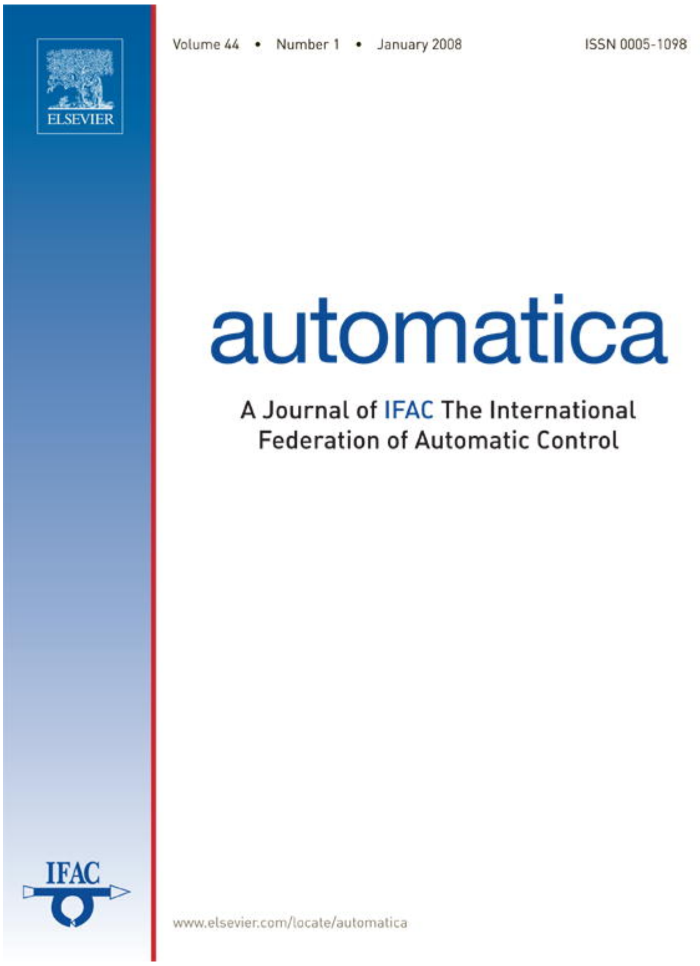
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Brief paper

MIMO controller synthesis with integral-action integrity ☆

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Abstract

A controller synthesis method is presented for closed-loop stability and asymptotic tracking of step input references with zero steady-state error. Integral-action is achieved in two design steps starting with any stabilizing controller and adding a PID-controller in a configuration that guarantees robust stability and tracking. The proposed design has integral-action integrity, where closed-loop stability is maintained even when any of the proportional, integral, or derivative terms are removed or the entire PID-controller is limited by a constant gain matrix. The integral constant can be switched off when integral-action is not wanted.

Keywords: Integral action; PID controllers; Steady-state tracking

1. Introduction

We consider integral-action controller design for linear, time-invariant (LTI) multi-input multi-output (MIMO) plants. Our goal is to achieve closed-loop stability and robust asymptotic tracking of step-input references with zero steady-state error. This objective is extended to type-m integral action in each output channel so that polynomial references up to order $m-1$ applied at each input would be asymptotically tracked.

The simplest controllers that achieve integral-action are in the proportional+integral+derivative (PID) form. However, closed-loop stability can be achieved using these low order controllers only for certain classes of plants, and many others cannot be stabilized using PID-controllers (Gündeş & Wai, 2005). The standard method of achieving integral-action is the well-known full-order observer-based integral-action controller design based on an augmented plant model, which uses linear quadratic regulator (LQR) or pole-placement methods to find state-feedback gains for the states of the integrators in addition to the states of the plant (Goodwin, Graebe, & Salgado, 2001). Although this method achieves both closed-loop stability and steady-state accuracy, the integrators cannot be completely switched off without affecting closed-loop stability. Furthermore, this standard method does not easily extend to higher integral-action type (Gündeş & Kabuli, 1998). In this paper we propose a two-step integral-action synthesis procedure that achieves robust tracking by adding a PID-controller over a previously designed stabilizing controller that is already present in the feedback loop. An initial stabilizing controller, which does not have integral-action, is designed (to be optimal and to satisfy given design objectives) for the original plant using any method (LQR, H_∞, etc.). Then an additional PID-controller is designed for a stable system (the numerator-matrix of the plant). The two controllers are then configured to achieve closed-loop stability and integral-action together. All integral-action controllers can be obtained by inclusion of a free controller parameter. The main advantage of this two-step approach is that the PID-controller block containing the integral-action designed in the second step can be switched off (taken out completely and the states are reset) without affecting closed-loop stability. The PID-controller can be designed with an additional property that we call integral-action integrity, where closed-loop stability is maintained even when any of the proportional, integral, or derivative terms are removed or the entire PID block is limited by a constant gain

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matrix. If the design requires a higher or lower integral-action type, the initial design can be easily modified by including incrementally designed additional integrator terms in the controller. High-order integral terms can be deleted to achieve a lower type all without re-designing the entire stabilization loop. This incremental feature of the design starting from stabilizing controllers for the original plant and adding on integrators as necessary makes it possible to compare the system performance for different integral-action types since all designs are based on the original plant instead of different augmented systems. In contrast with the standard approach to integral-action design for an augmented system, the design proposed here does not use an augmented system identification and does not need to re-identify the plant for a stabilizing controller without that integral-action component or a lower/higher-order integral-action component. Simulation comparisons of the proposed method with the standard augmentation-based method were given for a stable plant in Mete and Gündeş (2004). Since the performance of integral-action control depends on the system operating in a linear range and integral-action controllers suffer serious loss of performance due to integral windup, which occurs when the actuators in the control-loop saturate, it may be desirable to switch off the integral term while maintaining closed-loop stability for protection against windup (Doyle, Smith, & Enns, 1987; Kothare, Campo, Morari, & Nett, 1994; Kapoor, Teel, & Daoutidis, 1998). The methods proposed here simply design controllers whose integral-action components can be turned-off (or limited), and are not intended as alternate anti-windup schemes. When the integral-action is turned-off, the states in the part of the controller implementation that is taken out of service are all set to zero and the initial conditions and outputs are reset to zero.

