Simultaneous Tracking Controller Synthesis for MIMO Systems

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Abstract—Simultaneous stabilization of linear, time-invariant, multi-input multi-output stable plants is considered with asymptotic tracking of step-input references with zero steady-state error. Conditions are derived for existence of simultaneous integral-action controllers and PID-controllers. A systematic simultaneous PID synthesis method is proposed.

I. INTRODUCTION

We consider simultaneous stabilization of a finite class of linear, time-invariant (LTI) multi-input multi-output (MIMO) stable plants while achieving asymptotic tracking of step-input references with zero steady-state error. We derive conditions for existence of general integral-action controllers and particularly proportional-integral-derivative (PID) controllers that achieve simultaneous stabilization.

Simultaneous stabilization of three or more plants and strong stabilization are difficult problems even without the controller order restriction [1], [2], [6]. Rigorous PID design methods exist mostly for single-input single-output (SISO) systems (see e.g., [5]). Simultaneous PID designs for MIMO systems have not been explored extensively. The results here are necessary conditions for existence of simultaneous integral-action controllers based on the DC-gains of the plants (Lemma 3.1), and sufficient conditions and explicit PID synthesis (Proposition 3.1). For single-output systems, the sufficient conditions are also necessary. The freedom in the PID parameters may be used to satisfy additional performance criteria. The discussion is based on continuous-time systems; all results apply also to discrete-time systems with appropriate modifications. Notation: \( \mathcal{U} = \{ s \in \mathbb{C} \mid \Re(c(s)) \geq 0 \} \cup \{ \infty \} \) is the extended closed right-half complex plane; \( \mathbb{R}, \mathbb{R}_+ \) denote reals and positive reals; \( \mathbb{R}_p \) denotes real proper rational functions of \( s \); \( \mathbb{S} \subseteq \mathbb{R}_p \) is the stable subset with no poles in \( \mathcal{U} \); \( M(\mathcal{S}) \) is the set of matrices with entries in \( \mathbb{S} \); \( I_m \) is the (size \( m \)) identity matrix. The \( H_\infty \)-norm of \( M(s) \in M(\mathcal{S}) \) is \( \| M \| := \sup_{s \in \partial \mathcal{U}} \sigma(M(s)) \), where \( \sigma \) is the maximum singular value and \( \partial \mathcal{U} \) is the boundary of \( \mathcal{U} \). We drop \( s \) in transfer matrices such as \( G(s) \).

II. PRELIMINARIES

Consider the standard LTI, MIMO unity-feedback system

\[
\text{Sys}(G_j, C) \quad \text{in Fig. 1, where } G_j \in \mathbb{S}^{m \times u}, \quad j \in \{1, \ldots, k\}, \quad \text{and } C \in \mathbb{R}_p^{m \times m} \text{ denote the plant and the controller.}
\]

We assume \( \text{Sys}(G_j, C) \) is well-posed, \( G_j \) and \( C \) have no unstable hidden-modes, and \( \text{rank} G_j = m \). The objective is to design \( C \) achieving asymptotic tracking of step-input references with zero steady-state error for a finite class of stable plants \( G_j \) simultaneously. Let \( C = N_cD_c^{-1} \) be a right-coprime-factorization (RCF); \( N_c \in \mathbb{S}^{m \times m}, \quad D_c \in \mathbb{S}^{m \times m} \), \( \det D_c(\infty) \neq 0 \). Then \( C \) stabilizes \( G_j \in M(\mathcal{S}) \) if and only if \( M_j := D_c + G_jN_c \) is unimodular [6], [4]. Let the transfer-function from \( r \) to \( e \) be \( H^r \) and let the transfer-function from \( r \) to \( y \) be \( H^y \); then \( H^y = (I_m + G_jC)^{-1} = I_m - G_jC(I_m + G_jC)^{-1} =: I_m - G_jC_{\text{wu}} =: I_m - H^y \).

