Bicoprime Factorizations of the Plant and Their Relation to Right- and Left-Coprime Factorizations

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Abstract—In a general algebraic framework, starting with a bicoprime factorization \( P = N_pD_p^{-1}N_o \), we obtain a right-coprime factorization \( N_p, D_p^{-1} \), a left-coprime factorization \( D_p^{-1}N_o \), and the generalized Bezout identities associated with the pairs \((N_p, D_p)\) and \((D_p^{-1}, N_o)\). We express the set of all \( H \)-stabilizing compensators for \( P \) in the unity-feedback configuration \( S(P, C) \) in terms of \((N_p, D_p)\) and \((D_p^{-1}, N_o)\) and the elements of the Bezout identity. The state-space representation \( P = (C(sI - A))^{-1}B \) is included as an example.

INTRODUCTION

The set of all stabilizing compensators and achievable performance for a given plant \( P \) has been of great interest in the analysis and synthesis of linear time-invariant multinput multoutput (MIMO) systems. Stabilizing compensators were first characterized in [11] for continuous-time and discrete-time lumped systems. An algebraic approach that included distributed as well as lumped continuous-time and discrete-time systems was given in [2]. Algebraic formulations were used by many researchers; for a detailed review of the factorization approach and related topics until 1985, see [9] and the references therein.

The well-known class of all stabilizing compensators is based on a right-coprime factorization \( P = N_pD_p^{-1}N_o \) or a left-coprime factorization \( P = D_p^{-1}N_o \), as for the existence of a bicoprime factorization \( P = N_pD_p^{-1}N_o \) is often used, since a bicoprime-fraction representation (b.c.f.r.) is sometimes readily available (as in closed-loop input-output maps of MIMO feedback systems). For example, in decentralized control it is more convenient to factorize an \( m \)-channel plant as

\[
P = \begin{bmatrix} N_p & \vdots & \vdots & N_m \end{bmatrix} \begin{bmatrix} D_p^{-1} & \cdots & D_p^{-1} \end{bmatrix} \begin{bmatrix} N_p & \vdots & \vdots & N_m \end{bmatrix}^{-1}
\]

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REFERENCES


In the b.c.f.r. \((N_p, D, N_o)\), if \(N_p = I (N_p = I)\), then the b.c.f.r. reduces to a right-coprime-fractiion representation (r.c.f.r.) (a left-coprime-fractiion representation (l.c.f.r.), respectively). Reducing a b.c.f.r. to a r.c.f.r. or a l.c.f.r. is a difficult problem. As a special case, stable rational coprime factorizations were obtained in [8] from a stabilizable and detectable \(\text{state-space realization of } P = (C(sI - A))^{-1}B \); in this case it is possible to use constant state-feedback and output-injection to obtain stable matrices \((sI - A + BK)^{-1}B + (sI - A + PC)^{-1} \). Note that \(B\) and \(C\) are constant matrices. In general, all of \(N_p, D, N_o\) contain "dynamics," and we have a right-Bezout identity for \((N_p, D)\) and a left-Bezout identity for \((D, N_o)\); the purpose of this note is to use these Bezout identities appropriately to obtain the coprime factorizations.

In this note we use a completely general algebraic approach to obtain a r.c.f.r., a l.c.f.r., and the associated generalized Bezout identity from a b.c.f.r., the main result is Proposition 2.5. In order to motivate the connection between coprime factorizations, we analyze the unity-feedback system \(S(P, C)\), with \(P\) factorized as \(N_pD_p^{-1}, D_p^{-1}N_o\). We write the set of all stabilizing compensators in terms of the b.c.f.r. of \(P\) in Theorem 2.6. Finally, in Example 2.8, we apply Proposition 2.5 to the state-space representation and show that we obtain the same coprime factorizations as in [8].

Due to the general algebraic setting, our results apply to lumped or distributed, continuous-time or discrete-time systems.

We use the following symbols and abbreviations: "\(1/0\)" input-output, "\(a := b\)" is defined as \(b\), "\(\det A\)" the determinant of matrix \(A\), \(m[H]\) the set of matrices with elements in \(H\), \(\ldots\) the n \(\times\) n identity matrix.

I. ALGEBRAIC BACKGROUND

A. Notation [6], [9]

\(H\) is a principal ring (i.e., an entire commutative ring in which every ideal is principal).

