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Simultaneous Strong Stabilization and Tracking Controller Design for MIMO Systems

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Abstract— **A systematic controller synthesis method is presented for simultaneous stabilization with asymptotic tracking of step-input references for linear, time-invariant, multi-input multi-output plants. Necessary and sufficient conditions are derived for existence of simultaneous integral-action controllers and in particular, simultaneous PID-controllers. Strong simultaneous stabilization using low order PD-controllers as well as simultaneous stabilization with zero steady-state error using PID-controllers can be achieved for the class of plants under consideration.**

I. INTRODUCTION

Simultaneous stabilization problems arise in many practical control problems; for example, when designing a common controller for multiple operating points of the same system. We consider simultaneous stabilization of a finite class of linear, time-invariant (LTI) multi-input multi-output (MIMO) unstable or stable plants while achieving asymptotic tracking of step-input references with zero steady-state error. The simplest controllers that achieve integral-action are proportional+integral+derivative (PID) controllers, which are widely used and preferred for their simplicity. We derive conditions for existence of general integral-action controllers of any order, and particularly for existence of low-order controllers such as PID-controllers. We propose a systematic PID synthesis method for simultaneous stabilization of a finite number of plants.

Several rigorous PID design methods exist mostly for single-input single-output (SISO) systems (see [9], [1] and the references therein). Simultaneous stabilization while achieving asymptotic tracking and PID designs that achieve simultaneous closed-loop stability of MIMO systems have not been explored extensively. The problem of simultaneously stabilizing a class of (three or more) unstable plants is a difficult problem even without the order restriction imposed by PID [2], [3], [10]. In fact, since strong stabilizability is a necessary condition for existence of PID-controllers stabilizing a single plant, simultaneous strong stabilizability is necessary *but not sufficient* for simultaneous PID stabilizability of multiple plants.

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Simultaneous PID-control design for a set of stable MIMO plants satisfying a DC-gain condition was presented in [7]. This paper considers a more general class of plants that includes unstable as well as stable plants which may have at most one blocking-zero at infinity; there are no restrictions on the plant poles. It is shown that this class of (any finite number of) MIMO plants is strongly simultaneously stabilizable using PD-controllers and also simultaneously stabilizable with asymptotic tracking of step-input references using PID-controllers. The main results here are: 1) necessary conditions for existence of integral-action controllers (Lemma 1), and simultaneous integral-action controllers (Lemma 2), and 2) a systematic PD-controller and PID-controller synthesis method (Proposition 1). The systematic procedure for simultaneous PD/PID-controller synthesis is then applied to a simple example. The explicit designs allow freedom of choice of parameter values. Although the objective here is to achieve simultaneous closed-loop stability with tracking, the flexibility in the choice of the PID parameters offered by the design procedure may be used to satisfy additional performance criteria.

Although we discuss continuous-time systems here, all results apply also to discrete-time systems with appropriate modifications.

The following notation is used: $\mathcal{U}$ denotes the extended closed right-half plane, i.e., $\mathcal{U} = \{ s \in \mathbb{C} \mid \mathcal{R}e(s) \geq 0 \} \cup \{\infty\}$; $\mathbb{R}$, $\mathbb{R}_+$ denote real and positive real numbers; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in $\mathbf{S}$; I_m is the $m \times m$ identity matrix; we use I when the dimension is unambiguous. The H_∞-norm of $M(s) \in \mathcal{M}(\mathbf{S})$ is denoted by $\|M(s)\|$ (i.e., the norm $\|\cdot\|$ is defined as $\|M\| := \sup_{s\in\partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$). For simplicity, we drop (s) in transfer matrices such as $G(s)$ where this causes no confusion. We use coprime factorizations over $\mathbf{S}$; i.e., for $C \in \mathbf{R_p}^{m\times m}$, $C = ND^{-1}$ denotes a right-coprime-factorization (RCF), where $N \in \mathbf{S}^{m\times m}$, $D \in \mathbf{S}^{m\times m}$, $\det D(\infty) \neq 0$.

II. PRELIMINARIES

Consider the standard LTI, MIMO unity-feedback system $Sys(G_j, C)$ shown in Fig. 1, where $G_j \in \mathbf{R_p}^{m\times m}$, $j \in \{1, \ldots, k\}$, and $C \in \mathbf{R_p}^{m\times m}$ denote the plant's and the controller's transfer-functions, respectively. It is assumed that the feedback system is well-posed, G_j and C have no unstable hidden-modes, and each plant $G_j \in \mathbf{R_p}^{m\times m}$ is full normal rank. The objective is to design a single stabilizing controller C that achieves asymptotic tracking of step-input references with zero steady-state error for a finite class of plants G_j simultaneously.