Although continuous-time systems are discussed, all results apply also to discrete-time systems with appropriate modifications. *Notation*: $\mathcal{U}$ is the extended closed right-half plane, i.e., $\mathcal{U} = \{s \in \mathbb{C} \,|\, \mathcal{R}e(s) \geqslant 0\} \cup \{\infty\}$; $\mathbb{R}$, $\mathbb{R}_+$ denote real and positive real numbers; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in $\mathbf{S}$; I_n is the $n \times n$ identity matrix; we use I when the dimension is unambiguous. The H_∞-norm of $M(s) \in \mathcal{M}(\mathbf{S})$ is denoted by $\|M(s)\|$ (i.e., the norm $\|\cdot\|$ is defined as $\|M\| := \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$). For simplicity, we drop (s) in transfer matrices such as $G(s)$. We use coprime factorizations over $\mathbf{S}$; i.e., for $G \in \mathbf{R}_\mathbf{p}^{r\times q}$, $G = XY^{-1}$ denotes a right-coprime-factorization (RCF), $G = \widetilde{Y}^{-1}\widetilde{X}$denotes a left-coprime-factorization (LCF), where $X, \widetilde{X} \in \mathbf{S}^{r\times q}$, $Y \in \mathbf{S}^{q\times q}$, $\widetilde{Y} \in \mathbf{S}^{r\times r}$, $\det Y(\infty) \neq 0$, $\det \widetilde{Y}(\infty) \neq 0$.

2. Problem description and preliminaries

Consider the LTI, MIMO unity-feedback system $Sys(G, \widehat{C})$ in Fig. 1; $G \in \mathbf{R}_\mathbf{p}^{r\times q}$ and $\widehat{C} \in \mathbf{R}_\mathbf{p}^{q\times r}$ denote the plant's and the controller's transfer-functions. It is assumed that $Sys(G, \widehat{C})$ is well-posed, G and $\widehat{C}$ have no unstable hidden-modes, and $G \in$

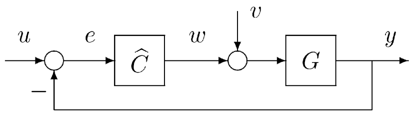

Fig. 1. Unity-feedback system $Sys(G, \widehat{C})$.

$\mathbf{R}_\mathbf{p}^{r\times q}$ is full normal rank. Let $H_{eu} = (I_r + G\widehat{C})^{-1} = I_r - G\widehat{C}(I_r + G\widehat{C})^{-1} =: I_r - GH_{wu} =: I_r - H_{yu}$ denote the (input-error) transfer-function from u to e.

Definition 1. (i) The system $Sys(G, \widehat{C})$ is called stable iff the closed-loop transfer-function from (u, v) to (y, w) is stable. (ii) The controller $\widehat{C}$ stabilizes G iff $\widehat{C}$ is proper and $Sys(G, \widehat{C})$ is stable. (iii) The stable system $Sys(G, \widehat{C})$ has integral-action iff H_{eu} has blocking-zeros at $s = 0$; it has type-m integral action in each output channel iff H_{eu} has (at least) m blocking-zeros at zero, i.e., $(s^{-(m-1)}H_{eu})(0) = 0$. (iv) The controller $\widehat{C}$ is called a controller with integral-action iff $\widehat{C}$ stabilizes G and D_c of any RCF $\widehat{C} = N_c D_c^{-1}$ has blocking-zeros at $s = 0$, i.e., $D_c(0) = 0$; $\widehat{C}$ is called a controller with type-m integral action iff $\widehat{C}$ stabilizes G and D_c has (at least) m blocking-zeros at $s = 0$, i.e., $(s^{-(m-1)}D_c)(0) = 0$.

Let $G = XY^{-1} = \widetilde{Y}^{-1}\widetilde{X}$ be any RCF, LCF of the plant, $\widehat{C} = N_c D_c^{-1} = \widetilde{D}_c^{-1}\widetilde{N}_c$ be any RCF, LCF of the controller. Then $\widehat{C}$ stabilizes G if and only if $M_L := \widetilde{Y} D_c + \widetilde{X} N_c$ is unimodular, equivalently, $M_R := \widetilde{D}_c Y + \widetilde{N}_c X$ is unimodular (Gündeş & Desoer, 1990; Vidyasagar, 1985). Suppose that $Sys(G, \widehat{C})$ is stable. Then the error $e(t)$ due to step inputs $u(t)$ goes to zero as $t \to \infty$ if and only if $H_{eu}(0) = 0$. Therefore, the stable system $Sys(G, \widehat{C})$ achieves asymptotic tracking of constant reference inputs with zero steady-state error if and only if it has integral-action; it achieves asymptotic tracking of polynomial references up to order $m - 1$iff it has (at least) type-m integral action (León de la Barra, Emami-Naeini, & Chincón, 1998). Write $H_{eu} = (I + G\widehat{C})^{-1} = I - G\widehat{C}(I + G\widehat{C})^{-1} = D_c\, M_L^{-1}\widetilde{Y} = I - XM_R^{-1}\widetilde{N}_c$. By Definition 1, $Sys(G, \widehat{C})$ has integral-action if and only if $H_{eu}(0) = (D_c M_L^{-1}\widetilde{Y})(0) = 0$. If $\widehat{C} = N_c D_c^{-1}$ is an integral-action controller, then $Sys(G, \widehat{C})$ has integral-action. For $H_{eu}(0) = (D_c M_L^{-1}\widetilde{Y})(0) = 0$, it is sufficient but not necessary to have $D_c(0) = 0$. For plants that have poles at $s = 0$, rank $\widetilde{Y}(0) < r$ and hence, the system may achieve integral-action even if $D_c(0) \neq 0$. For plants with no poles at $s = 0$,rank $\widetilde{Y}(0) = r$ implies $Sys(G, \widehat{C})$ has integral-action if and only if $\widehat{C} = N_c D_c^{-1}$ is an integral-action controller, i.e., $D_c(0) = 0$.