\(J \subseteq H\) is the group of units of \(H\).

\(I \subseteq H\) is a multiplicative subsystem, \(0 \in I\), \(I \subseteq H\) is symmetric about the real axis, and let \(x, y \in I\) imply \(xy^{-1} \in I\).

\(G = H/I := \{n/dx \in H, d \in I\}\) is the set of fractions of \(H\) associated with \(I\).

\(G_a\) (Jacobson radical of the ring \(G\)) := \{a \in G : (1 + xy)^{-1} \in G, \forall x \in G\}.

Note that if \(J \subseteq H\) is the set of units of \(G\) which are in \(H\), i.e., \(J \subseteq H\) is a multiplicative subsystem, \(0 \in J\) \(\subseteq H\), \(J \subseteq H\) is symmetric about the real axis, and \(x, y \in J\) imply \(xy^{-1} \in J\).

Let \(E, F \subseteq H\) such that \(E \subseteq F\) \(\subseteq H\). We choose \(T \subseteq E \cup F\) to be the multiplicative subset of \(E \cup F\) such that \(T \subseteq E \cup F\) implies that \(\phi(\omega)\) is a nonzero constant in \(\omega\); equivalently, \(T \subseteq E \cup F\) is the set of proper, but not strictly proper, real rational functions which are analytic in \(u\). Then \(R_a \subseteq H\) is the ring of proper rational functions \(R_a\). The set of strictly proper rational functions \(R_a\) is the Jacobson radical of the ring \(R_a\).

1.2. Example (Rational Functions in u): Let \(u \in \mathbb{R}\), be a closed subset of \(\mathbb{R}\), symmetric about the real axis, and let \(\emptyset \neq u \neq \mathbb{R}\); let \(\emptyset \cup u \neq \mathbb{R}\); the ring of proper scalar rational functions (with real coefficients) which are analytic in \(u\) is a principal ring; we denote it by \(R_a\).

Let \(H = R_a\). By definition of \(J, f \in J\) implies that \(f\) has neither poles nor zeros in \(u\). We choose \(T \subseteq E \cup F\) to be the multiplicative subset of \(E \cup F\) such that \(T \subseteq E \cup F\) implies that \(\phi(\omega)\) is a nonzero constant in \(\omega\); equivalently, \(T \subseteq E \cup F\) is the set of proper, but not strictly proper, real rational functions which are analytic in \(u\). Then \(R_a\) is the ring of proper rational functions \(R_a\). The set of strictly proper rational functions \(R_a\) is the Jacobson radical of the ring \(R_a\).

1.3. Definitions (Coprime Factorizations in H):

i) The pair \((N_p, D_p)\), where \(N_p, D_p \subseteq m(H)\), is called right-coprime (r.c.) if there exist \(U_p, V_p \subseteq m(H)\) such that

\[
V_pD_p + U_pN_p = I;
\]  

(1.1)

ii) the pair \((N_p, D_p)\) is called a right-coprime representation (r.c.r.) of \(P \subseteq m(H)\) if

\[
D_p = \text{square, } \det D_p \neq 0 \text{ and } P = N_pD_p^{-1};
\]  

(1.2)

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The pair \((N_p, D_p)\) is called a right-coprime-fraction representation (r.c.f.r.) of \(P \in m(H)\) if \((N_p, D_p)\) is a r.c.f.r. of \(P\) and \((N_p, D_p)\) is r.c.f.r. of \(P\).

The definitions of left-coprime (l.c.) fraction representation (l.f.r.) and left-coprime-fraction representation (l.c.f.r.) are duals of i), ii), and iii), respectively [9], [7], [5].

iv) The triple \((N_p, D, N_p, D, N_p)\) in \(m(H)\) is called a bicomponent-fraction representation (b.c.f.r.) of \(P \in m(H)\) if the pair \((N_p, D)\) is right-coprime, the pair \((D, N_p)\) is left-coprime, and the set \(D \in m(H)\) is called a b.c.f.r. of \(N_p, D, N_p)\) in \(H\) because \(H\) is a principal ring [9].

II. MAIN RESULTS

Consider the system \(S(P, C)\) in Fig. 1.