Let $G_j = Y_j^{-1} X_j$ be a left-coprime-factorization (LCF) and $C = ND^{-1}$ be a right-coprime-factorization (RCF), where $Y_j, X_j, D, N \in \mathbf{S}^{m\times m}$, $\det Y_j(\infty) \neq 0$, $\det D(\infty) \neq 0$. Then C stabilizes $G_j \in \mathcal{M}(\mathbf{R_p})$ if and only if

$$M_j := Y_j\, D + X_j\, N \tag{1}$$

is unimodular [10], [5]. Let the (input-error) transfer-function from r to e be denoted by H_j^{er} and let the (input-output) transfer-function from r to y be denoted by H_j^{yr} ; then

$$\begin{aligned} H_j^{er} &= (I + G_j C)^{-1} = I - G_j C (I + G_j C)^{-1} \\ &=: I - G_j H_j^{wu} =: I - H_j^{yr} \ . \end{aligned} \tag{2}$$

Definition 1: i) The system $Sys(G_j, C)$ is said to be stable iff the closed-loop transfer-function from (r, v) to (y, w) is stable.
ii) The controller C is said to simultaneously stabilize G_j for $j \in \{1, \ldots, k\}$ iff C is proper and the systems $Sys(G_j, C)$ are all stable.
iii) The stable systems $Sys(G_j, C)$ are said to have integral-action iff H_j^{er} has blocking-zeros at $s = 0$, $j \in \{1, \ldots, k\}$.
iv) The controller C is said to be a simultaneously stabilizing integral-action controller iff C stabilizes G_j for $j \in \{1, \ldots, k\}$, and the denominator-matrix D of any RCF $C = ND^{-1}$ has blocking-zeros at $s = 0$, i.e., $D(0) = 0$. ∎

Suppose that the system $Sys(G_j, C)$ is stable and that step input references are applied to the system. Then the steady-state error $e(t)$ due to step inputs at $r(t)$ goes to zero as $t \to \infty$ if and only if $H_j^{er}(0) = 0$. Therefore, by Definition 1, the stable system $Sys(G_j, C)$ achieves asymptotic tracking of constant reference inputs with zero steady-state error if and only if it has integral-action. Write H_j^{er} as:

$$H_j^{er} = (I + G_j C)^{-1} = D\ M_j^{-1} Y_j \ . \tag{3}$$

By (3) and Definition 1, $Sys(G_j, C)$ has integral-action if $C = ND^{-1}$ is an integral-action controller since $D(0) = 0$ implies $H_j^{er}(0) = (D M_j^{-1} Y_j)(0) = 0$.

The simplest integral-action controllers are in PID form. We consider the following (realizable) form of proper PID-controllers, where $K_P, K_I, K_D \in \mathbb{R}^{m\times m}$ are called the proportional constant, the integral constant, and the derivative constant, respectively [4], [8]:

$$C_{pid} = K_P + \frac{K_I}{s} + \frac{K_D\, s}{\tau\, s + 1} \ . \tag{4}$$

Due to implementation issues of the derivative action, a pole is typically added to the derivative term (with $\tau > 0$) so that the transfer-function C_{pid} in (4) is proper. The only $\mathcal{U}$-pole of the PID-controller in (4) is at zero. The constants K_P, K_D, K_I may be negative; in the scalar case, this would imply that the zeros of C_{pid} may be in the unstable region $\mathcal{U}$. The integral-action in the PID-controller is present when the integral constant K_I is nonzero. Subsets of the PID-controller in (4) are: proportional+integral (PI) $C_{pi} = K_P + \frac{K_I}{s}$ (when $K_D = 0$); proportional+derivative (PD) $C_{id} = K_P + \frac{K_D\, s}{\tau\, s+1}$ (when $K_I = 0$); integral+derivative (ID) $C_{id} = \frac{K_I}{s} + \frac{K_D\, s}{\tau\, s+1}$ (when $K_P = 0$); integral (I) $C_i = \frac{K_I}{s}$ (when $K_P = K_D = 0$); derivative (D) $C_d = \frac{K_D\, s}{\tau\, s+1}$ (when $K_P = K_I = 0$); proportional (P) $C_p = K_P$ (when $K_I = K_D = 0$).

III. MAIN RESULTS

In Section III-A we derive necessary conditions for existence of simultaneously stabilizing integral-action controllers. In Section III-B we propose explicit PID-controller design under a sufficient existence condition, which turns out to be a *necessary* and sufficient condition for SISO systems.

A. Existence conditions for integral-action controllers

Lemma 1-(a) states the basic necessary condition on each G_j for existence of integral-action controllers; Lemma 1-(b) states the condition for existence of PID-controllers:

Lemma 1: (*Necessary conditions for integral-action*): Let $G_j \in \mathbf{R_p}^{m\times m}$. Let rank$G(s) = m$.
a) If the system $Sys(G_j, C)$ has integral-action, then G_j does not have transmission-zeros at $s = 0$.
b) If there exists a PID-controller that stabilizes G_j, then G_j is strongly stabilizable. ∎

Although strong stabilizability is a *necessary* condition for PID stabilizability of each G_j, it is not *sufficient*. We consider a sub-class of strongly stabilizable plants, which are in fact PID-stabilizable. The plants in this class have *at most* one blocking-zero at infinity and no other transmission-zeros in $\mathcal{U}$; they may have any number of (transmission and blocking) zeros in the stable region $\mathbb{C} \setminus \mathcal{U}$. The poles are completely arbitrary; some plants in the class may be stable while others are unstable. We denote the class of plants by

$$\mathcal{G} = \mathcal{G}_\infty \cup \hat{\mathcal{G}}_o \subset \mathbf{R_p}^{m\times m}$$

where $\mathcal{G}_\infty$ is the subset of the class of plants with exactly *one* blocking-zero at infinity and $\hat{\mathcal{G}}_o$ is the subset with *no* zeros in the region of instability $\mathcal{U}$. More specifically, the plants G_j, $\hat{G}_k \in \mathbf{R_p}^{m\times m}$ can be expressed as follows:

For $G_j \in \mathcal{G}_\infty$ we have $Y_j := \frac{1}{s+a} G_j^{-1}$ stable for any $a \in \mathbb{R}_+$, i.e.,

$$G_j = Y_j^{-1} X = [\ \frac{1}{s+a} G_j^{-1}\ ]^{-1}\ [\ \frac{1}{s+a} I\ ]\ . \tag{5}$$

For $\hat{G}_k \in \hat{\mathcal{G}}_o$ we have $\hat{Y}_k := \hat{G}_k^{-1}$ stable, i.e.,

$$\hat{G}_k = \hat{Y}_k^{-1} \hat{X} = [\ \hat{G}_k^{-1}\ ]^{-1}\ [\ I\ ]\ . \tag{6}$$

Each individual plant G_j or $\hat{G}_k$ as described in (5) or (6) is PID stabilizable [6]. However, existence of a single integral-action controller that simultaneously stabilizes all plants $G_j \in \mathcal{G}_\infty$ requires additional necessary conditions as stated in Lemma 2.

Lemma 2: (*Necessary conditions for simultaneous integral-action*): Let $G_i, G_j \in \mathcal{G}_\infty$ be as in (5). If there exist simultaneously stabilizing integral-action controllers, then

$$\det [\, Y_j(\infty)\, Y_i(\infty)^{-1}\,] > 0\ , \text{ for all } G_j, G_i \in \mathcal{G}_\infty\ . \tag{7}$$

■

The necessary condition (7) requires $\det Y_j(\infty) = \det \frac{1}{s+a} G_j^{-1}|_{s\to\infty}$ to have the same sign for all $G_j \in \mathcal{G}_\infty$ for existence of a common stabilizing controller with integral-action, and in particular a simultaneous PID-controller. For SISO plants, this condition is in fact also sufficient for existence of PID-controllers as shown in Proposition 1 in Section III-B below. Note that the sign condition is only on $\det Y_j(\infty)$ of $G_j \in \mathcal{G}_\infty$; the sign of $\det \hat{Y}_k(\infty) = \det \hat{G}_k^{-1}(\infty)$ for $\hat{G}_k \in \hat{\mathcal{G}}_o$ need not satisfy a similar sign condition.

B. PD/PID-controller synthesis

For $G_j, \hat{G}_k \in \mathbf{R_p}^{m\times m}$ as in (5)-(6), Proposition 1 presents a method for designing PID-controllers that simultaneously stabilize all plants in the class $\mathcal{G} = \mathcal{G}_\infty \cup \hat{\mathcal{G}}_o$. A sufficient condition for existence of PD-controllers as well as PID-controllers simultaneously stabilizing all plants in $\mathcal{G}_\infty$ is that the eigenvalues of

$$W_j := Y_j(\infty)\, Y_1(\infty)^{-1} \tag{8}$$

are positive real for all $G_j \in \mathcal{G}_\infty$, where $G_1 = Y_1^{-1} X \in \mathcal{G}_\infty$. If this condition holds, then $\det Y_j(\infty) Y_1(\infty)^{-1} > 0$ implies the sign of $\det Y_j(\infty)$ is the same as the sign of $\det Y_1(\infty)$ and hence, the necessary condition (7) in Lemma 2 holds. Clearly, in the case of SISO systems, W_j is a scalar and hence, the necessary and sufficient condition for existence of PID-controllers that simultaneously stabilize all $G_j \in \mathcal{G}_\infty$ is that $W_j > 0$.

Note that the positive real condition on the eigenvalues of W_j in (8) is imposed only on the matrices $G_j, G_1 \in \mathcal{G}_\infty$; however, it is sufficient for simultaneous PD or PID stabilization of all plants $G_j, \hat{G}_k \in \mathcal{G}_\infty \cup \hat{\mathcal{G}}_o$. In fact, if the subset $\mathcal{G}_\infty$ is empty, then the plants $\hat{G}_k \in \hat{\mathcal{G}}_o$, which have no transmission or blocking-zeros in $\mathcal{U}$ as in (6), are simultaneously PD or PID-stabilizable without any additional assumptions.

Proposition 1 gives a systematic synthesis approach for the cases when
(i) $\mathcal{G}_\infty$ is non-empty (some plants are strictly-proper as in (5) and there may be other plants $\hat{G}_k$ as in (6) that have no zeros at infinity) and
(ii) $\mathcal{G}_\infty$ is empty (none of the plants is strictly-proper and $\mathcal{G} = \hat{\mathcal{G}}_o$).