Lemma 2.1 states two necessary conditions for integral-action. In Lemma 2.2, stabilizing controllers are decomposed into a sum of two components. A controller designed to stabilize the stable numerator X of the plant G can be added through a denominator factor to any controller that stabilizes G:

Lemma 2.1 (*Necessary conditions for integral-action*). *Let $G \in \mathbf{R}_\mathbf{p}^{r\times q}$. If the system $Sys(G, \widehat{C})$ has integral-action, then* (i) (*normal*) rank $G = r \leqslant m$; (ii) *G has no transmission-zeros at $s = 0$.*

Lemma 2.2 (*Two-step controller synthesis*). *Let* $G \in \mathbf{R_p}^{r\times q}$; *let* $G = XY^{-1}$ *be any RCF and* $G = \widetilde{Y}^{-1}\widetilde{X}$ *be any LCF of* G. *Suppose that* C_g *is any controller that stabilizes* G; *let* $C_g = \widetilde{D}_g^{-1}\widetilde{N}_g$ *be an LCF of* C_g *such that* $\widetilde{D}_g Y + \widetilde{N}_g X = I$. *Suppose that* C_x *is a controller that stabilizes* $X \in \mathcal{M}(\mathbf{S})$. *Then, for any* $Q \in \mathcal{M}(\mathbf{S})$ *such that* $\det(\widetilde{D}_g - Q\widetilde{X})(\infty) \neq 0$,

$$\widehat{C}_Q = (\widetilde{D}_g - Q\widetilde{X})^{-1}[(\widetilde{N}_g + Q\widetilde{Y}) + C_x] \tag{1}$$

also stabilizes G. *For* $Q = 0$, $\widehat{C}_Q$ becomes $\widehat{C} = \widetilde{D}_g^{-1}\widetilde{N}_g + \widetilde{D}_g^{-1}\,C_x = C_g + \widetilde{D}_g^{-1}\,C_x$.

The controller achieved as a sum in Lemma 2.2 is particularly useful when the controller is designed to satisfy a criterion such as asymptotic tracking. The poles at $s = 0$ can be designed into the term C_x so that the controller $\widehat{C}$ ends up with integral-action. Designing a special C_x with poles at a specific location (e.g., $s = 0$) for the numerator X, which is stable, is easier than designing one for the actual plant G, which may be unstable.

3. Main results

The simplest integral-action controllers are in PID form. We consider the following (realizable) form of proper PID-controllers; $K_P, K_I, K_D \in \mathbb{R}^{q\times r}$ are the proportional, the integral, and the derivative constant (Goodwin et al., 2001):

$$C_{pid} = K_P + \frac{K_I}{s} + \frac{K_D s}{\tau_d s + 1}. \tag{2}$$

A pole is typically added to the derivative term (with $\tau_d > 0$) so that the transfer-function C_{pid} in (2) is proper. The only $\mathcal{U}-$ pole of the PID-controller in (2) is at zero. The integral-action in the PID-controller is present when $K_I \neq 0$. Subsets of the PID-controller in (2) are: proportional + integral (PI) $C_{pi} = K_P + K_I/s$ (when $K_D = 0$); integral + derivative (ID) $C_{id} = K_I/s + K_D s/(\tau_d s + 1)$ (when $K_P = 0$); integral (I) $C_i = K_I/s$ (when $K_P = K_D = 0$); derivative (D) $C_d = K_D s/(\tau_d s + 1)$ (when $K_P = K_I = 0$); proportional (P) $C_p = K_P$ (when $K_I = K_D = 0$).

If the design also ensures that any subsets C_{pi}, C_{pd}, C_{id}, C_i, C_d, C_p of C_{pid} or all of the C_{pid} can be removed without affecting closed-loop stability, then this property of the PID-controller is called *integral-action integrity*. Integral-action is not maintained once $K_I = 0$,

Definition 2. The PID-controller in (2) has integral-action integrity iff $C_\Phi\Delta := [\Phi_P K_P + \Phi_I \frac{K_I}{s} + \Phi_D \frac{K_D s}{\tau_d s+1}]\Delta$ also maintains closed-loop stability for $\Phi_P \in \{I_q, 0\}$, $\Phi_D \in \{I_q, 0\}$, $\Phi_I \in \{I_q, 0\}$, and for all $\Delta = \mathrm{diag}[\delta_1, \delta_2, \ldots, \delta_r]$, $\delta_j \in (0, 1]$, $j = 1, \ldots, r$.

In Definition 2, Δ is a constant (possibly unknown) multiplicative perturbation. The channels of a PID-controller that has integral-action integrity can be scaled individually.

3.1. Two-step integral-action controller synthesis

Although PID-controllers are simple and achieve integral-action, some unstable plants cannot be stabilized using PID-controllers of the form (2). Consider the following example, where we introduce a two-step approach to design integral-action controllers for an unstable plant that is not stabilizable using PID-controllers. The two-step synthesis approach is formalized in Proposition 3 and Theorem 4 below.

Example 1. Consider the plant $G = (s-1)/(s+1)(s-2)$. which cannot be stabilized using PID-controllers and is not even strongly stabilizable (Vidyasagar, 1985). Consider e.g., $C_g = 9(s+1)/(s-5)$; every controller stabilizing this plant is unstable. The PID-controller $C_{pid} = 0.2 + 0.08s/(0.1s+1) + (-0.2/s)$ as in (2) is designed to stabilize $X = (s-1)/(s+1)^2$ of $G = XY^{-1}$. For every PID-controller that stabilizes X, $K_I < 0$ since $X(0) < 0$. By Lemma 2.2, $C_x = C_{pid}$ can be used together with C_g to introduce integral-action: The integral-action controller

$$\begin{aligned}\widehat{C}_{iQ} &= (\widetilde{D}_g - Q\widetilde{X})^{-1}(\widetilde{N}_g + Q\widetilde{Y} + C_{pid}) \\ &= \left(\frac{(s-5)}{(s+1)} - Q\frac{(s-1)}{(s+1)^2}\right)^{-1}\left(9 + Q\frac{(s-2)}{(s+1)} + C_{pid}\right)\end{aligned}$$

