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Simultaneous Tracking Controller Synthesis for MIMO Systems

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***Abstract*— Simultaneous stabilization of linear, time-invariant, multi-input multi-output stable plants is considered with asymptotic tracking of step-input references with zero steady-state error. Conditions are derived for existence of simultaneous integral-action controllers and PID-controllers. A systematic simultaneous PID synthesis method is proposed.**

I. INTRODUCTION

We consider simultaneous stabilization of a finite class of linear, time-invariant (LTI) multi-input multi-output (MIMO) stable plants while achieving asymptotic tracking of step-input references with zero steady-state error. We derive conditions for existence of general integral-action controllers and particularly proportional+integral+derivative (PID) controllers that achieve simultaneous stabilization.

Simultaneous stabilization of three or more plants and strong stabilization are difficult problems even without the controller order restriction [1], [2], [6]. Rigorous PID design methods exist mostly for single-input single-output (SISO) systems (see e.g., [5]). Simultaneous PID designs for MIMO systems have not been explored extensively. The results here are necessary conditions for existence of simultaneous integral-action controllers based on the DC-gains of the plants (Lemma 3.1), and sufficient conditions and explicit PID synthesis (Proposition 3.1). For single-output systems, the sufficient conditions are also necessary. The freedom in the PID parameters may be used to satisfy additional performance criteria. The discussion is based on continuous-time systems; all results apply also to discrete-time systems with appropriate modifications. *Notation*: $\mathcal{U} = \{ s \in \mathbb{C} \mid \mathcal{R}e(s) \geq 0 \} \cup \{\infty\}$ is the extended closed right-half complex plane; $\mathbb{R}$, $\mathbb{R}_+$ denote reals and positive reals; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in $\mathbf{S}$; I_m is the (size m) identity matrix. The H_∞-norm of $M(s) \in \mathcal{M}(\mathbf{S})$ is $\|M\| := \sup_{s\in\partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$. We drop (s) in transfer matrices such as $G(s)$.

II. PRELIMINARIES

Consider the standard LTI, MIMO unity-feedback system $Sys(G_j, C)$ in Fig. 1, where $G_j \in \mathbf{S}^{m\times u}$, $j \in \{1,\dots,k\}$, and $C \in \mathbf{R_p}^{u\times m}$ denote the plant and the controller. We assume $Sys(G_j, C)$ is well-posed, G_j and C have no unstable hidden-modes, and rank$G_j = m$. The objective is to design C achieving asymptotic tracking of step-input references with zero steady-state error for a finite class of stable plants G_j simultaneously. Let $C = N_c D_c^{-1}$ be a right-coprime-factorization (RCF); $N_c \in \mathbf{S}^{u\times m}$, $D_c \in \mathbf{S}^{m\times m}$, $\det D_c(\infty) \neq 0$. Then C stabilizes $G_j \in \mathcal{M}(\mathbf{S})$ if and only if $M_j := D_c + G_j N_c$ is unimodular [6], [4]. Let the transfer-function from r to e be H_j^{er} and let the transfer-function from r to y be H_j^{yr}; then $H_j^{er} = (I_m + G_j C)^{-1} = I_m - G_j C(I_m + G_j C)^{-1} =: I_m - G_j H_j^{wu} =: I_m - H_j^{yr}$.

Definition 2.1: i) The system $Sys(G_j, C)$ is stable iff the transfer-function from (r, v) to (y, w) is stable. *ii)* The controller C simultaneously stabilizes G_j for $j \in \{1,\dots,k\}$ iff C is proper and all $Sys(G_j, C)$ are stable. *iii)* The stable $Sys(G_j, C)$ has integral-action iff H_j^{er} has blocking-zeros at $s = 0$. *iv)* The controller C is a simultaneously stabilizing integral-action controller iff C stabilizes G_j for $j \in \{1,\dots,k\}$, and D_c of any RCF $C = N_c D_c^{-1}$ has blocking-zeros at $s = 0$, i.e., $D_c(0) = 0$. ▪