Proposition 1: (*PD/PID synthesis for simultaneous stabilization*): Let $G_j \in \mathcal{G}_\infty$ and $\hat{G}_k \in \hat{\mathcal{G}}_o$ be as in (5)-(6).

i) If $\mathcal{G}_\infty$ is non-empty, designate a plant $G_1 \in \mathcal{G}_\infty$ as the nominal plant. Let $\det Y_j(\infty) = \det \frac{1}{s+a} G_j^{-1}|_{s\to\infty}$. Suppose that all eigenvalues of $W_j := Y_j(\infty)\, Y_1(\infty)^{-1}$ are real and positive for all $G_j \in \mathcal{G}_\infty$. Then PD-controllers and PID-controllers exist that simultaneously stabilize all plants $G_j, \hat{G}_k \in \mathcal{G}_\infty \cup \hat{\mathcal{G}}_o$. Furthermore, they can be designed as follows:

a) PD-controllers: Choose any $K_D \in \mathbb{R}^{m\times m}$, $\tau \in \mathbb{R}_+$. Define Φ_j, $\hat{\Phi}_k \in \mathcal{M}(\mathbf{S})$ as

$$\Phi_j = (\ G_j^{-1} + \frac{K_D\, s}{\tau s + 1}\,)\, Y_1(\infty)^{-1} - s\, W_j\ , \tag{9}$$

$$\hat{\Phi}_k = (\ \hat{G}_k^{-1} + \frac{K_D\, s}{\tau s + 1}\,)\, Y_1(\infty)^{-1}\ . \tag{10}$$

Let $K_P = \alpha\, Y_1(\infty)$ and let C_{pd} be given by

$$C_{pd} = \alpha\, Y_1(\infty) + \frac{K_D\, s}{\tau s + 1}\ . \tag{11}$$

Then
1) C_{pd} in (11) stabilizes all $G_j \in \mathcal{G}_\infty$ for any $\alpha \in \mathbb{R}_+$ satisfying

$$\alpha > \max_{G_j \in \mathcal{G}_\infty} \|\ \Phi_j\ \|\ . \tag{12}$$

2) C_{pd} in (11) stabilizes all $G_j, \hat{G}_k \in \mathcal{G}_\infty \cup \hat{\mathcal{G}}_o$ for any $\alpha \in \mathbb{R}_+$ satisfying

$$\alpha > \max\{ \max_{G_j \in \mathcal{G}_\infty} \|\ \Phi_j\ \|\ ,\ \max_{\hat{G}_k \in \hat{\mathcal{G}}_o} \|\ \hat{\Phi}_k\ \|\ \}\ . \tag{13}$$

For $K_D = 0$, (11) is a P-controller.

b) PID-controllers: Choose any $K_D \in \mathbb{R}^{m\times m}$, $\tau \in \mathbb{R}_+$. Choose any $g \in \mathbb{R}_+$. Define Ψ_j , $\hat{\Psi}_k \in \mathcal{M}(\mathbf{S})$ as

$$\Psi_j = \frac{s}{s+g}\,(\, G_j^{-1} + \frac{K_D\, s}{\tau s+1}\,)\, Y_1(\infty)^{-1} - s\, W_j \ , \quad (14)$$

$$\hat{\Psi}_k = \frac{s}{s+g}\,(\, \hat{G}_k^{-1} + \frac{K_D\, s}{\tau s+1}\,)\, Y_1(\infty)^{-1} \ . \quad (15)$$

Let $K_P = \beta\, Y_1(\infty)$, $K_I = \beta\, g\, Y_1(\infty)$ and let C_{pid} be given by

$$C_{pid} \, = \, \beta\, Y_1(\infty) + \frac{K_D\, s}{\tau s+1} + \frac{\beta\, g\, Y_1(\infty)}{s} \ . \quad (16)$$

Then

1) C_{pid} in (16) stabilizes all $G_j \in \mathcal{G}_\infty$ for any $\beta \in \mathbb{R}_+$ satisfying

$$\beta > \max_{G_j \, \in \, \mathcal{G}_\infty} \ \| \ \Psi_j \ \| \ . \quad (17)$$

2) C_{pid} in (16) stabilizes all G_j , $\hat{G}_k \in \mathcal{G}_\infty \cup \hat{\mathcal{G}}_o$ for any $\beta \in \mathbb{R}_+$ satisfying

$$\beta > \max\{ \max_{G_j \, \in \, \mathcal{G}_\infty} \ \| \ \Psi_j \ \| \ , \ \max_{\hat{G}_k \, \in \, \hat{\mathcal{G}}_o} \ \| \ \hat{\Psi}_k \ \| \ \} \ . \quad (18)$$

For $K_D = 0$, (16) is a PI-controller.

ii) If $\mathcal{G}_\infty$ is empty, i.e., $\mathcal{G} = \hat{\mathcal{G}}_o$, then PD-controllers and PID-controllers exist that simultaneously stabilize all plants $\hat{G}_k \in \hat{\mathcal{G}}_o$. Furthermore, they can be designed as follows: Choose any $K_D \in \mathbb{R}^{m\times m}$, $\tau \in \mathbb{R}_+$. Choose any nonsingular $\hat{K}_P \in \mathbb{R}^{m\times m}$. Choose any $\hat{g} \in \mathbb{R}_+$. Let $K_P = \hat{\alpha}\, \hat{K}_P$, $K_I = \hat{\alpha}\, \hat{g}\, \hat{K}_P$; let C_{pd} be given by (19) and C_{pid} be given by (20):

$$C_{pd} \, = \, \hat{\alpha}\, \hat{K}_P + \frac{K_D\, s}{\tau s+1} \ , \quad (19)$$

$$C_{pid} \, = \, C_{pd} + \frac{K_I}{s} \, = \, \hat{\alpha}\, \hat{K}_P + \frac{K_D\, s}{\tau s+1} + \frac{\hat{\alpha}\, \hat{g}\, \hat{K}_P}{s} \ . \quad (20)$$