also stabilizes G for all $Q \in \mathcal{M}(\mathbf{S})$, including $Q = 0$. Since G has one zero at $s = 1$ between an integral-action controller's pole at $s = 0$ and the plant's pole at $s = 2$, every integral-action controller for G has an odd number of positive real-axis zeros in the interval $(0, 2)$. The controller $\widehat{C}_i = [(s+1)(0.92s^2 + 9.26s - 0.2)]/s(0.1s+1)(s-5)$ has a zero at $s = 0.0216$ in this interval. The PID-controller block in $\widehat{C}_{iQ}$ that introduces the integral-action is designed so that it can be partially or fully turned off, or scaled down while maintaining closed-loop stability. With $C_p = 0.2$, $C_d = 0.08s/(0.1s+1)$, $C_i = (-0.2/s)$, the system $Sys(G, \widehat{C}_{\Phi\Delta})$ is stable with $\widehat{C}_{\Phi\Delta} = (\widetilde{D}_g - Q\widetilde{X})^{-1}(\widetilde{N}_g + Q\widetilde{Y} + [\Phi_P C_p + \Phi_D C_d + \Phi_I C_i]\Delta)$, where each of Φ_P, Φ_D, Φ_I may be zero or identity, and $\Delta \in (0, 1]$. Any or all of the proportional, derivative, integral terms in the PID-controller portion of $\widehat{C}_{iQ}$ can be removed (multiplied by zero), or the terms that remain may be multiplied by a scaling factor $0 < \Delta \leqslant 1$. Closed-loop stability is maintained with any of the subsets of the PID-controller, but the integral-action is present only when the integral term is not multiplied by zero. □

Proposition 3 shows that stable systems can be stabilized using PID-controllers, with $K_I \neq 0$ if and only if the system has no transmission-zeros at $s = 0$, and proposes a method of selecting the proportional, integral, derivative constants for a PID-controller design with integral-action integrity.

Proposition 3 (*PID-controller synthesis with integrity for stable systems*). *Let* $X \in \mathbf{S}^{r\times q}$, rank $X(s) = r \leqslant q$. *If* K_I *is to be nonzero, also let* rank $X(0) = r$. *Let* $X(0)^I$ *be a right-inverse of* $X(0)$ (*if* $r = q$, *then* $X(0)^I = X(0)^{-1}$). *Let* $\Phi_P, \Phi_D, \Phi_I \in \{I_q, 0\}$. *Let* $\Delta = \mathrm{diag}[\delta_1, \delta_2, \ldots, \delta_r]$, $\delta_j \in (0, 1]$, $j = 1, \ldots, r$. *Choose any* $\hat{K}_P, \hat{K}_D \in \mathbb{R}^{q\times r}$, $\tau_d > 0$. (a) *Choose any* $\gamma \in \mathbb{R}_+$ *satisfying*

$$0 < \gamma < \left\| X\left(\hat{K}_P + \frac{\hat{K}_D s}{\tau_d s + 1}\right) + \frac{X(s)X(0)^I - I}{s} \right\|^{-1}. \tag{3}$$

Let $K_{\rm P}=\gamma\hat{K}_{\rm P}$, $K_{\rm D}=\gamma\hat{K}_{\rm D}$, $K_{\rm I}=\gamma X(0)^{\rm I}$; *then*

$$C_{\rm pid}=\gamma\hat{K}_{\rm P}+\frac{\gamma X(0)^{\rm I}}{s}+\frac{\gamma\hat{K}_{\rm D}s}{\tau_{\rm d}s+1} \tag{4}$$

is a PID-controller that stabilizes X. For $\hat{K}_{\rm D}=0$, (4) *is a PI-controller; for* $\hat{K}_{\rm P}=0$, (4) *is an ID-controller; for* $\hat{K}_{\rm D}=\hat{K}_{\rm P}=0$, (4) *is an I-controller*.

(b) *Choose any* $\gamma\in\mathbb{R}_+$ *satisfying*

$$\gamma<\min_{\Phi_{\rm P},\Phi_{\rm D},\Phi_{\rm I}}\left\|X\left(\Phi_{\rm P}\hat{K}_{\rm P}+\Phi_{\rm D}\frac{\hat{K}_{\rm D}s}{\tau_{\rm d}s+1}\right)+\frac{\Phi_{\rm I}(X(s)X(0)^{\rm I}-I)}{s}\right\|^{-1}. \tag{5}$$

With γ *satisfying* (5), $C_{\rm pid}$ *in* (4) *is a PID-controller that stabilizes X. Furthermore*, $Sys(X,C_\Phi\Delta)$ *is also stable for all* $\Delta={\rm diag}[\delta_1,\delta_2,\ldots,\delta_r]$, $\delta_j\in(0,1]$, $j=1,\ldots,r$, *where* $C_\Phi\Delta=[\Phi_{\rm P}\gamma\hat{K}_{\rm P}+\Phi_{\rm I}\frac{\gamma X(0)^{\rm I}}{s}+\Phi_{\rm D}\frac{\gamma\hat{K}_{\rm D}s}{\tau_{\rm d}s+1}]\Delta$.

Any subset of the PID-controller designed as in Proposition 3(b) also stabilizes X; i.e., $K_{\rm P}$, $K_{\rm I}/s$, $K_{\rm D}s/(\tau_{\rm d}s+1)$ each stabilize X individually, or in pairs, or all together. The proportional, integral, derivative blocks that remain in service can all be multiplied by the unknown diagonal multiplicative perturbation Δ that has "small gain".

Theorem 4 proposes a two-step approach for integral-action controller synthesis applicable to any general plant G. The two-step procedure starts with any stabilizing controller (without integral-action) for $G=XY^{-1}$ and then adds a PID-controller that was designed for the stable numerator X. Although this PID-controller for X can be designed using any method, we use the method of Proposition 3 that designs the PID-controller block with integral-action integrity. The closed-loop system remains stable even when any or all parts of the PID-controller is taken out of service or is perturbed by any diagonal constant matrix Δ such that $\|\Delta\|\leqslant 1$. Integral-action is present unless the integral term inside the PID block is taken out of service completely (switched-off). The integral-action in each channel can be scaled without affecting other channels since the individual channels may be multiplied by different factors.

