Appl. Comput. Math. **6** (2007), no.1, pp.18-26

RESILIENT PI AND PD CONTROLLER DESIGNS FOR A CLASS OF UNSTABLE PLANTS WITH I/O DELAYS*

H. ÖZBAY †, A. N. GÜNDEŞ ‡, §

Abstract. In [8] we obtained stabilizing PID controllers for a class of MIMO unstable plants with time delays in the input and output channels (I/O delays). Using this approach, for plants with one unstable pole, we investigate resilient PI and PD controllers. Specifically, for PD controllers, optimal derivative action gain is determined to maximize the allowable controller gain interval. For PI controllers, optimal proportional gain is determined to maximize a lower bound of the largest allowable integral action gain.
Key words:PID Control, Time Delay, Unstable Systems

1. Introduction

PID controllers are still very popular in many control applications thanks to their simple structure, [1, 5]. Design of PID controllers for delay systems is still an active research area, see for example the recent book [15], and its references. In this paper we consider unstable plants with time delays. It is clear that, even for delay-free systems, not all unstable plants are stabilizable by a PID controller (strong stabilizability is a necessary condition for stabilization by a PID controller, and there are bounds on the order of strongly stabilizing controllers, [8, 17, 19]). Moreover, right half plane poles and zeros in the plant transfer matrix, as well as time delays in the input and/or output channels (I/O delays) of the plant, impose additional restrictions on the feedback controllers, see e.g. [6, 7, 11, 18, 21].

Recently, PID controllers are designed in [20] under specified gain margin and sensitivity constraints, and in [14] under an $\mathcal{H}_\infty$ performance condition. PID controller tuning rules are also discussed in [9, 16] under different optimality conditions. For SISO unstable systems with delays PID controller tuning has been studied in [10, 13]. An extension of predictive control is used in [3] to derive PID controllers for a class of MIMO unstable plants with delays.

In a recent work [8] obtained PID controllers from a small gain argument for a class of MIMO unstable plants with delays in the input and output channels (I/O delays). In this paper we use the results of [8] for plants with one unstable pole, and investigate stabilizing PI and PD controllers with the largest allowable interval for the controller gain. This is an important problem to study, because sensitivity of the closed loop stability to perturbations in the controller coefficients can be minimized this way, and hence resilient PI and PD controllers (see e.g. [15] and its references for a discussion of this issue) can be obtained. There are many important practical examples of plants with single unstable pole and time delays, see e.g. [2, 10, 13, 15, 18] and their references.

Remaining parts of the paper are organized as follows. Preliminary results from [8] are summarized in Section 2. Main results on PD controller design are given in Section 3, and the results on PI controller are given in Section 4; concluding remarks are made in Section 5.

* This work was supported in part by the European Commission under contract no. MIRG-CT-2004-006666 and by TÜBİTAK BAYG and EEEAG under grant no. EEEAG-105E065 and EEEAG-105E156.
†Dept. Electrical & Electronics Eng., Bilkent Univ., Ankara, 06800 Turkey, hitay@bilkent.edu.tr
Dept. Electrical & Computer Eng., Univ. of California, Davis, CA 95616, U.S.A., angundes@ucdavis.edu
§*Manuscript received May 22, 2007.*

2. Problem Definition and Preliminary Results

Consider the linear time invariant (LTI) feedback system shown in Figure 1, where C is the controller to be designed and $G_\Lambda := \Lambda_o G \Lambda_i$ is the plant with r inputs and r outputs. Here G is the delay free part of the system which is assumed to be finite dimensional. Time delays in the input and output channels of the plant are represented by their transfer matrices as $\Lambda_\bullet = diag\left[e^{-sh_1^\bullet}, \ldots, e^{-sh_r^\bullet}\right]$, where, $h_j^\bullet$ is the j^{th} channel input (when $\bullet = i$) or output (when $\bullet = o$) delay, for $1 \le j \le r$.