Then for any $\hat{\alpha} \in \mathbb{R}_+$ satisfying

$$\hat{\alpha} > \max_{\hat{G}_k \in \hat{\mathcal{G}}_o} \ \| \ (\ \hat{G}_k^{-1} + \frac{K_D\, s}{\tau s+1}\,)\, \hat{K}_P^{-1} \ \| \ , \quad (21)$$

the PD-controller C_{pd} in (19) stabilizes all $\hat{G}_k \in \hat{\mathcal{G}}_o$ and the PID-controller C_{pid} in (20) stabilizes all $\hat{G}_k \in \hat{\mathcal{G}}_o$. For $K_D = 0$, (19) is a P-controller and (20) is a PI-controller. ■

In Example 1 we apply the systematic design procedure of Proposition 1 to a class of SISO plants with no zeros at $s = 0$.

Example 1: Consider the class of strictly-proper (relative-degree one) plants $\mathcal{G}_\infty = \{\, G_1, G_2, G_3\,\}$, and the class of non-strictly-proper (relative-degree zero) plants $\hat{\mathcal{G}}_o = \{\, \hat{G}_1, \hat{G}_2\,\}$, where

$$G_1 = \frac{1}{4(s-3)} \ , \quad G_2 = \frac{(s+9)}{(s-2)(s-5)} \ ,$$

$$G_3 = \frac{s+17}{8(s^2+6s+25)} \ ,$$

$$\hat{G}_1 = \frac{-(s^2+8s+30)}{5(s+1)(s-9)} \ ,$$

$$\hat{G}_2 = \frac{(s+2)^2(s+3)(s+5)}{10(s-4)^2(s-6)^2} \ .$$

Since the plants are SISO, by Lemma 2 and Proposition 1, the necessary and sufficient condition for existence of PID-controllers that simultaneously stabilize all plants in the class $\mathcal{G}_\infty$ is that $Y_j(\infty) Y_i(\infty)^{-1} > 0$, where $Y_j(\infty) = \frac{1}{(s+a)} G_j^{-1}(s)|_{s\to\infty}$. Now $Y_1(\infty) = 4$, $Y_2(\infty) = 1$, $Y_3(\infty) = 8$ are all positive, and hence the necessary and sufficient condition holds. Note that $\hat{Y}_k(\infty) = \hat{G}_k^{-1}(\infty)$ need not have the same sign for all $\hat{G}_k \in \hat{\mathcal{G}}_o$; in this example $\hat{Y}_1(\infty) = -0.2$ and $\hat{Y}_2(\infty) = 10$.

We follow Proposition 1-(i-a) to design PD-controllers. Choose $K_D = 4$, $\tau = 0.05$. With $W_1 = 1$, $W_2 = 1/4$, $W_3 = 2$, $\| \Phi_j \|$ in (9) are $\| \Phi_1 \| = 17$, $\| \Phi_2 \| = 16$, $\| \Phi_3 \| = 2.9558$. Also, $\| \hat{\Phi}_k \|$ in (10) are computed as $\| \hat{\Phi}_1 \| = 18.75$, $\| \hat{\Phi}_2 \| = 24.0$. Let C_{pd} be as in (11), i.e.,

$$C_{pd} = 4\,\alpha + \frac{4s}{0.05s+1} \ .$$

Then by (12), C_{pd} simultaneously stabilizes the plants in $\mathcal{G}_\infty = \{\, G_1, G_2, G_3\,\}$ for $\alpha > 17$. By (13), C_{pd} simultaneously stabilizes the plants in $\mathcal{G}_\infty \cup \hat{\mathcal{G}}_o = \{\, G_1, G_2, G_3, \hat{G}_1, \hat{G}_2\,\}$ for $\alpha > 24.0$.

Now we follow Proposition 1-(i-b) to design PID-controllers with the same K_D and τ as in the PD design and we choose $g = 30$. We compute $\| \Psi_j \|$ in (14) and $\| \hat{\Psi}_k \|$ in (15) as $\| \Psi_1 \| = 15.08$, $\| \Psi_2 \| = 8.5$, $\| \Psi_3 \| = 62.0$, $\| \hat{\Psi}_1 \| = 18.75$, $\| \hat{\Psi}_2 \| = 22.5$. Let C_{pid} be as in (16), i.e.,

$$C_{pid} = 4\,\beta + \frac{4s}{0.05s+1} + \frac{120\,\beta}{s} \ .$$

Then by (17), C_{pid} simultaneously stabilizes the plants in $\mathcal{G}_\infty = \{\, G_1, G_2, G_3\,\}$ for $\beta > 62$. By (18), C_{pid} simultaneously stabilizes the plants in $\mathcal{G}_\infty \cup \hat{\mathcal{G}}_o = \{\, G_1, G_2, G_3, \hat{G}_1, \hat{G}_2\,\}$ for $\beta > 62$.