Theorem 4 (*Two-step controller synthesis with integral-action integrity*). *Let* $G\in\mathbf{R_p}^{r\times q}$, ${\rm rank}\,G(s)=r\leqslant q$, *and let G have no transmission-zeros at* $s=0$. *Let* $G=XY^{-1}$ *be any RCF and* $G=\widetilde{Y}^{-1}\widetilde{X}$ *be any LCF of G. Let* $X(0)^{\rm I}$ *be any right-inverse of* $X(0)$. *Let* $\Phi_{\rm P},\Phi_{\rm D},\Phi_{\rm I}\in\{I_q,0\}$. *Let* $\Delta={\rm diag}[\delta_1,\delta_2,\ldots,\delta_r]$, $\delta_j\in(0,1]$, $j=1,\ldots,r$. *Choose any controller* $C_g\in\mathbf{R_p}^{q\times r}$ *that stabilizes G; let* $C_g=\widetilde{D}_g^{-1}\widetilde{N}_g$ *be an LCF of* C_g *such that* $\widetilde{D}_gY+\widetilde{N}_gX=I$. *Let a PID-controller* $C_{\rm pid}$ *that stabilizes* $X\in\mathbf{S}^{r\times q}$ *be designed as in Proposition* 3(*b*), *where* $\gamma\in\mathbb{R}_+$ *satisfies* (5). *Then for any* $Q\in\mathbf{S}^{q\times r}$ *such that* $\det(\widetilde{D}_g-Q\widetilde{X})(\infty)\neq 0$, *all integral-action controllers that stabilize G are given by*

$$\widehat{C}_{{\rm i}Q}=(\widetilde{D}_g-Q\widetilde{X})^{-1}[(\widetilde{N}_g+Q\widetilde{Y})+C_{\rm pid}]. \tag{6}$$

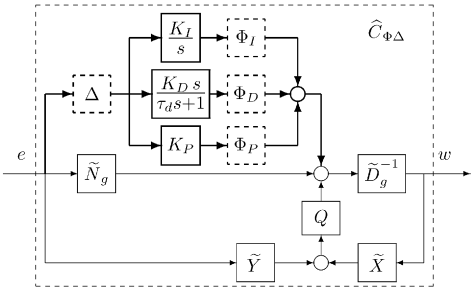

Fig. 2. The controller $\widehat{C}_{\Phi\Delta}$ as in Theorem 4.

For $Q=0$, (6) *becomes* $\widehat{C}_{\rm i}=\widetilde{D}_g^{-1}(\widetilde{N}_g+C_{\rm pid})=C_g+\widetilde{D}_g^{-1}C_{\rm pid}$. *Furthermore*, $Sys(G,\widehat{C}_{\Phi\Delta})$ *is stable for all* Δ *where* $\widehat{C}_{\Phi\Delta}=(\widetilde{D}_g-Q\widetilde{X})^{-1}(\widetilde{N}_g+Q\widetilde{Y}+C_\Phi\Delta)=(\widetilde{D}_g-Q\widetilde{X})^{-1}(\widetilde{N}_g+Q\widetilde{Y}+[\Phi_{\rm P}\gamma\hat{K}_{\rm P}+\Phi_{\rm I}(\gamma X(0)^{\rm I}/s)+\Phi_{\rm D}\gamma\hat{K}_{\rm D}s/(\tau_{\rm d}s+1)]\Delta)$.

The block-diagram of $Sys(G,\widehat{C}_{\Phi\Delta})$ is shown in Fig. 2. The main advantage of the two-step design proposed in Theorem 4 is that the integral-action is added as part of a separate module, which can be removed or limited without affecting closed-loop stability. Even if the plant is directly stabilizable by PID-controllers, this two-step approach may be preferred to switch the integrators off, e.g., to prevent integrator windup.

The two-stage design in Theorem 4 is extended in Corollary 5 to type-m integral-action achieving asymptotic tracking of polynomial references up to order $m-1$. As in the type-1 case, the block C_m that provides type-m integral-action can be turned-off completely without losing closed-loop stability:

Corollary 5 (*Extension to controller synthesis with type-m integral-action integrity*). *Under the assumptions of Theorem* 4, *let* $C_{\rm pid}$ *be a PID-controller that stabilizes* $X\in\mathbf{S}^{r\times q}$ *designed as in Proposition* 3(a). *If* $q>1$, *define* $G_1:=XC_{\rm pid}(I+XC_{\rm pid})^{-1}$; *if* $q>2$, *for* $2\leqslant v\leqslant q-1$, *define* $G_v:=XC_{\rm pid}\frac{1}{s^{v-1}}\prod_{j=2}^{v}k_j(I+XC_{\rm pid}+XC_{\rm pid}\sum_{j=2}^{v}\frac{1}{s^{j-1}}\prod_{\ell=2}^{j}k_\ell)^{-1}\in\mathbf{S}^{r\times r}$. *For* $2\leqslant v\leqslant q$, *choose any* $k_v\in\mathbb{R}_+$ *satisfying* (7) *and let* C_m *be as in* (8):

$$0<k_v<\left\|\frac{G_{v-1}(s)-I}{s}\right\|^{-1}, \tag{7}$$

$$C_m=C_{\rm pid}+C_{\rm pid}\sum_{j=2}^{m}\frac{1}{s^{j-1}}\prod_{\ell=2}^{j}k_\ell. \tag{8}$$

Then with $\Phi=I_q$, *all type-m integral-action controllers that stabilize G are given by*

$$\widehat{C}_{{\rm i}Q}=(\widetilde{D}_g-Q\widetilde{X})^{-1}[(\widetilde{N}_g+Q\widetilde{Y})+\Phi C_m], \tag{9}$$

for any $Q\in\mathbf{S}^{q\times r}$ *such that* $\det(\widetilde{D}_g-Q\widetilde{X})(\infty)\neq 0$. *The controller* $\widehat{C}_{{\rm i}Q}$ *stabilizes G also when* $\Phi=0$.