Suppose that $Sys(G_j, C)$ is stable. Then the steady-state error $e(t)$ due to step inputs at $r(t)$ goes to zero as $t \to \infty$ if and only if $H_j^{er}(0) = 0$. By Definition 2.1, the stable $Sys(G_j, C)$ achieves asymptotic tracking of constant reference inputs with zero steady-state error if and only if it has integral-action. Write $H_j^{er} = (I_m + G_j\, C)^{-1} = D_c M_j^{-1} = I_m - G_j\, N_c\, M_j^{-1}$. Then $Sys(G_j, C)$ has integral-action if and only if $C = N_c D_c^{-1}$ is an integral-action controller since $H_j^{er}(0) = (D_c M_j^{-1})(0) = 0$ if and only if $D_c(0) = 0$. The simplest integral-action controllers are in the proper (realizable) PID form $C_{pid} = K_P + \frac{K_I}{s} + \frac{K_D\, s}{\tau\, s+1}$, where $K_P, K_I, K_D \in \mathbb{R}^{u\times m}$ are the proportional, integral, and derivative constants [3]. Due to implementation issues, a pole is typically added to the derivative term (with $\tau > 0$) so that the transfer-function C_{pid} is proper. The only $\mathcal{U}$-pole of C_{pid} is at $s = 0$. The constants K_P, K_D, K_I may be negative; in the scalar case, this means the zeros of C_{pid} may be in $\mathcal{U}$. The integral-action in C_{pid} is present when $K_I \neq 0$.

III. MAIN RESULTS

We derive necessary conditions for existence of simultaneously stabilizing integral-action controllers. We propose explicit PID-controller design under a sufficient condition, which is necessary for systems with a single-output.

Lemma 3.1: (*i*) *(Necessary condition for integral-action)*: Let $G_j \in \mathbf{S}^{m\times u}$. If the system $Sys(G_j, C)$ has integral-action, then rank$G_j(0) = m \leq u$, i.e., G_j has no transmission-zeros at $s = 0$. (*ii*) *(Necessary conditions for simultaneous integral-action)*: Let $G_j \in \mathbf{S}^{m\times u}$, $j \in \{1,\dots,k\}$. Let rank$G_j(0) = m \leq u$. Let $G_j(0)^I \in \mathbb{R}^{u\times m}$ denote a right-inverse of $G_j(0)$. *a)* Suppose that for all $i, j \in \{1,\dots,k\}$, $G_i(z_o) - G_j(z_o) = 0$ for some $s = z_o \in \mathbb{R} \cap \mathcal{U}$. If there exist simultaneously stabilizing integral-action controllers, then

$$\det [\, G_j(0)\, G_i(0)^I \,] \;>\; 0 \;, \quad \text{for all } i, j \in \{1,\dots,k\} \,. \quad (1)$$

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b) Suppose that each $G_j(z_j) = 0$ for some $z_j \in \mathbb{R} \cap \mathcal{U}$, $j \in \{1, \ldots, k\}$. If there exist simultaneously stabilizing PID-controllers, then (1) holds for all $i, j \in \{1, \ldots, k\}$. ∎

Proposition 3.1: (*Simultaneous PID-controller synthesis*): Let $G_j \in \mathbf{S}^{m\times u}$, rank$G_j(0) = m \leq u$, $j \in \{1, \ldots, k\}$. Designate an arbitrary plant as G_1. Let $G_1(0)^I \in \mathbb{R}^{u\times m}$ denote a right-inverse of $G_1(0)$. If all eigenvalues of $G_j(0)G_1(0)^I$ are real and positive for $j \in \{2, \ldots, k\}$, then simultaneously stabilizing PID-controllers exist and can be designed as follows: Let $F = 0$ for a PD-controller, $F = I$ for a PID-controller. Choose any $\hat{K}_P, \hat{K}_D \in \mathbb{R}^{u\times m}$, $\tau > 0$. Define $\hat{C} := \hat{K}_P + \frac{\hat{K}_D\, s}{\tau\, s+1} + \frac{G_1(0)^I}{s} F$. Choose any $\beta \in \mathbb{R}_+$ satisfying $0 < \beta < \min_{j\in\{1,\ldots,k\}} \| \frac{s\, G_j\, \hat{C} - G_j(0)\, G_1(0)^I\, F}{s} \|^{-1}$. Let $K_P = \beta\hat{K}_P$, $K_D = \beta\hat{K}_D$, $K_I = \beta G_1(0)^I$. Then a PID-controller that simultaneously stabilizes all G_j is $C_{pid} = \beta\,\hat{C} = \beta\,\hat{K}_P + \frac{\beta\,\hat{K}_D\, s}{\tau\, s+1} + \frac{\beta\, G_1(0)^I}{s} F$, which is a PD-controller for $F = 0$, a PI-controller for $\hat{K}_D = 0$, an ID-controller for $\hat{K}_P = 0$, an I-controller for $\hat{K}_D = \hat{K}_P = 0$. ∎