To design PD/PID-controllers that would only stabilize the plants $\hat{G}_1, \hat{G}_2 \in \hat{\mathcal{G}}_o$, we could also follow Proposition 1-(ii). Choosing $K_D = 4$, $\tau = 0.05$, $\hat{K}_P = 20$, the controllers C_{pd} and C_{pid} as in (19)-(20) are

$$C_{pd} = 20\,\hat{\alpha} + \frac{4s}{0.05s+1} \ ,$$

$$C_{pid} = 20\,\hat{\alpha} + \frac{4s}{0.05s+1} + \frac{20\,\hat{\alpha}\,\hat{g}}{s} \ ,$$

where $\hat{g} > 0$ can be chosen arbitrarily. Then by (21), for $\hat{\alpha} > \max\{\, 3.75, 4.8\,\} = 4.8$, C_{pd} and C_{pid} are both stabilizing controllers for $\hat{\mathcal{G}}_o = \{\, \hat{G}_1, \hat{G}_2\,\}$. ■

IV. CONCLUSIONS

We showed that a class of stable or unstable plants $\{G_1, \ldots, G_j, \hat{G}_1, \ldots, \hat{G}_k\}$ which do not have more than one blocking-zero at infinity can be simultaneously stabilized using low-order PD-controllers or integral-action PID-controllers under the sufficient condition that the eigenvalues of $W_j = Y_j(\infty)Y_1(\infty)^{-1}$ are all positive real for the plants G_j. The sufficient positivity condition is also necessary for SISO systems. We presented a systematic method to design PD-controllers as well as PID-controllers for the plant class under consideration. This synthesis method allows a wide range of choices for the PID parameters.

The restriction in the plant class we considered is that the plants may only have blocking or transmission zeros in the region of stability and they may have up to one blocking-zero at infinity; in the SISO case this means they may have relative degree ≤ 1. Since PID-controllers do not necessarily exist for unstable plants and since simultaneous stabilization of three or more unstable plants is an extremely difficult problem even without restrictions on the controller order, expanding the plant classes is a challenge. Future studies will include plants with higher order blocking-zeros at infinity and those with other right-half-plane transmission-zeros.

APPENDIX (PROOFS)

Proof of Lemma 1:

a) Let $G_j = \tilde{X}_j\tilde{Y}_j^{-1}$ be an RCF of G_j. If $Sys(G_j, C)$ is stable, then by (2), $H_j^{wu} \in \mathcal{M}(\mathbf{S})$ and $H_j^{yr} = G_j H_j^{wu} = \tilde{X}_j\tilde{Y}_j^{-1}H_j^{wu} \in \mathcal{M}(\mathbf{S})$ implies $\tilde{Y}_j^{-1}H_j^{wu} =: R_j \in \mathcal{M}(\mathbf{S})$. If $Sys(G_j, C)$ has integral-action, then $H_j^{er}(0) = 0$ implies $H_j^{yr}(0) = I = \tilde{X}_j(0)R_j(0)$. Therefore, $\text{rank}\tilde{X}_j(0) = m$, equivalently, G_j has no transmission-zeros at $s = 0$.

b) Write $C_{pid} = ND^{-1}$ as

$$C_{pid} = ND^{-1} = [\,(\,K_P + \frac{K_D\, s}{\tau s+1}\,)\frac{s}{s+e} + \frac{K_I}{s+e}\,][\frac{s\,I}{s+e}]^{-1}$$

$$= [\frac{s}{s+e}C_{pid}\,]\,[\frac{s\,I}{s+e}]^{-1} \qquad (22)$$

(for any $e \in \mathbb{R}_+$). For all $z > 0$, det $D(z) = $ det $\frac{z}{z+e}I > 0$. If C_{pid} stabilizes G_j, by (1) M_j unimodular implies det $M_j(z) = \det Y_j(z)$ det $D(z)$ has the same sign for all $z \in \mathcal{U}$ such that $X_j(z) = 0$; equivalently, $\det Y_j(z)$ has the same sign at all blocking-zeros of G_j, i.e., G_j has the parity-interlacing-property; therefore G_j is strongly stabilizable [10]. ■

Proof of Lemma 2:

a) Let $C = ND^{-1}$ be an integral-action controller simultaneously stabilizing all plants $G_j \in \mathcal{G}_\infty$, where G_i, G_j are any two arbitrary plants in the class; therefore they are in the form (5). Since C has integral-action, the denominator D can be written as $D =: \frac{s}{s+e}D_c$ for $e \in \mathbb{R}_+$, where $D_c \in \mathcal{M}(\mathbf{S})$. By (1), $M_i = \frac{s}{s+e}Y_i\,D_c + XN$ and $M_j = \frac{s}{s+e}Y_j\,D_c + XN$ are unimodular. But $M_i(0) = X(0)N(0) = M_j(0)$ implies $\det M_i(s)$ has the same sign as $\det M_j(s)$ for all $s \in \mathcal{U}$ and in particular, for $s = \infty$. Since $X(\infty) = 0$, $\det M_j(\infty) = \det Y_j(\infty)\det D_c(\infty)$ has the same sign as $\det M_i(\infty) = \det Y_i(\infty)\det D_c(\infty)$ implies $\det Y_j(\infty)$ has the same sign as $\det Y_i(\infty)$ and hence, condition (7) follows. ■