**Definition 2.1:** i) The system \( \text{Sys}(G_j, C) \) is stable iff the transfer-function from \( (r, v) \) to \( (y, w) \) is stable. ii) The controller \( C \) simultaneously stabilizes \( G_j \) for \( j \in \{1, \ldots, k\} \) iff \( C \) is proper and all \( \text{Sys}(G_j, C) \) are stable. iii) The stable \( \text{Sys}(G_j, C) \) has integral-action iff \( H^y \) has blocking-zeros at \( s = 0 \). iv) The controller \( C \) is a simultaneously stabilizing integral-action controller iff \( C \) stabilizes \( G_j \) for \( j \in \{1, \ldots, k\} \), and \( D_c \) of any RCF \( C = N_cD_c^{-1} \) has blocking-zeros at \( s = 0 \), i.e., \( D_c(0) = 0 \).

Suppose that \( \text{Sys}(G_j, C) \) is stable. Then the steady-state error \( e(t) \) due to step inputs at \( t = 0 \) to zero as \( t \to \infty \) and only if \( H^y(0) = 0 \). By Definition 2.1, the stable \( \text{Sys}(G_j, C) \) achieves asymptotic tracking of constant reference inputs with zero steady-state error if and only if it has integral-action. Write \( H^y = (I_m + G_jC)^{-1} = D_cM_c^{-1} = I_m - G_jN_cM_c^{-1} \). Then \( \text{Sys}(G_j, C) \) has integral-action if and only if \( C = N_cD_c^{-1} \) is an integral-action controller since \( H^y(0) = (D_cM_c^{-1})(0) = 0 \) if and only if \( D_c(0) = 0 \). The simplest integral-action controllers are in the proper (realizable) PID form \( C_{\text{pid}} = K_p + K_i r + K_ds^2 + K_ds + 1 \), where \( K_p, K_i, K_D \in \mathbb{R}^{u \times m} \) are the proportional, integral, and derivative constants [3]. Due to implementation issues, a pole is typically added to the derivative term (with \( r > 0 \)) so that the transfer-function \( C_{\text{pid}} \) is proper. The only \( \mathcal{U} \)-pole of \( C_{\text{pid}} \) is at \( s = 0 \). The constants \( K_p, K_D, K_I \) may be negative; in the scalar case, this means the zeros of \( C_{\text{pid}} \) may be in \( \mathcal{U} \). The integral-action in \( C_{\text{pid}} \) is present when \( K_I \neq 0 \).

III. MAIN RESULTS

We derive necessary conditions for existence of simultaneously stabilizing integral-action controllers. We propose explicit PID-controller design under a sufficient condition, which is necessary for systems with a single-output.

**Lemma 3.1:** (i) (Necessary condition for integral-action): Let \( G_j \in \mathbb{S}^{m \times u} \). If the system \( \text{Sys}(G_j, C) \) has integral-action, then \( \text{rank} G_j(0) = m \leq u \), i.e., \( G_j \) has no transmission-zeros at \( s = 0 \). (ii) (Necessary conditions for simultaneous integral-action): Let \( G_j \in \mathbb{S}^{m \times u}, \quad j \in \{1, \ldots, k\}, \) denote a right-inverse of \( G_j(0) \). a) Suppose that for all \( i, j \in \{1, \ldots, k\}, \) \( G_i(z_0) - G_j(z_0) = 0 \) for some \( s = z_0 \in \mathbb{R} \cap \mathcal{U} \). If there exist simultaneously stabilizing integral-action controllers, then

\[
\det [G_j(0)G_i(0)^T] > 0, \quad \text{for all } i, j \in \{1, \ldots, k\}. \quad (1)
\]

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b) Suppose that each \( G_j(z_j) = 0 \) for some \( z_j \in \mathbb{R} \cap \mathcal{U}, j \in \{1, \ldots, k\} \). If there exist simultaneously stabilizing PID-controllers, then (1) holds for all \( i, j \in \{1, \ldots, k\} \).

**Proposition 3.1: (Simultaneous PID-controller synthesis):**

Let \( G_j(z) \in \mathbb{S}^{n \times n}, \text{rank}(G_j(0)) = m \leq u, j \in \{1, \ldots, k\} \).