Remarks. ((1) *All achievable closed-loop transfer-functions using integral-action controllers*): With the initial controller

C_g, the achievable closed-loop (input–output) transfer-function of the stable $Sys(G, \widehat{C}_{iQ})$ is $H^g_{yu} := GC_g(I + GC_g)^{-1} = X(\widetilde{N}_g + Q\widetilde{Y})$ and the corresponding input-error transfer-function is $H^g_{eu} := (I + GC_g)^{-1} = (D_g - XQ)\widetilde{Y}$. With the integral-action controller $\widehat{C}_{\Phi\Delta} = (\widetilde{D}_g - Q\widetilde{X})^{-1}\,(\widetilde{N}_g + Q\widetilde{Y} + [\Phi_{\rm P}\gamma\hat{K}_{\rm P} + \Phi_{\rm I}(\gamma X(0)^{\rm I}/s) + \Phi_{\rm D}(\gamma\hat{K}_{\rm D}s/\tau_{\rm d}s + 1)]\Delta)$ in Theorem 4, all achievable $H_{eu} = (I_r + G\widehat{C}_{iQ})^{-1} = I - H_{yu} = (D_g - XQ)\widetilde{Y} - XC_\Phi\Delta(I + XC_\Phi\Delta)^{-1}(D_g - XQ)\widetilde{Y}$ becomes

$$H_{eu} = (I + XC_\Phi\Delta)^{-1}H^g_{eu}. \tag{10}$$

When $\Phi_{\rm I} = I$, $\Delta \neq 0$, the integral term is present in $C_\Phi\Delta$ and $(I + XC_\Phi\Delta)^{-1}(0) = 0$, which implies integral-action. Achievable input-error maps with or without integral-action can be easily compared using (10). With $\widehat{C}_{\Phi\Delta}$, all achievable $H_{yu} = I - H_{eu} = (I + XC_\Phi\Delta)^{-1}X(\widetilde{N}_g + Q\widetilde{Y}) + (I + XC_\Phi\Delta)^{-1}XC_\Phi\Delta = I - (I + XC_\Phi\Delta)^{-1}H^g_{eu} = X(\widetilde{N}_g + Q\widetilde{Y}) + XC_\Phi\Delta(I + XC_\Phi\Delta)^{-1}(D_g - XQ)\widetilde{Y}$ becomes

$$H_{yu} = H^g_{yu} + XC_\Phi\Delta(I + XC_\Phi\Delta)^{-1}H^g_{eu}. \tag{11}$$

Again, setting $\Phi_{\rm I} = I$ or $\Phi_{\rm I} = 0$, achievable input–output maps with or without integral-action can be easily compared using (11). Similarly, with the type-m integral-action controller $\widehat{C}_{iQ}$ in (9), $H_{yu} = (I + XC_m)^{-1}H^g_{yu} + (I + XC_m)^{-1}XC_m = H^g_{yu} + XC_m(I + XC_m)^{-1}H^g_{eu}$, where C_m can be completely removed. The achievable transfer-functions H_{yu} and H^g_{yu} can be compared to evaluate the system performance with or without the integral-action, which introduces to H^g_{yu} the additional term $XC_{\rm pid}(I + XC_{\rm pid})^{-1}H^g_{eu}$ (or $XC_m(I + XC_m)^{-1}H^g_{eu}$). The following complementary sensitivity and sensitivity bound comparisons are obtained for the system with and without the integral-action: $\|H_{yu}\| = \|H^g_{yu} + XC_{\rm pid}(I + XC_{\rm pid})^{-1}H^g_{eu}\| \leqslant \|H^g_{yu}\| + \|XC_{\rm pid}(I + XC_{\rm pid})^{-1}\|\|H^g_{eu}\|$, $\|H_{eu}\| = \|H^g_{eu} - XC_{\rm pid}(I + XC_{\rm pid})^{-1}H^g_{eu}\| \leqslant \|(I + XC_{\rm pid})^{-1}\|\|H^g_{eu}\|$.

(2) (*Two-step integral-action synthesis with an observer-based controller*): The integral-action controllers in (6) can also be expressed using a state-space representation (A, B, C, D) of the plant G, where $A \in \mathbb{R}^{n\times n}$, (A, B) is stabilizable and (C, A) is detectable. Let $K \in \mathbb{R}^{q\times n}$ and $L \in \mathbb{R}^{n\times r}$ be such that $F_K := (sI - A + BK)^{-1} \in \mathcal{M}(\mathbf{S})$ and $F_L := (sI - A + LC)^{-1} \in \mathcal{M}(\mathbf{S})$. In terms of this state-space representation, an RCF and LCF $G = XY^{-1} = \widetilde{Y}^{-1}\widetilde{X}$ are given by $Y = I - KF_KB$, $X = (C - DK)F_KB + D$, $\widetilde{Y} = I - CF_LL$, $\widetilde{X} = CF_L(B - LD) + D$ Vidyasagar, 1985. With $D_g = I + (C - DK)F_KL$, $N_g = KF_KL$, $\widetilde{D}_g = I + KF_L(B - LD)$, $\widetilde{N}_g = KF_LL$ satisfying $\widetilde{D}_gY + \widetilde{N}_gX = I$, a stabilizing controller is$C_g = \widetilde{D}_g^{-1}\widetilde{N}_g = N_gD_g^{-1} = K(sI - A + BK + L[C - DK])^{-1}L$. The PID-controller $C_{\rm pid}$ is designed for the stable numerator $X = (C - DK)F_KB + D$, where $X(0) = (C - DK)(-A + BK)^{-1}B + D$. If $C_{\rm pid}$ is designed following Proposition 3(b), then with $\widehat{C}_{\Phi\Delta}$ as in Theorem 4, $Sys(G, \widehat{C}_{\Phi\Delta})$ is stable. With the full-order observer-based controller C_g, the expression (6) for the integral-action controllers of Theorem 4 is obtained as $\widehat{C}_{iQ} = [I + KF_L(B - LD) - Q(\widehat{C}F_L(B - LD) + D)]^{-1}[KF_LL + Q(I - \widehat{C}F_LL) + C_{\rm pid}]$,