1. Assumptions: A) \(P \in G^{n \times n}\). Let \((N_p, D_p)\) be a r.c.f.r., \((D_p, N_p)\) be a l.c.f.r., \((N_p, D, N_p)\) be a b.c.f.r. of \(P\), where \(N_p \in H^{n \times n}, D_p \in H^{p \times p}, N_p \in H^{p \times p}, D_p \in H^{n \times n}, N_p \in H^{n \times n}, D_p \in H^{n \times n}, N_p \in H^{n \times n}, D_p \in H^{p \times p}\).

B) \(C \in G^{r \times s}\). Let \((D_z, N_z)\) be a l.c.f.r. and \((N_z, D, N_z)\) be a r.c.f.r. of \(C\), where \(D_z \in H^{n \times n}, N_z \in H^{n \times n}, D_z \in H^{n \times n}, D_z \in H^{n \times n}\).

If \(P\) satisfies assumption A) we have the following generalization of Bezout identities.

1) For the r.c. pair \((N_p, D_p)\) and the l.c. pair \((D_z, N_z)\), where \(P = N_pD_p^{-1} = D_z^{-1}N_z\), there are matrices \(V_p, U_p, \bar{X}, \bar{Y}, \bar{U}, \bar{V} \in m(H)\) such that

\[
\begin{bmatrix}
V_p & U_p \\
-D_p & D_p \\
-N_p & D_p \\
\end{bmatrix}
= \begin{bmatrix}
I_p & 0 \\
0 & I_p \\
0 & I_p \\
\end{bmatrix}
\]

(2.1)

\((N_p, D_p)\), \((D_z, N_z)\) is called a doubly-coprime factorization of \(P\).

2) For the b.c.f.r. \((N_p, D, N_p)\) we have two generalization of Bezout identities: for the r.c. pair \((N_p, D)\), there are matrices \(V_p, U_p, \bar{X}, \bar{Y}, \bar{U}, \bar{V} \in m(H)\) such that

\[
\begin{bmatrix}
V_p & U_p \\
-D & D \\
-N_p & -N_p \\
\end{bmatrix}
= \begin{bmatrix}
I_p & 0 \\
0 & I_p \\
0 & I_p \\
\end{bmatrix}
\]

(2.2)

for the l.c. pair \((D, N_p)\) there are matrices \(V_p, U_p, X, Y, U, V \in m(H)\) such that

\[
\begin{bmatrix}
D & -N_p \\
U & V \\
\end{bmatrix}
= \begin{bmatrix}
I_p & 0 \\
0 & I_p \\
\end{bmatrix}
\]

(2.3)

Let

\[
y := \begin{bmatrix}
y_m \\
y_i \\
\end{bmatrix}, \quad u := \begin{bmatrix}
u_i \\
\end{bmatrix}
\]

the map \(H_{w_m} = y\) is called the I/O map.

2.2. Definition (H-Stability): The system \(S(P, C)\) is said to be H-stable if \(H_{w_m} \in m(H)\).

2.3. Definition (H-Stabilizing Compensator): 1) \(C\) is called an H-stabilizing compensator for \(P\) if \(C \in G^{n \times n}\) satisfies assumption B) and the system \(S(P, C)\) is H-stable. 2) The set

\[
S(P) := \{ C : C \text{ H-stabilizes} P \}
\]

(2.4)

is called the set of all H-stabilizing compensators for \(P\).

We analyze the system \(S(P, C)\) by factorizing \(P\) and \(C\) as in the four cases below; the first two analyses give the well-known set \(S(P)\) of all H-stabilizing compensators in terms of familiar r.c.f.r. and l.c.f.r. of \(P\) [9], [10], [3].

2.4. Analysis: Case 1: Let \(P = N_pD_p^{-1}\) and let \(C = D_z^{-1}N_z\), where \((N_p, D_p)\) is r.c.f.r. and \((D_z, N_z)\) is l.c.f.r. (see Fig. 2). \(S(P, C)\) is then described by (2.5), (2.6)

\[
[D_z D_p + N_z N_p I_z - (\bar{D}_z - N_z)] \begin{bmatrix}
u_i \\
u_i \\
\end{bmatrix}
\]

(2.5)

\[
[D_z D_p + N_z N_p I_z - (\bar{D}_z - N_z)] \begin{bmatrix}
u_i \\
u_i \\
\end{bmatrix}
\]

(2.6)

\(S(P, C)\) is H-stable if and only if \([\bar{D}_z - N_z, N_z] \in m(H)\) is H-unimodular [9], [10], [5]. It is well known (see, for example, [9], [3], [5], [7]) that the set \(S(P)\) of all H-stabilizing compensators is given by

\[
S(P) = \{ (V_p \cdot Q^P, U_p \cdot Q^P) : Q \in H^{n \times n} \}
\]

(2.7)

where \(V_p, U_p, N_p, D_p\) as in (2.1).