Designate an arbitrary plant as \( G_i \). Let \( G_i(0)^{i} \in \mathbb{R}^{n \times m} \) denote a right-inverse of \( G_i(0) \). If all eigenvalues of \( G_i(0)G_i(0)^{i} \) are real and positive for \( j \in \{2, \ldots, k\} \), then simultaneously stabilizing PID-controllers exist and can be designed as follows: Let \( F = 0 \) for a PD-controller, \( F = 1 \) for a PID-controller. Choose any \( K_{P}^\ast, K_D \in \mathbb{R}^{n \times m}, \tau > 0 \). Define \( \hat{C} := K_P + \frac{K_D}{\tau + I} + \frac{G_i(0)^{i}}{s} F \). Choose any \( \beta \in \mathbb{R}_+ \) satisfying \( 0 < \beta < \min_j \{\{1, \ldots, k\} \} \parallel s \hat{G}_i \hat{C} - G_i(0)G_i(0)^{i} F \parallel^{-1} \). Let \( K_P = \beta K_P^\ast, K_D = \beta K_D, K_I = \beta G_i(0)^{i} \). Then a PID-controller that simultaneously stabilizes all \( G_j \) is \( \hat{C} \) and \( \beta \hat{C} = \beta K_P^\ast, \beta K_D, \beta K_I = \beta G_i(0)^{i} \). This is a PID-controller for \( F = 0 \), a PI-controller for \( K_D = 0 \), an ID-controller for \( K_P = 0 \), and an I-controller for \( \hat{K}_D^\ast = \beta K_P^\ast \).

**Corollary 3.1: (Necessary and sufficient existence conditions for simultaneous PID-controllers):** Let \( G_j(0) \leq 0 \), \( j \in \{1, \ldots, k\} \). a) Suppose that for all \( i, j \in \{1, \ldots, k\}, G_i(z_i) - G_j(z_j) = 0, \) for some \( s = z_o \in \mathbb{R} \cap \mathcal{U} \). There exist simultaneously stabilizing integral-action controllers if and only if \( G_j(0)G_i(0)^{i} \) > 0, for all \( i, j \in \{1, \ldots, k\} \). b) Suppose that each \( G_j(z_j) = 0 \) for some \( z_j \in \mathbb{R} \cap \mathcal{U} \), then simultaneously stabilizing PID-controllers exist and can be designed as follows: Let \( F = 0 \) for a PD-controller, \( F = 1 \) for a PID-controller. Choose any \( K_P^\ast, K_D \in \mathbb{R}^{n \times m}, \tau > 0 \). Define \( \hat{C} := K_P + \frac{K_D}{\tau + I} + \frac{G_i(0)^{i}}{s} F \). Choose any \( \beta \in \mathbb{R}_+ \) satisfying \( 0 < \beta < \min_j \{\{1, \ldots, k\} \} \parallel s \hat{G}_i \hat{C} - G_i(0)G_i(0)^{i} F \parallel^{-1} \). Let \( K_P = \beta K_P^\ast, K_D = \beta K_D, K_I = \beta G_i(0)^{i} \). Then a PID-controller that simultaneously stabilizes all \( G_j \) is \( \hat{C} \) and \( \beta \hat{C} = \beta K_P^\ast, \beta K_D, \beta K_I = \beta G_i(0)^{i} \). This is a PID-controller for \( F = 0 \), a PI-controller for \( K_D = 0 \), an ID-controller for \( K_P = 0 \), and an I-controller for \( \hat{K}_D^\ast = \beta K_P^\ast \).

**IV. CONCLUSIONS**

We showed that a class of stable plants \( \{G_1, \ldots, G_k\} \) with blocking-zeros in \( \mathcal{U} \) can be simultaneously stabilized using low-order integral-action (PID) controllers only if \( \text{det}(G_j(0)G_i(0)^{i}) > 0 \) for all \( i, j \in \{1, \ldots, k\} \). If the eigenvalues of \( G_j(0)G_i(0)^{i} \) are real and positive for some arbitrary \( G_1 \) of the class, then there exist simultaneous PID-controllers. The necessary conditions and the sufficient conditions coincide for systems with only one output. Under the sufficient condition of positive eigenvalues for the Dc-gain matrix, we presented a PID synthesis method, which allows a wide range of choices for the PID parameters.