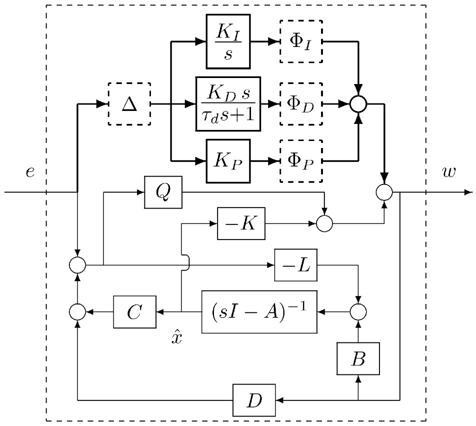

Fig. 3. The controller $\widehat{C}_{\Phi\Delta}$ using full-order observer-based controller C_g.

where $Q \in \mathbf{S}^{q\times r}$ is such that $\det(I - Q(\infty)D) \neq 0$. With $Q = 0$, the integral-action controller $\widehat{C}_{iQ}$ becomes $\widehat{C}_{\rm i} = C_g + [I + KF_L(B - LD)]^{-1}\,C_{\rm pid}$. The block diagram of $Sys(G, \widehat{C}_{\Phi\Delta})$ is shown in Fig. 3, with the full-order observer-based controller C_g.

4. Conclusions

We proposed a systematic integral-action synthesis procedure in Theorem 4, which achieves robust asymptotic tracking of step references in two steps: First, an initial stabilizing controller that does not have integral-action is designed for the original plant using any method. Second, a PID-controller is designed for the stable numerator-matrix of the plant using Proposition 3, which defines the proportional, integral and derivative constants explicitly. The two controllers acting together achieve closed-loop stability and integral-action. The PID-controller block designed in the second step can be removed and even limited by a constant gain matrix, multiplying the channels with arbitrary constants varying between zero and one. The design for the PID-controller in Proposition 3 also allows any or all of the (MIMO) terms $K_{\rm P}$, $K_{\rm I}$, $K_{\rm D}$ to be zero without losing stability. The significance of this integral-action integrity property is that it is possible to completely remove the integral term together with or separately from the proportional and derivative terms. The integral-action type can be iteratively increased or higher integral terms can be decreased by using the same stabilizing controller designed in the first step without plant augmentation or re-design.

Appendix

Proof of Lemma 2.1. If $Sys(G, \hat{C})$ is stable, $H_{eu}(0) = I_r - GH_{wu}(0) = 0$ implies $GH_{wu}(0) = I_r$; i.e., (normal) rank $(GH_{wu}) = r \leqslant \min\{\text{rank}\, G, \text{rank}\, H_{wu}\}$ implies $r \leqslant \text{rank}\, G \leqslant \min\{r, q\}$. By $H_{eu} = D_c\,M_L^{-1}\widetilde{Y} = I - XM_R^{-1}\widetilde{N}_c$, $H_{eu}(0) = 0$ implies $X(0)M_R^{-1}(0)\widetilde{N}_c(0) = I_r$; hence, rank $X(0) =$ rank $\widetilde{N}_c(0) = r$.

Proof of Lemma 2.2. The controller $C_g = N_g D_g^{-1}$ stabilizes $G = \widetilde{Y}^{-1}\widetilde{X}$ if and only if $M_g := \widetilde{Y} D_g + \widetilde{X} N_g$ is unimodular. Now C_x stabilizes $X \in \mathcal{M}(\mathbf{S})$ if and only if $C_x(I + XC_x)^{-1} \in \mathcal{M}(\mathbf{S})$, which implies $(I + XC_x)^{-1} \in \mathcal{M}(\mathbf{S})$. Define $D_c := (I + XC_x)^{-1} D_g \in \mathcal{M}(\mathbf{S})$, $N_c := [N_g + YC_x(I + XC_x)^{-1} D_g] \in \mathcal{M}(\mathbf{S})$. Since $(C_g X + Y) = \tilde{D}_g^{-1}$, for $Q = 0$, write $\hat{C}_Q = \hat{C} = N_c D_c^{-1}$. Then $\widetilde{Y} D_c + \widetilde{X} N_c = \widetilde{Y}(I + XC_x)^{-1} D_g + \widetilde{X} N_g + \widetilde{X} Y C_x (I + XC_x)^{-1} D_g = \widetilde{Y}(I + XC_x)^{-1} D_g + \widetilde{X} N_g + \widetilde{Y} X C_x (I + XC_x)^{-1} D_g = \widetilde{Y} D_g + \widetilde{X} N_g = M_g$ is unimodular; hence, $\widehat{C} = N_c D_c^{-1}$ stabilizes G. Since all stabilizing controllers for G can be expressed as $(\widetilde{D}_g - Q\widetilde{X})^{-1}(\widetilde{N}_g + Q\widetilde{Y})$, (1) is also a stabilizing controller for G for any $Q \in \mathcal{M}(\mathbf{S})$ such that $\det(\widetilde{D}_g - Q\widetilde{X})$ is biproper (Gündeş & Desoer, 1990; Vidyasagar, 1985).

