j = \begin{bmatrix} \dfrac{s+4}{s+1} & \dfrac{s-1}{s+1} \\ \dfrac{20}{s+6} & g_j \end{bmatrix}, \quad g_1 = -0.5, \quad g_2 = 2,$$

$$g_3 = 10, \quad g_4 = 0.$$

The plants have no blocking-zeros; G_1 has a transmission-zero at $s = 0.318 \in \mathscr{U}$ (and one at $s = -50.318 \notin \mathscr{U}$); G_2 has transmission-zeros at $s = 0 \pm j\sqrt{34} \in \mathscr{U}$; G_3 has no transmission-zeros in $\mathscr{U}$, and G_4 has transmission-zeros at $s = 1, \infty \in \mathscr{U}$. The eigenvalues of $G_j(0)G_1(0)^{-1}$ are all positive, and hence, the class satisfies the sufficient conditions for existence of simultaneously stabilizing PID-controllers given in Proposition 1. Following the synthesis method, we choose $\hat{K}_P = \begin{bmatrix} 0.1 & 0 \\ 1.8 & -0.4 \end{bmatrix}$, $\hat{K}_D = 0.1I_2$, $\tau = 0.05$. We take $\beta = 0.04 < \min\{0.2215, 0.2043, 0.0415, 0.2202\}$ satisfying (5). The corresponding simultaneous PID-controller $C_{\text{pid}} = \beta\hat{K}_P + \beta(\hat{K}_D s/\tau s+1) + (\beta G_1(0)^{-1}/s)$, which is one among many that can be designed using the procedure of

Proposition 1, is

$$C_{\text{pid}} = \begin{bmatrix} \dfrac{0.084s^2+0.065s-0.3}{s(s+20)} & \dfrac{0.6}{s} \\ \dfrac{-1.44s-2}{s} & \dfrac{0.064s^2-0.2s+2.4}{s(s+20)} \end{bmatrix}.$$

4. Conclusions

We showed that a class of stable plants $\{G_1, \ldots, G_k\}$ with blocking-zeros on the extended positive real-axis can be simultaneously stabilized using low-order integral-action (PID) controllers only if $\det[G_j(0)G_i(0)^I] > 0$ for all $i, j \in \{1, \ldots, k\}$. We derived a sufficient condition which is not too far from this necessary condition: If the eigenvalues of $G_j(0)G_1(0)^I$ are all real and positive for some arbitrary member G_1 of the class, then there exist simultaneous PID-controllers. In fact, the necessary conditions and the sufficient conditions coincide for plants with only one output (although they may have multiple inputs). Under the sufficient condition of positive eigenvalues for the DC-gain matrix, we presented a PID synthesis method, which allows a wide range of choices for the PID parameters. We applied this systematic simultaneous PID-controller design to several MIMO plant classes.

We only considered stable plant classes for simultaneous integral-action control here. Since PID-controllers do not necessarily exist for unstable plants and since simultaneous stabilization of three or more unstable plants is an extremely difficult problem even without restrictions on the controller order, unstable plant classes would be very challenging to tackle. Sufficient conditions for simultaneous PID stabilization of unstable plant classes are being explored as extensions of the present results.

Appendix

Proof of Lemma 1. The stability of $Sys(G_j, C)$ implies $H_j^{er}(0) = I_{n_y} - G_j H_j^{wr}(0) = 0$, i.e., $G_j H_j^{wr}(0) = I_{n_y}$. Therefore, $\text{rank}[G_j(0)H_j^{wr}(0)] = n_y \leqslant \min\{\text{rank}\, G_j(0), \text{rank}\, H_j^{wr}(0)\}$ implies $n_y \leqslant \text{rank}\, G_j(0) \leqslant \min\{n_y, n_u\}$ and hence, $\text{rank}\, G_j(0) = n_y$. □