Case 2: Now let \(P = N_pD_p^{-1}\) and let \(C = D_z^{-1}N_z\), where \((D_z, N_z)\) is l.c.f.r. and \((D_z, N_z)\) is r.c.f.r. (see Fig. 3). \(S(P, C)\) is then described by (2.8), (2.9)

\[
[D_z D_p + N_z N_p I_z - (\bar{D}_z - N_z)] \begin{bmatrix}
u_i \\
u_i \\
\end{bmatrix}
\]

(2.8)

\[
[D_z D_p + N_z N_p I_z - (\bar{D}_z - N_z)] \begin{bmatrix}
u_i \\
u_i \\
\end{bmatrix}
\]

(2.9)

\(S(P, C)\) is H-stable if and only if \([\bar{D}_z - N_z, N_z] \in m(H)\) is H-unimodular (which is equivalent to \([\bar{D}_z - N_z, N_z] \in m(H)\) is H-unimodular). The set \(S(P)\) of all H-stabilizing compensators is given by

\[
S(P) = \{ (\bar{U}_p + D_p \bar{Q}_p - Q P, N_p \bar{Q}_p) : Q \in H^{n \times n} \}
\]

(2.10)

where \(\bar{U}_p, \bar{V}_p, N_p, D_p\) as in (2.1).

Case 3: Now let \(P = N_pD_p^{-1}\) and let \(C = D_z^{-1}N_z\), where \((N_p, D, N_p)\) is a b.c.f.r. and \((D_z, N_z)\) is l.c.f.r. (see Fig. 4). \(S(P, C)\) is then described by (2.11), (2.12)

\[
[D_z D_p + N_z N_p I_z - (\bar{D}_z - N_z)] \begin{bmatrix}
u_i \\
u_i \\
\end{bmatrix}
\]

(2.11)

\[
[D_z D_p + N_z N_p I_z - (\bar{D}_z - N_z)] \begin{bmatrix}
u_i \\
u_i \\
\end{bmatrix}
\]

(2.12)

Equations (2.11), (2.12) are of the form \(D_p \xi = N_p \eta, N_p \xi = y\), where \((N_p, D_p)\) is a r.c.f.r. pair and \((D_p, N_p)\) is a l.c.f.r. pair. \(N_p, D_p, N_p, D_p \in m(H)\). The system \(S(P, C)\) is H-stable if and only if \(D_p \xi \in m(H)\); equivalently, \(S(P, C)\) is H-stable if and only if

\[
D_p = \begin{bmatrix}
D & -N_p \\
N_p & \bar{D}_z \\
\end{bmatrix}
\]

(2.13)

is H-unimodular.

Let

\[
R := \begin{bmatrix}
V_p & X \\
-U_p & Y \\
\end{bmatrix}
\]

(2.14)

by (2.3), \(R \in m(H)\) is H-unimodular. Postmultiply \(D_p\) by \(R\)
\[ D_H \text{ is H-unimodular if and only if } D_H R = H\text{-unimodular}; \text{ hence, (2.13)} \]

holds if and only if

\[ \tilde{N}_p \tilde{N}_p X + \tilde{D}_H Y = : = D_{H*} \text{ is H-unimodular.} \quad (2.15) \]

The set \( S(P) \) of all H-stabilizing compensators is then the set of all \( \tilde{D}_H^{-1} \tilde{N}_c \) such that (2.15) is satisfied.

Case 4: Finally, let \( P = N_p D^{-1} N_p \) and let \( C = N_c D^{-1} \), where \( (N_p, D, N_c) \) is a b.c.f.r. and \( (N_p, D) \) is r.c. (see Fig. 5). \( S(P, C) \) is then described by (2.16), (2.17)

\[
\begin{bmatrix}
D & -N_p N_c \\
N_p & D_c
\end{bmatrix}
\begin{bmatrix}
\tilde{e}_1 \\
\tilde{e}_2
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_1 \\
\tilde{u}_2
\end{bmatrix},
\]

(2.17)