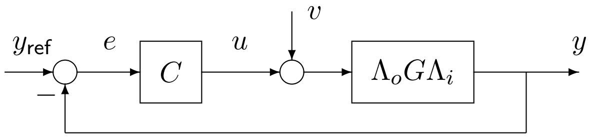

Figure 1. Feedback system $Sys(G_\Lambda, C)$ with $G_\Lambda = \Lambda_o G \Lambda_i$.

Transfer matrix H_{cl} from (y_{ref}, v) to (u, y) is

$$\mathbf{H}_{cl} = \begin{bmatrix} C(I+G_\Lambda C)^{-1} & -C(I+G_\Lambda C)^{-1}G_\Lambda \\ G_\Lambda C(I+G_\Lambda C)^{-1} & (I+G_\Lambda C)^{-1}G_\Lambda \end{bmatrix}. \tag{1}$$

We consider the proper form of PID controllers, [5],

$$C(s) = C_{pid}(s) = K_p + \frac{K_i}{s} + \frac{K_d\ s}{\tau_d s + 1}\ , \tag{2}$$

where K_p, K_i, K_d are real matrices and $\tau_d > 0$. But we will restrict ourselves to PI and PD controllers, i.e., $C_{pi} = K_p + \frac{K_i}{s}$ and $C_{pd} = K_p + \frac{K_d\ s}{\tau_d s+1}$ respectively.

Definition. The feedback system $Sys(G_\Lambda, C)$ is stable if all entries of H_{cl} are in $\mathcal{H}_\infty$. We define S_{pid}, S_{pi}, S_{pd} to be the sets of all PID, PI and PD (respectively) controllers stabilizing the feedback system $Sys(G_\Lambda, C)$.
Assumptions.
(A1) G admits a coprime factorization in the form $G(s) = Y(s)^{-1}X(s) = X(s)Y(s)^{-1}$ where $X \in \mathcal{H}_\infty^{r\times r}$, and $Y(s) = \frac{(s-p)}{(as+1)}I$. Here $p \ge 0$ is the unstable pole of the plant, and $a > 0$ is arbitrary.
(A2) $X(0) = (s-p)G(s)|_{s=0}$ is nonsingular.

Proposition 2.1. [8] *Consider the plant $G_\Lambda = \Lambda_o G \Lambda_i$, where G satisfies* **(A1)** *and* **(A2)**. *i) PD-design: Choose any $\hat{K}_d \in \mathbb{R}^{r\times r}$, and $\tau_d > 0$. Define $\hat{C}_{pd} := X(0)^{-1} + \frac{\hat{K}_d s}{\tau_d s+1}$ and*

$$\Phi_\Lambda := s^{-1}\left((s-p)G_\Lambda(s)\hat{c}_{pd}(s) - I\right)$$
$$\widetilde{\Phi}_\Lambda := s^{-1}\left(\widehat{c}_{pd}(s)(s-p)G_\Lambda(s) - I\right).$$

If $0 \le p < \max\{\|\Phi_\Lambda\|_\infty^{-1}, \|\widetilde{\Phi}_\Lambda\|_\infty^{-1}\}$, then for any $\alpha > 0$ satisfying

$$0 < \ \alpha \ < \max\{\|\Phi_\Lambda\|_\infty^{-1} - p\ ,\ \|\widetilde{\Phi}_\Lambda\|_\infty^{-1} - p\ \}\ , \tag{3}$$

the controller $\hat{C}_{pd}(s) = (\alpha + p)\hat{C}_{pd}(s)$ is in S_{pd}.

ii) PID-design : Let C_{pd} be as above, and define $H_{pd} := G_\Lambda(I+C_{pd}G_\Lambda)^{-1}$, $\Upsilon := \frac{H_{pd}(s)H_{pd}(0)^{-1}-I}{s}$, $\widetilde{\Upsilon} := \frac{H_{pd}(0)^{-1}H_{pd}(s)-I}{s}$. Then, for any $\gamma \in \mathbb{R}$ satisfying

$$0 < \gamma < \ \max\{\|\Upsilon\|_\infty^{-1}\ ,\ \|\widetilde{\Upsilon}\|_\infty^{-1}\}, \tag{4}$$

the PID-controller given in (5) is in S_{pid},

$$C_{pid}(s) = C_{pd}(s) + \frac{\gamma\alpha X(0)^{-1}}{s}\ . \tag{5}$$

If (3) and (4) are satisfied for $\hat{K}_d = 0$ then (5) with $\hat{K}_d = 0$ is a PI controller in S_{pi}.

$$\Lambda_G(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{180}{(s+6)(s+30)} \end{bmatrix}$$

This result appears in [8] for systems with possibly uncertain time delays, but for our purposes fixed time delays version stated above is sufficient. Now consider the plants with input delays only satisfying the following structural assumption.