Corollary 3.1: (*Necessary and sufficient existence conditions for simultaneous PID-controllers*): Let $G_j \in \mathbf{S}^{1\times u}$, $G_j(0) \neq 0$, $j \in \{1, \ldots, k\}$. *a)* Suppose that for all $i, j \in \{1, \ldots, k\}$, $G_i(z_o) - G_j(z_o) = 0$, for some $s = z_o \in \mathbb{R} \cap \mathcal{U}$. There exist simultaneously stabilizing integral-action controllers if and only if $G_j(0)G_i(0)^I > 0$, for all $i, j \in \{1, \ldots, k\}$. *b)* Suppose that each $G_j(z_j) = 0$ for some $z_j \in \mathbb{R}\cap\mathcal{U}$, $j \in \{1, \ldots, k\}$. There exist simultaneously stabilizing PID-controllers if and only if $G_j(0)G_1(0)^I > 0$, for all $j \in \{2, \ldots, k\}$. ∎

IV. CONCLUSIONS

We showed that a class of stable plants $\{G_1, \ldots, G_k\}$ with blocking-zeros in $\mathcal{U}$ can be simultaneously stabilized using low-order integral-action (PID) controllers only if $\det[\,G_j(0)\, G_i(0)^I\,] > 0$ for all $i, j \in \{1, \ldots, k\}$. If the eigenvalues of $G_j(0)G_1(0)^I$ are all real and positive for some arbitrary G_1 of the class, then there exist simultaneous PID-controllers. The necessary conditions and the sufficient conditions coincide for systems with only one output. Under the sufficient condition of positive eigenvalues for the DC-gain matrix, we presented a PID synthesis method, which allows a wide range of choices for the PID parameters.

APPENDIX: PROOFS

Proof of Lemma 3.1: i) The stability of $Sys(G_j\,, C)$ implies $H_j^{er}(0) = I_m - G_j H_j^{wu}(0) = 0$, i.e., $G_j H_j^{wu}(0) = I_m$. Therefore, rank$[G_j(0)H_j^{wu}(0)] = m \leq \min\{\text{rank}G_j(0), \text{rank}H_j^{wu}(0)\}$ implies $m \leq \text{rank}G_j(0) \leq \min\{m, u\}$ and hence, rank$G_j(0) = m$. ii) it a) Let $C = N_c D_c^{-1}$ be an integral-action controller simultaneously stabilizing the class $\{G_1, \ldots, G_i, \ldots, G_j, \ldots, G_k\}$, where G_i, G_j are two arbitrary plants. Since C has integral-action, $D_c =: \frac{s}{s+a}\hat{D}_c$ for any $a \in \mathbb{R}_+$, where $\hat{D}_c \in \mathcal{M}(\mathbf{S})$. Then $M_i = \frac{s}{s+a}\hat{D}_c + G_i N_c$ and $M_j = \frac{s}{s+a}\hat{D}_c + G_j N_c$ are unimodular. By assumption, $G_i(z_o) = G_j(z_o)$ for the same $z_o \in \mathbb{R}_+ \cup \infty$ implies $M_i(z_o) - M_j(z_o) = [\,G_i(z_o) - G_j(z_o)\,]N_c(z_o) = 0$, i.e., $M_i(z_o) = M_j(z_o)$. Since $\det M_i(z_o) = \det M_j(z_o)$ at some $z_o \in \mathcal{U}$, $\det M_i(s)$ has the same sign as $\det M_j(s)$ for all $s \in \mathcal{U} \cap \mathbb{R}$. At $s = 0$, $M_i(0) = G_i(0)N_c(0)$ implies $N_c(0) = G_i(0)^I M_i(0)$ and hence, $M_j(0) = G_j(0)N_c(0) = G_j(0)G_i(0)^I M_i(0)$. The conclusion follows since $\det M_j(0) = \det[\,G_j(0)G_i(0)^I\,]\det M_i(0)$, with $\det M_j(0)$ having the same sign as $\det M_i(0)$, implies (1). *b)* Let C_{pid} be a simultaneously stabilizing PID-controller. Write $C_{pid} = N_c D_c^{-1} = (\,\frac{s}{s+a}\, C_{pid})(\,\frac{s}{s+a} I_m)^{-1}$ for any $a > 0$; i.e., $N_c = [K_P + \frac{K_D\, s}{\tau s+1}]\frac{s}{s+a} + \frac{K_I}{s+a}$. Then $M_i = \frac{s}{s+a}I + G_i N_c$, $M_j = \frac{s}{s+a}I + G_j N_c$ are unimodular. By assumption, $G_i(z_i) = 0$ for some $z_i \in \mathbb{R}_+ \cup \infty$ and $G_j(z_o)$ for same $z_j \in \mathbb{R}_+ \cup \infty$ implies $\det M_i(z_i) = \det \frac{z_i}{z_i+a} > 0$ and $\det M_j(z_j) = \det \frac{z_j}{z_j+a} > 0$. Since $\det M_i(s)$ has the same sign for all $s \in \mathcal{U} \cap \mathbb{R}$, $\det M_i(0) > 0$; similarly, $\det M_j(0) > 0$. At $s = 0$, $M_i(0) = G_i(0)N_c(0) = G_i(0)a^{-1}K_i$ implies $K_i = aG_i(0)^I M_i(0)$ and hence, $M_j(0) = G_j(0)N_c(0) = G_j(0)G_i(0)^I M_i(0)$. The conclusion follows since $\det M_j(0) = \det[\,G_j(0)G_i(0)^I\,]\det M_i(0)$, with $\det M_j(0) > 0$ and $\det M_i(0) > 0$, implies (1). ∎