Following similar steps as in Case 3 of the analysis, we conclude that \( S(P, C) \) is H-stable if and only if

\[ \tilde{D}_H := 
\begin{bmatrix}
D & -N_p N_c \\
N_p & D_c
\end{bmatrix}
\text{ is H-unimodular.} \quad (2.18) \]

Let

\[ L := 
\begin{bmatrix}
V_p & U_p \\
-X & \tilde{Y}
\end{bmatrix}
\]

by (2.2), \( L \in m(H) \) is H-unimodular; and hence, \( \tilde{D}_H \) is H-unimodular if and only if \( L \tilde{D}_H \) is H-unimodular. The set \( S(P) \) of all H-stabilizing compensators is then the set of all \( N_c D^{-1} \) such that

\[ X N_p N_c + Y D = : = D_{H*} \text{ is H-unimodular.} \quad (2.19) \]

We obtain a r.c.f.r. \( (\tilde{N}_p, D_p) \) and a l.c.f.r. \( (\tilde{D}_p, \tilde{N}_p) \) for \( P \) from the given b.c.f.r. \( (N_p, D, N_c) \) in Proposition 2.5 below; using the relationship between these coprime-factorizations, the set of all H-stabilizing compensators is given by (2.7) and equivalently, by (2.10).

2.5 Proposition: Let \( P \in m(G) \). Let \( (N_p, D, N_c) \) be a b.c.f.r. of \( P \); hence, (2.2), (2.3) hold. Under these conditions,

\[ (N_p, D_p) := (N_p X, Y) \text{ is a r.c.f.r. of } P, \quad (2.20) \]

\[ (\tilde{D}_p, \tilde{N}_p) := (\tilde{Y}, X N_p) \text{ is a l.c.f.r. of } P, \quad (2.21) \]

where \( X, Y, \tilde{X}, \tilde{Y} \in m(H) \) are defined in (2.2), (2.3).

Comments: 1) Using (2.2), (2.3) we obtain a generalized Bezout identity for the doubly-coprime pair \( (N_p X, Y), (\tilde{Y}, X N_p) \)

\[
\begin{bmatrix}
V + U_p N_p & U_p U_p \\
-X N_p & \tilde{Y}
\end{bmatrix}
\begin{bmatrix}
Y & -U_p \tilde{Y} \\
N_p X & \tilde{V} + U_p U_p \tilde{Y}
\end{bmatrix}
= 
\begin{bmatrix}
I_p & 0 \\
0 & I_p
\end{bmatrix},
\]

(2.22)

Note the similarity between (2.1) and (2.22). We refer to the matrices on the left-hand side of (2.22) as \( M^+ \) and \( M^- \), respectively, (2.22) then reads

\[ MM^+ = I_p. \quad (2.23) \]

2) If, instead of \( N_p D^{-1} N_p \), the plant is given by \( P = N_p D^{-1} N_p + E \), where \( E \in m(H) \), then a r.c.f.r. and a l.c.f.r. are given by

\[ (N_p, D_p) := (N_p X + E Y, \tilde{Y}), (\tilde{D}_p, \tilde{N}_p) := (\tilde{Y}, X N_p + \tilde{E} Y), \]

\[ V_p := V + U_p N_p E - U_p U_p E, U_p := U_p U_p, \]

(2.18)

where

\[ V_p := V + U_p N_p E, U_p := U_p U_p, \quad \tilde{V}_p := \tilde{V} + N_p V_p \tilde{E}. \quad (2.25) \]

Now \( P \in m(G) \). Postmultiplying \( P \) by \( Y \) and using \( U_p N_p Y = DX \) from the Bezout equation (2.3), we obtain

\[ P Y = N_p D^{-1} N_p Y = N_p Y \in m(G). \quad (2.26) \]

Premultiplying \( P \) by \( \tilde{Y} \) and using \( \tilde{Y} N_p \tilde{Y} = \tilde{X} D \) from the Bezout equation (2.2), we obtain

\[ \tilde{P} = \tilde{P} Y N_p D^{-1} N_p Y = \tilde{X} N_p Y \in m(G), \quad (2.27) \]

By (2.26), \( N_p := N_p X \in m(G) \) and hence, \( U_p N_p := U_p U_p \tilde{X} N_p \). Since (2.26), (2.24) implies that \( det (V_p D_p) = det (I_p - U_p N_p) \) and hence, \( det V_p \in I \) and \( det D_p := det \tilde{Y} \in I \). From (2.22)–(2.24), since \( det M \in I \) and \( det Y \in I \), we obtain \( det \tilde{Y} = det M = det |Y^{-1} I \) \( = \tilde{Y} \in I \).