Proof of Proposition 3. We prove (b); (a) is a special case when $\Phi_{\mathrm{P}} = \Phi_{\mathrm{D}} = \Phi_{\mathrm{I}} = I_q$. Write $C_{\mathrm{pid}} = [(s + a)^{-1} s C_{\mathrm{pid}}][(s + a)^{-1} s I_r]^{-1}$ for any $a > 0$. If $\Phi_{\mathrm{I}} = I_q$, define $M_\Delta := (sI + \gamma\Delta)^{-1} sI + (sI + \gamma\Delta)^{-1} s X C_{\mathrm{pid}} \Delta = (sI + \gamma\Delta)^{-1} sI + (sI + \gamma\Delta)^{-1} \gamma s X(\Phi_{\mathrm{P}} \hat{K}_{\mathrm{P}} + \Phi_{\mathrm{D}} \hat{K}_{\mathrm{D}} s/(\tau_{\mathrm{d}} s + 1) + X(0)^{\mathrm{I}}/s)\Delta = I + (sI + \gamma\Delta)^{-1} \gamma s[\, X(\Phi_{\mathrm{P}} \hat{K}_{\mathrm{P}} + \Phi_{\mathrm{D}} \hat{K}_{\mathrm{D}} s/(\tau_{\mathrm{d}} s + 1)) + (XX(0)^{\mathrm{I}} - I)/s]\Delta$. Since $\|\Delta\| \leqslant 1$ and $\|(sI + \gamma\Delta)^{-1} s\| = 1$, M_Δ is unimodular for $\gamma > 0$ satisfying (5); hence, $Sys(X, C_\Phi\Delta)$is stable for $\Phi_{\mathrm{P}}, \Phi_{\mathrm{D}} \in \{0, I_q\}$. If $\Phi_{\mathrm{I}}=0$, then $M_{\Delta pd} = I + \gamma X(\Phi_{\mathrm{P}} \hat{K}_{\mathrm{P}} + \Phi_{\mathrm{D}} \frac{\hat{K}_{\mathrm{D}} s}{\tau_{\mathrm{d}} s + 1})\Delta$ is unimodular since $\gamma < \|(\Phi_{\mathrm{P}} \hat{K}_{\mathrm{P}} + \Phi_{\mathrm{D}} \hat{K}_{\mathrm{D}} s/(\tau_{\mathrm{d}} s + 1))X\|^{-1}$ and $\|\Delta\| \leqslant 1$. Hence, $Sys(X, C_\Phi \Delta)$ is stable without the integral term in C_{pid} for $\Phi_{\mathrm{P}}, \Phi_{\mathrm{D}} \in \{0, I_q\}$.

Proof of Theorem 4. By Lemma 2.2, with $C_x = C_{\mathrm{pid}}$, $\widehat{C}_{\mathrm{i}Q}$ in (6) stabilizes G and has integral-action due to C_{pid}. By Proposition 3, the PID-controller for X has integral-action integrity, i.e., $C_\Phi \Delta$ also stabilizes X. Therefore, $Sys(G, \widehat{C}_{\Phi\Delta})$ is stable with $\widehat{C}_{\Phi\Delta}$ as in Eq. (6).

Proof of Corollary 5. Since C_g is a stabilizing controller, (9) stabilizes G when $\Phi = 0$. When $\Phi = I$, write $C_{\mathrm{pid}} = [(s/(s + a))C_{\mathrm{pid}}][sI/(s + a)]^{-1}$ for any $a \in \mathbb{R}_+$. Define

$$\frac{(s+a)}{(s+\gamma)} M_1 := \frac{sI}{s+\gamma} + \frac{s}{s+\gamma} X C_{\mathrm{pid}}$$
$$= I + \frac{\gamma s}{s+\gamma} \left[X(\hat{K}_{\mathrm{P}} + \frac{\hat{K}_{\mathrm{D}}\, s}{\tau_{\mathrm{d}} s + 1}) + \frac{XX(0)^{\mathrm{I}} - I}{s} \right].$$

Then M_1 is unimodular for $\gamma \in \mathbb{R}_+$ as in (3). If $m > 1$, $G_1 = M_1^{-1}(s/(s + a)) X C_{\mathrm{pid}} = (I + XC_{\mathrm{pid}})^{-1} X C_{\mathrm{pid}} \in \mathcal{M}(\mathbf{S})$. With $G_1(0) = I$, for any k_2 satisfying (Eq. (7)),

$$M_2 := \frac{sI}{s+a} + M_1^{-1} \frac{k_2 s X C_{\mathrm{pid}}}{(s+a)^2} = \frac{1}{(s+a)}[sI + k_2 G_1]$$

is unimodular since

$$\frac{(s+a)}{(s+k_2)} M_2 = I + \frac{k_2 s}{(s+k_2)} \frac{(XX(0)^{\mathrm{I}} - I)}{s}.$$

If $q > 2$, for $v = 2$, G_v becomes

$$G_2 = (M_1 M_2)^{-1} \frac{k_2 s X C_{\mathrm{pid}}}{(s+a)^2} = (I + G_1 \frac{k_2}{s})^{-1} G_1 \frac{k_2}{s}$$
$$= \left(I + XC_{\mathrm{pid}} + XC_{\mathrm{pid}} \frac{k_2}{s} \right)^{-1} XC_{\mathrm{pid}} \frac{k_2}{s} \in \mathcal{M}(\mathbf{S})$$

since $(M_1 M_2)$ is unimodular. With $G_2(0) = I$, for any k_3 satisfying (7),

$$M_3 := \frac{sI}{s+a} + (M_1 M_2)^{-1} \frac{k_2 k_3 s X C_{\mathrm{pid}}}{(s+a)^3}$$
$$= \frac{1}{(s+a)}[sI + k_3 G_2]$$

is unimodular. If $m > 3$, continue similarly; define

$$G_v = \left(\prod_{j=1}^{v} M_j \right)^{-1} \frac{s X C_{\mathrm{pid}} \prod_{j=1}^{v} k_j}{(s+a)^v}$$

for $v = 3, \ldots, m-1$; G_v is stable since $(\prod_{j=1}^{v} M_j)$ is unimodular. With $G_v(0) = I$, for any k_v satisfying (7),

$$M_{v+1} := \frac{sI}{s+a} + \left(\prod_{j=1}^{v} M_j \right)^{-1} \frac{s X C_{\mathrm{pid}} \prod_{j=2}^{v+1} k_j}{(s+a)^{v+1}}$$
$$= \frac{1}{(s+a)}[sI + k_{v+1} G_v]$$

is unimodular. Finally,

$$\left(\prod_{j=1}^{m} M_j \right) = \frac{s^m}{(s+a)^m} I + \frac{s^m}{(s+a)^m} X C_m$$

is unimodular and therefore, C_m stabilizes X. By Lemma 2.2, with $C_x = C_m$, $\widehat{C}_{\mathrm{i}Q}$ in Eq. (9) stabilizes G and has type-m integral-action due to C_m.

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