Assumption (A3). $G_\Lambda(s) = G(s)\Lambda_i(s)$, with $G(s) = \frac{1}{s-p}G_0\Lambda_G(s)$ where G_0 is a non-singular constant matrix and $\Lambda_G(s)$ is a stable diagonal matrix with $\Lambda_G(0) = I$, i.e., $\Lambda_G(s) = diag[g_1(s), \ldots, g_r(s)]$, where $g_1(s), \ldots, g_r(s)$ are stable proper transfer functions with $g_j(0) = 1$, for all $j = 1, \ldots, r$. Note that with **A3** we have $X(0) = G_0$ and earlier assumptions **A1** and **A2** are satisfied. Moreover, this assumption results in a diagonal structure in the input sensitivity matrix, as demonstrated below. An example for **A3** is the transfer matrix of a distillation column with input channel delays, [4], $G_\Lambda(s) = \frac{1}{s}\ G_0\ \Lambda_G(s)\Lambda_i(s)$, where G_0 $= \begin{bmatrix} 3.04 & -278.2/180 \\ 0.052 & 206.6/180 \end{bmatrix}$, $\Lambda_G(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{180}{(s+6)(s+30)} \end{bmatrix}$.

2.1. **PD Control of Systems With Input Delays.** Let **A3** hold, and define $\hat{K}_d = \tilde{K}^i_d X(0)^{-1} = \tilde{K}^i_d G_0^{-1}$. Then, the PD controller of Proposition 2.1 can be re-written as $C_{pd}(s) = (\alpha + p)\left(I + \tilde{K}^i_d \frac{s}{\tau_d s+1}\right) G_0^{-1}$. Then choosing $\tilde{K}^i_d := diag[q^i_1, \ldots, q^i_r]$, we have a diagonal input sensitivity matrix $S_i(s) = (I + L_i(s))^{-1}$, where

$$L_i(s) = \frac{(\alpha+p)}{(s-p)}\left(I + \tilde{K}^i_d \frac{s}{\tau_d s+1}\right)\Lambda_G(s)\Lambda_i(s).$$

Proposition 2.1 gives a lower bound on the largest controller gain interval: $p < (\alpha + p) < \|\tilde{\Phi}_\Lambda\|_\infty^{-1}$. For the purpose of designing a resilient controller, we would like to maximize the size of this interval. That is equivalent to minimizing

$$\mu^i := \|\tilde{\Phi}_\Lambda\|_\infty = \|\frac{\Lambda_{Fi}(s) - I}{s} + \tilde{K}^i_d \frac{\Lambda_{Fi}(s)}{\tau_d s + 1}\|_\infty \tag{6}$$

where $\Lambda_{Fi} := \Lambda_G\Lambda_i$. Therefore, in Section 3 we will study the problem of minimizing μ^i defined by (7) over the free parameters $q^i_1, \ldots, q^i_r$, where $f^i_j(s) := g_j(s)e^{-h^i_j s}$

$$\mu^i = \max_j \|\frac{f^i_j(s) - 1}{s} + q^i_j\ \frac{f^i_j(s)}{\tau_d s + 1}\|_\infty. \tag{7}$$

We should point out that with the dual structural assumption $G_\Lambda(s) = \Lambda_o(s)G(s)$, with $G(s) = \frac{1}{s-p}\Lambda_G(s)G_0$ where G_0 and $\Lambda_G(s)$ are as in **A3**, a similar problem can be defined for the output delay case, where $\|\Phi_\Lambda\|_\infty$ is minimized. The case where both input and output delays exist is more difficult, but if either output or input delays are equalized in all the channels, then that would lead to the same problem of minimizing either $\|\Phi_\Lambda\|_\infty$ or $\|\tilde{\Phi}_\Lambda\|_\infty$, see [12].