At this point we know that \( Y^{-1} \in m(G) \) and \( \tilde{Y}^{-1} \in m(G) \). By (2.26),

\[ P = N_p X Y^{-1} \quad (2.28) \]

and similarly, by (2.27),

\[ \tilde{P} = \tilde{Y}^{-1} \tilde{X} N_p \quad (2.29) \]

Finally, since (2.28) and (2.24) hold and since \( Y \in I \), with \( N_p, D_p \in m(H) \), \( (N_p X, Y) := (N_p, D_p) \), is a r.c.f.r. of \( P \). Similarly, from (2.29), (2.24), and \( \tilde{Y} \in I \), with \( D_p, \tilde{N}_p \in m(H) \), \( (\tilde{Y}, X N_p) := (\tilde{D}_p, \tilde{N}_p) \), is a l.c.f.r. of \( P \).

Comment: If \( P \in m(G) \) but not \( m(G), \) (2.20), (2.21) still give a r.c.f.r. and a l.c.f.r. of \( P \), respectively. The only difference in this case is in showing that \( Y \in I \) and \( \tilde{Y} \in I \). Consider the Bezout equation (2.2); since \( P \in m(G) \), \( det V_p \) is not necessarily \( I \). Choose \( T \in m(H) \) such that \( det (V_p - T \tilde{X} \tilde{E}) \in I \) [9]; then by (2.2),

\[
\begin{bmatrix}
V_p - T \tilde{X} & U_p U_p \tilde{Y} \\
-X \tilde{Y} & N_p \tilde{Y} - N_p T
\end{bmatrix}
= 
\begin{bmatrix}
I_p & 0 \\
0 & I_p
\end{bmatrix},
\]

(2.30)
Since det $D \in I$, from (2.30) we get det $(V - T \tilde{D}) \tilde{D} = \det (U_\rho - (U_\rho + T \tilde{D})N_\Phi) = \det (I - N_\Phi(U_\rho + T \tilde{D})) \equiv I$, equivalently, det $(V - T \tilde{D}) \equiv I$. So by (2.22), since det $M \in J$, we obtain det $Y = \det \tilde{Y} \det M^{-1} \equiv I$.

2.6. Theorem (Set of All H-Stabilizing Compensators): Let $P \in m(G)$ and let $(N_\rho, D, N_p)$ be a b.c.f.r. of $P$. Then $\mathcal{S}(P) = \{(U_\rho, \rho + \Sigma Q)(P + \Sigma, V_\rho, \rho - \Sigma XQ) = I \in m(H)\}$; where $\mathcal{S}(P)$ is entirely similar.

Proof: From (2.32), (2.33) holds.

2.7. Corollary: Let $C_1, C_2 \in \mathcal{S}(P)$; then $C_1 = C_2$ if and only if $Q_1 = Q_2$. Equivalently, the map $\mathcal{Q} \to \mathcal{Q}$, $\mathcal{Q} \in m(H)$, $\mathcal{C}$ is onto.

From (2.32), $C_1 = D_1^{-1} \tilde{N}_1$ is in the set $\mathcal{S}(P)$ in (2.32) for some $Q \in H^{\rho + 4}$. (In fact, there is a unique $Q$ for each $C$; we prove this in Corollary 2.7.)

Proof: Let $\mathcal{S}(P)$ be given as in (2.32); the proof for (2.33) is entirely similar.