Proof of Proposition 1:

i) a) Let $C_{pd} = C_{pd}I^{-1}$ be as in (11). For $G_j \in \mathcal{G}_\infty$ as in (5), where $X = \frac{1}{s+a}I$, M_j in (1) becomes

$$M_j = XC_{pd} + Y_j = X\alpha Y_1(\infty) + Y_j + X\frac{K_D\,s}{\tau\,s+1}$$

$$= \frac{(W_j s + \alpha I)}{(s+a)}\;[\,(W_j s + \alpha I)^{-1}\alpha\,I$$

$$+\,(W_j s + \alpha I)^{-1}(\,(s+a)Y_j + \frac{K_D\,s}{\tau\,s+1})\,Y_1(\infty)^{-1}]\,Y_1(\infty)$$

$$= \frac{(W_j s + \alpha I)}{(s+a)}[\,I + (W_j s + \alpha I)^{-1}\Phi_j\,]\,Y_1(\infty)\;.$$

By assumption, W_j has real positive eigenvalues implies $(W_j s + \alpha I)^{-1} \in \mathcal{M}(\mathbf{S})$; then $\|(W_j s + \alpha I)^{-1}\| = 1/\alpha$. If α satisfies (12), then

$$\|\,(W_j s + \alpha I)^{-1}\Phi_j\,\| \leq \|\,(W_j s + \alpha I)^{-1}\|\;\|\Phi_j\,\|$$

$$= \frac{1}{\alpha}\,\|\Phi_j\,\| \,<\, 1$$

implies M_j is unimodular for each $G_j \in \mathcal{G}_\infty$ and hence, $S(G_j, C_{pd})$ is stable. Now for $\hat{G}_k \in \hat{\mathcal{G}}_o$ as in (6), where $\hat{X} = I$, $\hat{Y}_k = \hat{G}_k^{-1}$, we have

$$\hat{M}_k = \hat{X}C_{pd} + \hat{Y}_k = \alpha Y_1(\infty) + \hat{Y}_k + \frac{K_D\,s}{\tau\,s+1}$$

$$= [\,I + \frac{1}{\alpha}(\hat{G}_k^{-1} + \frac{K_D\,s}{\tau\,s+1})Y_1(\infty)^{-1}\,]\,\alpha Y_1(\infty)$$

$$= [\,I + \frac{1}{\alpha}\,\hat{\Phi}_k\,]\,\alpha Y_1(\infty)\;.$$

If $\alpha > \max_{\hat{G}_k \in \hat{\mathcal{G}}_o} \|\,\hat{\Phi}_k\,\|$, then $\|\,\frac{1}{\alpha}\hat{\Phi}_k\,\| < 1$ implies $\hat{M}_k$ is unimodular for each $\hat{G}_k \in \hat{\mathcal{G}}_o$ and hence, $S(\hat{G}_k, C_{pd})$ is stable. Therefore, if α satisfies (13), then C_{pd} stabilizes all plants in the entire class $\mathcal{G} = \mathcal{G}_\infty \cup \hat{\mathcal{G}}_o$. If $\hat{\mathcal{G}}_o$ is empty, then α only satisfies (12).

b) Let C_{pid} be as in (16) and write $C_{pid} = [\frac{s}{s+e}C_{pid}]\,[\frac{sI}{s+e}]^{-1}$ as in (22). For $G_j \in \mathcal{G}_\infty$, M_j becomes

$$M_j = X\, C_{pid}\frac{s}{s+e} + Y_j\,\frac{s}{s+e}$$

$$= \frac{(s+g)}{(s+a)}\Big[\frac{1}{(s+g)}\Big(K_P + \frac{gK_P}{s} + \frac{K_D\, s}{\tau\, s+1} + (s+a)Y_j\Big)\Big]\frac{s}{(s+e)}$$

$$= \frac{(s+g)}{(s+a)}\Big[\frac{\beta\, Y_1(\infty)}{s+e} + \frac{1}{(s+g)}\Big(G_j^{-1} + \frac{K_D\, s}{\tau\, s+1}\Big)\,\frac{s}{(s+e)}\,\Big]$$

$$= \frac{(s+g)}{(s+a)}\,\frac{(W_j s + \beta I)}{(s+e)}\,\Big[\,(W_j s+\beta I)^{-1}\beta\, I$$

$$+(W_j s+\beta I)^{-1}\Big(\,G_j^{-1} + \frac{K_D\, s}{\tau\, s+1}\Big)\frac{s}{(s+g)}\,Y_1(\infty)^{-1}\Big]\,Y_1(\infty)$$

$$= \frac{(s+g)(W_j s+\beta I)}{(s+a)(s+e)}\,[\,I + (W_j s+\beta I)^{-1}\Psi_j\,]\,Y_1(\infty)\ .$$