2.2. **PI Control of Systems With Input or Output Delays.** Now consider PI controllers with the proportional part $C_p = (\alpha + p)X(0)^{-1}$, where α satisfies (3). The PI controller is then in the form

$$C_{pi}(s) = (\alpha + p)X(0)^{-1} + \frac{\gamma\alpha}{s}X(0)^{-1} \tag{8}$$

where γ satisfies (4). Recall that, under the structural assumption **A3**, we have $X(0) = G_0$. An interesting problem in this case is to find the largest allowable interval for γ, for a fixed α satisfying (3).

Note that in this case $H_{pd}(s) = H_p(s) = G_\Lambda(I + C_pG_\Lambda)^{-1} = (I + G_\Lambda C_p)^{-1}G_\Lambda$. As in the above discussion on PD controller design we will assume that **A3** holds and α is in the interval

$0 < \alpha < \|\tilde{\Phi}_\Lambda\|_\infty^{-1} - p$. In this case, since the derivative term is absent, we have $\tilde{\Phi}_\Lambda = \frac{\Lambda(s)-I}{s}$, where $\Lambda = \Lambda_G \Lambda_i$. Then a lower bound for the maximum interval for the allowable "integral action gain" γ is found from (4) where $\widetilde{\Upsilon} = \frac{\alpha\Lambda(s)((s-p)I+(\alpha+p)\Lambda(s))^{-1}-I}{s}$. It is easy to see that in the dual case, where output delays are considered, and the added restriction $0 < \alpha < \|\Phi_\Lambda\|_\infty^{-1} - p$, we have $\Upsilon = \frac{\alpha\Lambda(s)((s-p)I+(\alpha+p)\Lambda(s))^{-1}-I}{s}$, where $\Lambda = \Lambda_o \Lambda_G$. Thus, it is interesting to study the upper bound $\gamma_{\max}$ for γ where

$$\gamma_{\max} := \| \frac{\frac{\alpha}{s-p}\Lambda(s)(I + \frac{\alpha+p}{s-p}\Lambda(s))^{-1} - I}{s} \|_\infty^{-1} \tag{9}$$

as a function of α satisfying

$$0 < \alpha < \| \frac{\Lambda(s) - I}{s} \|_\infty^{-1} - p \tag{10}$$

where $\Lambda(s) = \Lambda_G(s)\Lambda_i(s)$ for the input delay case and $\Lambda(s) = \Lambda_o(s)\Lambda_G(s)$ for the output delay case.

3. Optimal Derivative Action Gain

Recall from (7) that we are interested in solving the following problem: given $h > 0$ and a stable transfer function $g(s)$ with $g(0) = 1$, let $f(s) = g(s)e^{-hs}$, and find $q \in \mathbb{R}$ such that μ is minimized, where

$$\mu = \| \frac{f(s) - 1}{s} + q \, \frac{f(s)}{\tau_d s + 1} \|_\infty \ , \ \tau_d \to 0. \tag{11}$$

We shall denote the optimal solution by q^{opt}. This is a single parameter scalar function $\mathcal{H}_\infty$ norm minimization problem and it can be solved numerically using brute force search. More precisely, such an algorithm would perform the following steps:

- **0.** Choose the candidate values of $q = q_1, \ldots, q_N$, over which the optimization is to be done, and the frequency values $\omega = \omega_1, \ldots, \omega_M$ over which the norm (cost function) is to be computed.
- **1.** For $k = 1, \ldots, N$ and $\ell = 1, \ldots, M$ compute $\Psi(q_k, \omega_\ell) := |\frac{f(j\omega_\ell)-1}{j\omega_\ell} + q_k \, \frac{f(j\omega_\ell)}{j\tau_d\omega_\ell+1}|$.
- **2.** Define $\mu(q_k) := \max_\ell \Psi(q_k, \omega_\ell)$.
- **3.** Optimal q is $q^{\text{opt}} = \arg\ \min_k \mu(q_k)$.