Proof of Proposition 3.1: Write $C_{pid} = N_c D_c^{-1} = [C_{pid}\, D_c][I - \frac{a}{s+a} F]^{-1}$ for any $a \in \mathbb{R}_+$. Then C_{pid} stabilizes $G_j \in \mathcal{M}(\mathbf{S})$ if and only if $M_j := D_c + G_j\, N_c$ is unimodular. By assumption, $\Theta_j := G_j(0)G_1(0)^I$ has positive real eigenvalues. Since $a > 0, \beta > 0$ and $(sI + \beta\Theta_j)^{-1} \in \mathcal{M}(\mathbf{S})$, M_j is unimodular if and only if $\hat{M}_j := M_j\,[\,I + (aI - \beta\Theta_j)\,(sI + \beta\Theta_j)^{-1}\, F\,]$ is unimodular. Note that $\hat{M}_j := M_j$ for $F = 0$, and $\hat{M}_j := M_j\,(s+a)\,(sI + \beta\Theta_j)^{-1}$ for $F = I$. Define $\tilde{D} := I - \beta\Theta_j\,(\,sI + \beta\Theta_j)^{-1}F$, $\tilde{N} := C_{pid}\,\tilde{D}$; i.e., $\tilde{D} = I$ for $F = 0$ and $\tilde{D} = s\,I\,(\,sI + \beta\Theta_j)^{-1}$ for $F = I$. Then $\hat{M}_j$ can be written as $\hat{M}_j = \tilde{D} + G_j\,\tilde{N} = I + \beta\,[\,G_j\,(\hat{K}_P + \frac{\hat{K}_D s}{\tau\, s+1})\tilde{D} + \frac{(\,G_j\, G_1(0)^I - \Theta_j\,)}{s}\, sI(sI + \beta\Theta_j)^{-1}\, F\,]$. Since $\|(sI + \beta\Theta_j)^{-1}\, s\,\| = 1$, $\hat{M}_j$ is unimodular. Hence, the systems $Sys(G_j\,, C)$ are stable for $j \in \{1, \ldots, k\}$. ∎

REFERENCES

[1] V. Blondel, M. Gevers, R. Mortini, and R. Rupp, "Simultaneous stabilization of three or more plants: Conditions on the positive real axis do not suffice," *SIAM Journal on Control and Optimization*, vol. 32, no. 2, pp. 572-590, 1994.

[2] V. D. Blondel, *Simultaneous Stabilization of Linear Systems*, Springer Verlag, Heidelberg, 1994.

[3] G. C. Goodwin, S. F. Graebe, M. E. Salgado, *Control System Design*, Prentice Hall, 2001.

[4] A. N. Gündeş, C. A. Desoer, *Algebraic Theory of Linear Feedback Systems with Full and Decentralized Compensators,* Lect. Notes in Contr. and Inform. Sciences, 142, Springer, 1990.

[5] G. J. Silva, A. Datta, S. P. Bhattacharyya, "New results on the synthesis of PID controllers," *IEEE Trans. Automatic Control*, vol. 47: 2, pp. 241-252, 2002.

[6] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, M.I.T. Press, 1985.

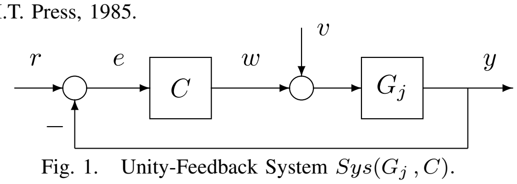

Fig. 1. Unity-Feedback System $Sys(G_j\,, C)$.