**APPENDIX: PROOFS**

**Proof of Lemma 3.1:** i) The stability of \( \text{Sys}(G_j, C) \) implies \( H_j^{sr}(0) = I_m - G_jH_j^{uw}(0) = 0, \) i.e., \( G_jH_j^{uw}(0) = I_m \). Therefore, \( \text{rank}(G_jH_j^{uw}(0)) = m \leq \min\{\text{rank}G_j(0), \text{rank}H_j^{uw}(0)\} \leq \min\{m, u\} \) and hence, \( \text{rank}(G_j(0)) = m \). ii) a) Let \( C = N_D^{-1} \) be an integral-action controller simultaneously stabilizing the class \( \{G_1, \ldots, G_i, \ldots, G_j, \ldots, G_k\} \), where \( G_i, G_j \) are two arbitrary plants. Since \( C \) has integral-action, \( D_c = \frac{a}{s+a} D_c \) for any \( a \in \mathbb{R}_+ \), where \( D_c \in \mathcal{M}(S) \). Then \( M_i = \frac{a}{s+a} D_c + G_i N_c \) and \( M_j = \frac{a}{s+a} D_c + G_j N_c \) are unimodular. By assumption, \( G_i(z_0) = G_j(z_0) \) for the same \( z_0 \in \mathbb{R}_+ \) and \( \infty \) implies \( M_i(z_0) = M_j(z_0) = [G_i(z_0) - G_j(z_0)]N_c(z_0) = 0, \) i.e., \( M_i(z_0) = M_j(z_0) \). Since \( \text{det}(M_i(z_0)) = \text{det}(M_j(z_0)) \) at some \( z_0 \in \mathcal{U} \), \( \text{det}(M_i(s)) \) has the same sign as \( \text{det}(M_j(s)) \) for all \( s \in \mathcal{U} \). At \( s = 0, M_i(0) = G_i(0)N_c(0) \) implies \( N_c(0) = G_i(0)^{i}M_i(0) \) and hence, \( M_j(0) = G_j(0)N_c(0) = G_j(0)(G_j(0)^{i}M_i(0)). \) The conclusion follows since \( \text{det}(M_j(0)) = \text{det}(G_j(0)(G_j(0)^{i}M_i(0))) \) with \( \text{det}(M_j(0)) > 0 \) and \( \text{det}(M_i(0)) > 0 \), implies (1). b) Let \( C_{\text{pid}} \) be a simultaneously stabilizing PID-controller. Write \( C_{\text{pid}} = N_D^{-1} = (\frac{1}{s+a} + C_{\text{pid}}(\frac{1}{s+a} I_{m})^{-1} \) for any \( a > 0 \), i.e., \( N_c = [K_P + \frac{K_D}{\tau + I} + \frac{K_I}{\tau + I}] \). Then \( M_i = \frac{a}{s+a} I + G_i N_c, M_j = \frac{a}{s+a} I + G_j N_c \) are unimodular. By assumption, \( G_i(z_0) = 0 \) for some \( z_0 \in \mathbb{R}_+ \). And \( G_j(z_o) \) for some \( z_o \in \mathbb{R}_+ \) implies \( \text{det}(M_i(z_0)) = \frac{a}{s+a} > 0 \) and \( \text{det}(M_j(z_0)) = \frac{a}{s+a} > 0 \). Since \( \text{det}(M_i(s)) \) has the same sign for all \( s \in \mathcal{U} \), \( \text{det}(M_i(s)) > 0 \), similarly, \( \text{det}(M_j(s)) > 0 \). At \( s = 0, M_i(0) = G_i(0)N_c(0) = G_i(0)^{i}K_i \) implies \( K_i = G_i(0)^{i}M_i(0) \) and hence, \( M_i(0) = G_i(0)N_c(0) = G_i(0)^{i}M_i(0) \). The conclusion follows since \( \text{det}(M_i(0)) = \text{det}(G_i(0)(G_i(0)^{i}M_i(0))) \) with \( \text{det}(M_i(0)) > 0 \) and \( \text{det}(M_i(0)) > 0 \), implies (1).