As an example, consider the distillation column transfer matrix given in Section 2, where $g_1(s) = 1$ and $g_2(s) = \frac{180}{(s+6)(s+30)}$. Optimal derivative gains are computed in [8] (see Figure 4 of [8]) using the numerical procedure given above. However, this procedure is sensitive to the number of grid points chosen for q and ω. So, it would be useful if one could derive a closed form expression for the solution, at least for the simplest case $g(s) = 1$, i.e. $f(s) = e^{-hs}$. It turns out that this is possible, and we claim that $q^{\text{opt}}(h) = \frac{\sin(2.33)}{2.33}\ h = 0.31\ h$ for $f(s) = e^{-hs}$. In the rest of this section we discuss how q^{opt} can be computed directly for a class of functions f.

Note that (11) is a min-max problem

$$\mu = \min_{q\in\mathbb{R}} \ \max_{\omega\in\mathbb{R}} \ \Psi(q, \omega) \tag{12}$$

where $\Psi(q, \omega) = |\frac{f(j\omega)-1}{j\omega} + q \, \frac{f(j\omega)}{j\tau_d\omega+1}|$, $\tau_d \to 0$. Let us now consider the max-min problem where minimization over q is done for each fixed ω. In this case, it is easy to show that optimal q is

$$q_{\text{opt}}(\omega) = -\frac{1}{\omega} \, \frac{\sin(\phi(\omega))}{\rho(\omega)} \tag{13}$$

where $\rho(\omega) = |f(j\omega)|$ is the magnitude and $\phi(\omega) = \angle f(j\omega)$ is the phase of $f(j\omega)$. Inserting (13) into $\Psi(q, \omega)$ we obtain

$$\Psi(q_{\text{opt}}(\omega), \omega) = \left| \frac{\rho(\omega) - \cos(\phi(\omega))}{\omega} \right| =: \eta(\omega). \tag{14}$$

Therefore, solution of the max-min problem is

$$q_o = -\frac{1}{\omega_o} \frac{\sin(\phi(\omega_o))}{\rho(\omega_o)} \tag{15}$$

where ω_o is maximizing $\eta(\omega)$. It is very easy to find q_o, we only need to search for ω_o. Whereas the min-max problem requires two dimensional search.

Example. Consider $f(s) = e^{-hs}$, $h > 0$. Then $\rho(\omega) = 1$ and $\phi(\omega) = -h\omega$. Hence $\eta(\omega) = |\frac{1-\cos(h\omega))}{\omega}|$. It is easy to show that the ω value maximizing this function is the solution of $\cos(h\omega) + (h\omega)\sin(h\omega) = 1$. That gives $h\omega_o = 2.33$ rad., $q_o = 0.31\ h$, and it matches $q^{\text{opt}}(h)$.

Now it remains to be shown that q_o given in (15) is equal to the solution q^{opt} of the original problem defined by (12), at least for a large class of functions $f(s)$, including the above example. For this purpose, we need to show that the pair (ω_o, q_o) is a saddle point for the min-max problem (12), i.e. the following inequalities hold

$$\Psi(q_o, \omega) \leq \Psi(q_o, \omega_o) \leq \Psi(q, \omega_o)\ \forall\ q, \omega \in \mathbb{R}\ . \tag{16}$$

First note that by the definition of $q_{\text{opt}}(\omega)$ we have $\Psi(q_{\text{opt}}(\omega), \omega) \leq \Psi(q, \omega)$ for all $q \in \mathbb{R}$ and $\omega \in \mathbb{R}$. In particular, setting $\omega = \omega_o$ in this inequality we obtain the second part of (16), namely

$$\Psi(q_o, \omega_o) \leq \Psi(q, \omega_o)\ \forall\ q \in \mathbb{R}\ . \tag{17}$$