Following similar steps as in part (a) above, $(W_j s+\beta I)^{-1} \in \mathcal{M}(\mathbf{S})$; then $\|(W_j s+\beta I)^{-1}\| = 1/\beta$. If β satisfies (17), then

$$\|\,(W_j s+\beta I)^{-1}\Psi_j\,\| \le \|\,(W_j s+\beta I)^{-1}\|\,\|\Psi_j\,\|$$

$$= \frac{1}{\beta}\,\|\Psi_j\,\| \ <\ 1$$

implies M_j is unimodular for each $G_j \in \mathcal{G}_\infty$ and hence, $S(G_j, C_{pid})$ is stable. Now for $\hat{G}_k \in \hat{\mathcal{G}}_o$ we have

$$\hat{M}_k = \hat{X}\, C_{pid}\frac{s}{s+e} + \hat{Y}_k\,\frac{s}{s+e}$$

$$= [K_P + \frac{g\,K_P}{s} + \frac{K_D\, s}{\tau\, s+1} + \hat{Y}_k\,]\frac{s}{s+e}$$

$$= \frac{(s+g)}{(s+e)}K_P + (\,\hat{Y}_k + \frac{K_D\, s}{\tau\, s+1}\,)\,\frac{s}{s+e}$$

$$= [\,I + \frac{1}{\beta}\frac{s}{(s+g)}(\hat{G}_k^{-1} + \frac{K_D\, s}{\tau\, s+1})Y_1(\infty)^{-1}\,]\beta Y_1(\infty)\frac{(s+g)}{(s+e)}$$

$$= [\,I + \frac{1}{\beta}\,\hat{\Psi}_k\,]\,\beta Y_1(\infty)\frac{(s+g)}{(s+e)}\ .$$

If $\beta > \max_{\hat{G}_k \in \hat{\mathcal{G}}_o} \|\,\hat{\Psi}_k\,\|$, then $\|\,\frac{1}{\beta}\hat{\Psi}_k\,\| \ <\ 1$ implies $\hat{M}_k$ is unimodular for each $\hat{G}_k \in \hat{\mathcal{G}}_o$ and hence, $S(\hat{G}_k, C_{pid})$ is stable. Therefore, if β satisfies (18), then C_{pid} stabilizes all plants in the entire class $\mathcal{G} = \mathcal{G}_\infty \cup \hat{\mathcal{G}}_o$. If $\hat{\mathcal{G}}_o$ is empty, then β only satisfies (17).

ii) Let C_{pd} be as in (19). For $\hat{G}_k \in \hat{\mathcal{G}}_o$ we have

$$\hat{M}_k = \hat{X}C_{pd} + \hat{Y}_k = \hat{\alpha}\hat{K}_P + \frac{K_D\, s}{\tau\, s+1} + \hat{G}_k^{-1}$$

$$= [\,I + \frac{1}{\hat{\alpha}}(\hat{G}_k^{-1} + \frac{K_D\, s}{\tau\, s+1})\hat{K}_P^{-1}\,]\,\hat{\alpha}\hat{K}_P\ .$$

For $\hat{\alpha}$ satisfying (21), $\|\,\frac{1}{\hat{\alpha}}(\hat{G}_k^{-1} + \frac{K_D\, s}{\tau\, s+1})\hat{K}_P^{-1}\,\| < 1$ implies $\hat{M}_k$ is unimodular and hence, C_{pd} stabilizes all $\hat{G}_k \in \hat{\mathcal{G}}_o$. Now let C_{pid} be as in (20). Then

$$\hat{M}_k = \hat{X}C_{pid}\frac{s}{s+e} + \hat{Y}_k\,\frac{s}{s+e}$$

$$= [(1 + \frac{\hat{g}}{s})\,\hat{\alpha}\,\hat{K}_P + \frac{K_D\, s}{\tau\, s+1} + \hat{G}_k^{-1}\,]\frac{s}{s+e}$$

$$= \frac{(s+\hat{g})}{(s+e)}\,\hat{\alpha}\hat{K}_P + (\,\hat{G}_k^{-1} + \frac{K_D\, s}{\tau\, s+1}\,)\frac{s}{s+e}$$

$$= [\,I + \frac{1}{\hat{\alpha}}\frac{s}{(s+\hat{g})}(\hat{G}_k^{-1} + \frac{K_D\, s}{\tau\, s+1})\hat{K}_P^{-1}\,]\,\hat{\alpha}\hat{K}_P\,\frac{(s+\hat{g})}{(s+e)}\ .$$

For $\hat{\alpha}$ satisfying (21),

$$\|\,\frac{1}{\hat{\alpha}}\frac{s}{(s+\hat{g})}(\hat{G}_k^{-1} + \frac{K_D\, s}{\tau\, s+1})\hat{K}_P^{-1}\,\|$$

$$\le \frac{1}{\hat{\alpha}}\ \|\,\frac{s}{(s+\hat{g})}\,\|\ \|\,(\hat{G}_k^{-1} + \frac{K_D\, s}{\tau\, s+1})\hat{K}_P^{-1}\,\|$$

$$\le \frac{1}{\hat{\alpha}}\,\|\,(\hat{G}_k^{-1} + \frac{K_D\, s}{\tau\, s+1})\hat{K}_P^{-1}\,\|\ <\ 1$$

implies $\hat{M}_k$ is unimodular and hence, C_{pid} stabilizes all $\hat{G}_k \in \hat{\mathcal{G}}_o$. ∎

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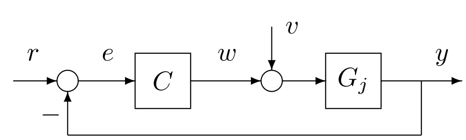

Fig. 1. Unity-Feedback System $Sys(G_j\,,C)$.