Proof of Lemma 2. (a) Let $C = N_c D_c^{-1}$ be an integral-action controller simultaneously stabilizing the class $\{G_1, \ldots, G_i, \ldots, G_j, \ldots, G_k\}$, where G_i, G_j are two arbitrary plants in the class. Since C has integral-action, the denominator D_c can be written as $D_c =: (s/s+a)\hat{D}_c$ for any $a \in \mathbb{R}_+$, where $\hat{D}_c \in \mathscr{M}(\mathbf{S})$. By (1), $M_i = (s/s+a)\hat{D}_c + G_i N_c$ and $M_j = (s/s+a)\hat{D}_c + G_j N_c$ are unimodular. By assumption, $G_i(z_0) = G_j(z_0)$ for the same $z_0 \in \mathbb{R}_+ \cup \infty$ implies $M_i(z_0) - M_j(z_0) = [G_i(z_0) - G_j(z_0)]N_c(z_0) = 0$, i.e., $M_i(z_0) = M_j(z_0)$. Since $\det M_i(z_0) = \det M_j(z_0)$ at some point $z_0 \in \mathscr{U}$, $\det M_i(s)$ has the same sign as $\det M_j(s)$ for all $s \in \mathscr{U} \cap \mathbb{R}$. In particular, at $s=0$, $M_i(0) = G_i(0)N_c(0)$implies $N_c(0) = G_i(0)^I M_i(0)$ and hence, $M_j(0) = G_j(0)N_c(0) = G_j(0)G_i(0)^I M_i(0)$. The conclusion follows since $\det M_j(0) = \det[G_j(0)G_i(0)^I] \det M_i(0)$, with $\det M_j(0)$ having the same sign as $\det M_i(0)$, implies (4).

(b) Now let C_{pid} be a PID-controller simultaneously stabilizing the class. Write C_{pid} as $C_{\text{pid}} = N_c D_c^{-1} = ((s/s+a)C_{\text{pid}})((s/s+a)I_{n_y})^{-1}$ for any $a > 0$; i.e., $N_c = [K_P + (K_D s/\tau s + 1)](s/s+a) + (K_I/s+a)$. By (1), $M_i = (s/s+a)I + G_i N_c$ and $M_j = (s/s+a)I + G_j N_c$ are unimodular. By assumption, $G_i(z_i) = 0$ for some $z_i \in \mathbb{R}_+ \cup \infty$ and $G_j(z_j) = 0$ for some $z_j \in \mathbb{R}_+ \cup \infty$ implies $\det M_i(z_i) = \det(z_i/z_i + a)I > 0$ and $\det M_j(z_j) = \det(z_j/z_j + a)I > 0$. Since $\det M_i(s)$ has the same sign for all $s \in \mathscr{U} \cap \mathbb{R}$, $\det M_i(0) > 0$; similarly, $\det M_j(0) > 0$. At $s = 0$, $M_i(0) = G_i(0)N_c(0) = G_i(0)a^{-1}K_i$ implies $K_i = aG_i(0)^I M_i(0)$ and hence, $M_j(0) = G_j(0)N_c(0) = G_j(0)G_i(0)^I M_i(0)$. The conclusion follows since $\det M_j(0) = \det[G_j(0)G_i(0)^I] \det M_i(0)$, with $\det M_j(0) > 0$ and $\det M_i(0) > 0$,implies (4). □

Proof of Proposition 1. Write $C_{\text{pid}} = ((s/s+a)C_{\text{pid}})((s/s+a)I_{n_y})^{-1}$ for any $a > 0$. By assumption, $\Theta_j := G_j(0)G_1(0)^I$ has positive real eigenvalues. For $j \in \{1, \ldots, k\}$, define $M_j := (s+a)^{-1}sI_{n_y} + G_j(s+a)^{-1}sC_{\text{pid}}$. Since $a > 0$, $\beta > 0$ and $(sI + \beta\Theta_j)^{-1} \in \mathscr{M}(\mathbf{S})$, M_j is unimodular if and only if $\hat{M}_j := (s+a)(sI + \beta\Theta_j)^{-1}M_j$ is unimodular, which can be written as $\hat{M}_j = (sI+\beta\Theta_j)^{-1}sI + (sI+\beta\Theta_j)^{-1}sG_jC_{\text{pid}} = (sI+\beta\Theta_j)^{-1}sI + (sI+\beta\Theta_j)^{-1}s\beta G_j[\hat{K}_P + (\hat{K}_D s/\tau s + 1) + (G_1(0)^I/s)] = I + (sI+\beta\Theta_j)^{-1}s\beta[G_j(\hat{K}_P + (\hat{K}_D s/\tau s + 1)) + (G_j(s)G_1(0)^I - \Theta_j)/s]$. Since $\|(sI+\beta\Theta_j)^{-1}s\| = 1$, $\hat{M}_j$ is unimodular for $\beta > 0$ satisfying (5). Hence, by (1), the systems $Sys(G_j, C)$ are stable for all $j \in \{1, \ldots, k\}$. □

Proof of Corollary 1. When $n_y = 1 \leqslant n_u$, $G_j(0)G_1(0)^I$ is a scalar. The necessity of the conditions in Corollary 1(a–b) follow from Lemma 2. The sufficiency follows from Proposition 1 since the eigenvalue of $G_j(0)G_1(0)^I \in \mathbb{R}$ is itself. □

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