For the first inequality of (16), when $\tau_d = 0$, we have $\Psi(q_o, \omega) = |\Psi(q_{\text{opt}}(\omega), \omega) + \Delta_q(\omega)\ f(j\omega)|$, where $\Delta_q(\omega) = q_o - q_{\text{opt}}(\omega)$. We claim that

$$|\Psi(q_o, \omega)|^2 = |\eta(\omega)|^2 + |\Delta_q(\omega)|^2 |\rho(\omega)|^2\ \forall\ \omega. \tag{18}$$

To see this let us define the real and imaginary parts $R(\omega) + jI(\omega) := \frac{f(j\omega)-1}{j\omega} + q_{\text{opt}}(\omega)\ f(j\omega)$. Similarly, let $R_f(\omega) + jI_f(\omega) := f(j\omega)$ be the real and imaginary parts of f. With these definitions we have $R_f R + I_f I = 0$, which implies (18).

Assumption A4. The function $f(s)$ is such that $\Gamma(\omega) := \eta_o^2 - \eta^2(\omega) - |\Delta_q(\omega)|^2 \rho^2(\omega) \geq 0\ \forall\ \omega$ where $\eta(\omega)$ is defined by (14), $\eta_o = \max_\omega \eta(\omega)$, and equations (13) and (15) define $\Delta_q(\omega) = q_o - q_{\text{opt}}(\omega)$.

Now with **A4**, (18) and $\eta_o = \Psi(q_o, \omega_o)$, we have $\Psi(q_o, \omega) \leq \Psi(q_o, \omega_o)\ \forall\ \omega \in \mathbb{R}$ which is the first part of (16). In summary, we have proven the following result.

Proposition 3.1. *Let $f(s) = g(s)e^{-hs}$, with $g \in \mathcal{H}_\infty$, $g(0) = 1$ and $h > 0$, satisfy* **A4**. *Then, $q^{\text{opt}} = q_o$ where*

$$\begin{aligned} q^{\text{opt}} &= \arg \min_{q \in \mathbb{R}} \| \frac{f(s) - 1}{s} + q\ \frac{f(s)}{\tau_d s + 1} \|_\infty\ ,\ \tau_d \to 0 \\ q_o &= -\frac{1}{\omega_o} \frac{\sin(\phi(\omega_o))}{\rho(\omega_o)}, \end{aligned}$$

where ω_o is maximizing $\eta(\omega) := |\frac{\rho(\omega) - \cos(\phi(\omega))}{\omega}|$.

Example. Consider the first channel in the distillation column example, where $f(s) = e^{-hs}$, $h > 0$. Figure 2 shows Γ/h versus ω. Since $\Gamma(\omega) \geq 0$ for all ω, **A4** is satisfied, hence the formula $q_{\text{opt}} = 0.31\ h$ is valid. Now for the second channel in the distillation column example, $f(s) = \frac{180}{(s+6)(s+30)} e^{-hs}$, Figure 3 illustrates that **A4** is satisfied. Figure 4 shows q_{opt} and μ versus h for this example. We observe that, as h increases μ increases, which means the allowable

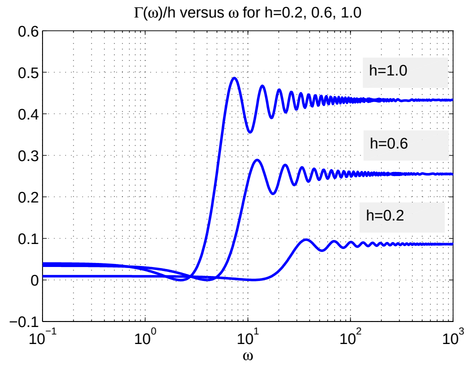

FIGURE 2. $\Gamma(\omega)/h$ versus ω for $f(s) = e^{-hs}$.

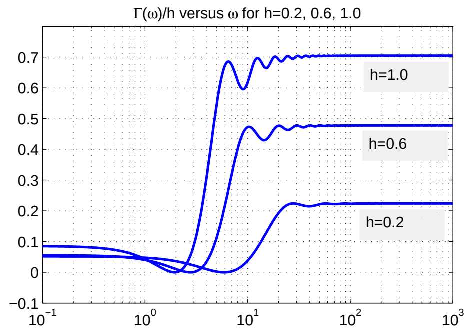

FIGURE 3. $\Gamma(\omega)/h$ versus ω for $f(s) = \frac{e^{-hs}180}{(s+6)(s+30)}$.

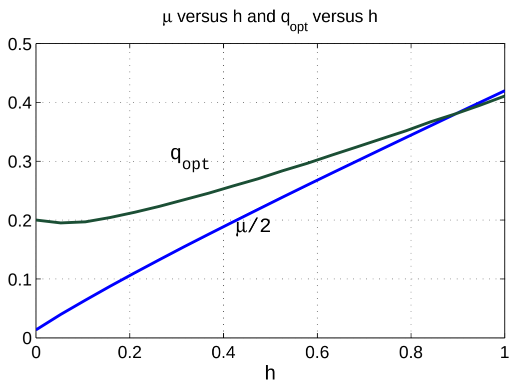

FIGURE 4. $q_{\rm opt}$ and μ versus h.

interval for the control gain shrinks with increasing h. Note that $q_{\rm opt}$ in Figure 4 is in perfect agreement with Figure 4 of [8].

An interesting problem arising in this context is to characterize the class of functions $f(s) = g(s)e^{-hs}$, $g \in \mathcal{H}_\infty$, $g(0) = 1$, $h > 0$, satisfying **A4**. At the moment we do not have a definite answer to this question. As shown for the distillation column example, **A4** holds for many interesting classes of f. In particular, it holds for all f in the form $f(s) = \frac{e^{-hs}}{1+\tau s}$, and $f(s) = e^{-hs}\frac{1-\tau s}{1+\tau s}$, for all $\tau \geq 0$ and $h > 0$. But, there are also many important functions for which it does not hold. For example, $f(s) = e^{-s}\frac{1-s}{1+\tau s}$ satisfies **A4** when $\tau \geq 0.25$; but **A4** is violated when $\tau \le 0.2$. Similarly, **A4** holds for $f(s) = e^{-s}\frac{1+s}{1+\tau s}$ when $\tau \le 1.02$, but it is violated when $\tau \ge 1.05$.

4. Integral Action Gain in PI Controller

We now study the bound $\gamma_{\max}$ on the integral action gain γ defined by (9), where $\Lambda(s)$ is a given diagonal matrix in the form $diag[f_1(s), \dots, f_r(s)]$ with $f_k(s) = g_k(s)e^{-h_k s}$, $g_k \in \mathcal{H}_\infty$, $g_k(0) = 1$, $h_k > 0$, and α satisfies (10) which is equivalent to $p < \alpha + p < \min_k \|\frac{f_k(s)-1}{s}\|_\infty^{-1}$. Clearly,

$$\gamma_{\max}^{-1} = \max_k \| \frac{\frac{\alpha}{s-p} f_k(s)(1 + \frac{\alpha+p}{s-p} f_k(s))^{-1} - 1}{s} \|_\infty. \tag{19}$$

Let us define

$$\theta := \max_k \theta_k \text{ where } \theta_k := \| \frac{f_k(s) - 1}{s} \|_\infty \ . \tag{20}$$

Then, a necessary condition for the results stated in Proposition 2.1 is $0 < \alpha\theta < 1 - p\theta$. After a simple algebra, it can be shown that (19) implies

$$\gamma_\star := \alpha \ \frac{1 - (\alpha + p)\ \theta}{1 + p\ \theta} \le \gamma_{\max} \ . \tag{21}$$

The lower bound $\gamma_\star$, found in (21) for $\gamma_{\max}$, is between 0 and α, and it decreases with increasing θ. Note that θ^{-1} is also an upper bound for the proportional gain $(\alpha + p)$. Therefore, the level of difficulty in controlling the system increases with increasing θ. The other difficulty comes from the $\mathbb{C}_+$ pole of the plant: as p increases $\gamma_\star$ decreases.

Example. Let $f_k(s) = e^{-h_k s}$. Then, $\theta_k = h_k$, and θ is the largest time delay in the system. Now consider $f_1(s) = e^{-h_1 s}$, and $f_2(s) = \frac{180}{(s+6)(s+30)} e^{-h_2 s}$. In this case we have $\theta_1 = h_1$, and $\theta_2 = 0.2 + h_2$. Since the norm in (20) is attained at $\omega = 0$ for both f_1 and f_2 and the phase of $f_2(j\omega)$ near $\omega \approx 0$ is $-0.2\ \omega$, we can see θ_2 as the "effective time delay" in the second channel. Then, $\theta = \max\{h_1, 0.2 + h_2\}$ is the largest effective time delay.

In the light of (21) an interesting problem to study is to find the optimal α maximizing $\gamma_\star$, subject to $0 < \alpha\theta < 1 - p\theta$. It is easy to see that in this sense the optimal α is

$$\alpha_\star = \frac{1 - p\theta}{2\theta} \tag{22}$$

and the corresponding maximal $\gamma_\star$ is

$$\gamma_{\star,\max} = \frac{\alpha_\star}{2} \ \frac{(1 - p\theta)}{(1 + p\theta)}. \tag{23}$$

Equations (22) and (23) show once again that the difficulty level increases with increasing $p\theta$, where p is the right half plane pole and θ can be seen as the maximal "effective time delay" in the system.

5. Conclusions

PI and PD controller design problems are studied for unstable systems with delays in the input/output channels. The results of [8] are used for plants with single right half plane pole. For PD controller design, optimal derivative action gain is determined for maximizing the interval for the overall controller gain. For PI controller design, optimal proportional gain is calculated for maximizing the interval for the integral action gain. With these results resilient PI and PD controllers can be designed for the class of plants considered. Examples illustrated difficulty of controller design for plants whose products of unstable pole with effective time delay are large.

Acknowledgement: The authors would like to thank Prof. A. B. Özgüler for fruitful discussions.

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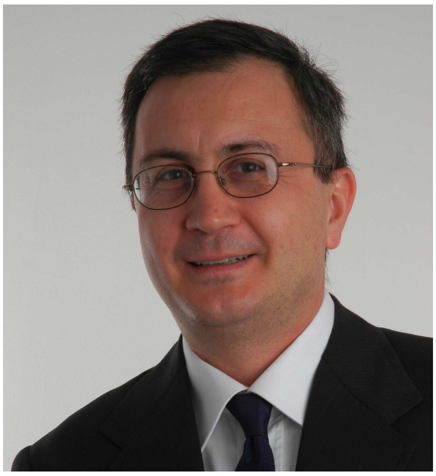

H. Özbay -received his B.S., M.Eng. and PhD degrees from Middle East Technical University (Ankara, Turkey, 1985), McGill University (Montreal, Canada, 1987), and University of Minnesota, (Minneapolis, USA, 1989), respectively. He was with the University of Rhode Island (1989-1990), and with the The Ohio State University, Department of Electrical and Computer Engineering, (1991-2006). Professor zbay joined the Electrical and Electronics Engineering Department of Bilkent University in 2002 when he was on leave from the Ohio State University. He served as an Associate Editor on the Editorial Board of the IEEE Transactions on Automatic Control (1997-1999), and was a member of the Board of Governors of the IEEE Control Systems Society (1999). Currently he is an Associate editor of Automatica, and vice-chair of the Technical Committee Networked Control Systems of IFAC. Dr. zbay has authored/co-authored many technical papers in the area of robust control of infinite dimensional systems, and engineering applications of this theory. He is a senior member of IEEE and its Control Systems Society.

A. N. Gündeş - received the BS, MS, PhD (1988) degrees in Electrical Engineering and Computer Sciences from the University of California, Berkeley. Since 1988 she has been with the University of California at Davis, where she is a professor of Electrical and Computer Engineering. She served as an Associate Editor of the IEEE Transactions on Automatic Control (1993-1996) and has been an Associate Editor of the Journal of Applied and Computational Mathematics since 2001. She was a recipient of the NSF Young Investigator Award (NYI) in 1992.