2.8. Example: Let $H = R_\alpha(s)$ as in Example 1.2. Let $P = \tilde{B}_0(s)^*\tilde{F}_0(s)^*$ be represented by its state-space representation $X = A + Bu, y = Cx$, where $(C, A, B)$ is stabilizable and detectable in $\tilde{u}$. Then $P = (s + a)^{-1}(s + a)^{-1}C(s + a)^{-1}(s + a)^{-1}I = B_1$, where $a \in \mathbb{R}^*$, $a > 0$. The pair $(0 + a)^{-1}(s + a)^{-1}(s + a)^{-1}I$ is r.c. in $R_\alpha(s)$, the pair $(0 + a)^{-1}(s + a)^{-1}I$ is r.c. in $R_\alpha(s)$. Therefore, $\det(S(\tilde{P})) = (s + a)^{-1}(s + a)^{-1}C, (s + a)^{-1}(s + a)^{-1}I$. But $C, D_2$ is a b.c.f.r. of $P$. Choose $K \in H^{\rho + 4}$ and $F \in H^{\rho + 4}$ such that $A \in H^{\rho + 4}$ and $F \in H^{\rho + 4}$ have all eigenvalues in $0 + a$. Let $G_0 := (s + a + K)^{-1}$, and let $G_0 := (s + a + K)^{-1}$. Then $s = (s + a)^{-1}(s + a)^{-1}I = (s + a)^{-1}(s + a)^{-1}I = (s + a)^{-1}(s + a)^{-1}I = m(\tilde{P}(s))$. From this special b.c.f.r. (2.2), (2.3) and (2.2) become

\[
\begin{align*}
\begin{bmatrix}
(s + a)G_0 & (s + a)G_0F \\
-(s + a)^{-1}C & I_{s}^{} & I_{s}^{}
\end{bmatrix}
&= I_{s}^{}, \\
\begin{bmatrix}
(s + a)^{-1}(s + a) - B \\
-(s + a)^{-1}K & I_{s}^{} & -G_{k}^{}
\end{bmatrix}
&= I_{s}^{}, \\
\begin{bmatrix}
I_{s}^{} & K
\end{bmatrix}
&= I_{s}^{}, \\
\begin{bmatrix}
-C_{f} & B \\
-C_{f} & I_{s}^{} & -G_{k}^{}
\end{bmatrix}
&= I_{s}^{},
\end{align*}
\]

2.8. Example: Let $H = R_\alpha(s)$ as in Example 1.2. Let $P = \tilde{B}_0(s)^*\tilde{F}_0(s)^*$ be represented by its state-space representation $X = A + Bu, y = Cx$, where $(C, A, B)$ is stabilizable and detectable in $\tilde{u}$. Then $P = (s + a)^{-1}(s + a)^{-1}C(s + a)^{-1}(s + a)^{-1}I = B_1$, where $a \in \mathbb{R}^*$, $a > 0$. The pair $(0 + a)^{-1}(s + a)^{-1}(s + a)^{-1}I$ is r.c. in $R_\alpha(s)$, the pair $(0 + a)^{-1}(s + a)^{-1}I$ is r.c. in $R_\alpha(s)$. Therefore, $\det(S(\tilde{P})) = (s + a)^{-1}(s + a)^{-1}C, (s + a)^{-1}(s + a)^{-1}I$. But $C, D_2$ is a b.c.f.r. of $P$. Choose $K \in H^{\rho + 4}$ and $F \in H^{\rho + 4}$ such that $A \in H^{\rho + 4}$ and $F \in H^{\rho + 4}$ have all eigenvalues in $0 + a$. Let $G_0 := (s + a + K)^{-1}$, and let $G_0 := (s + a)^{-1}(s + a)^{-1}I = (s + a)^{-1}(s + a)^{-1}I = (s + a)^{-1}(s + a)^{-1}I = m(\tilde{P}(s))$. From this special b.c.f.r. (2.2), (2.3) and (2.2) become

\[
\begin{align*}
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\end{bmatrix}
&= I_{s}^{}, \\
\begin{bmatrix}
(s + a)^{-1}(s + a) - B \\
-(s + a)^{-1}K & I_{s}^{} & -G_{k}^{}
\end{bmatrix}
&= I_{s}^{}, \\
\begin{bmatrix}
I_{s}^{} & K
\end{bmatrix}
&= I_{s}^{}, \\
\begin{bmatrix}
-C_{f} & B \\
-C_{f} & I_{s}^{} & -G_{k}^{}
\end{bmatrix}
&= I_{s}^{},
\end{align*}
\]

Note that (2.22S) gives the coprime factorizations obtained in [8, eq. (1)-(4)].

III. CONCLUSIONS

From a given b.c.f.r. $(N_\rho, D, N_p)$ for $P \in m(G)$, we obtain a r.c.f.r. $(N_\rho, D, N_p)$, a b.c.f.r. $(D_2, N_\rho)$, and the associated generalized Bezout identities. We find the class of all H-stabilizing compensators for $P$ based on $(N_\rho, D, N_p)$ with $V, U, U_\rho, \tilde{X}, \tilde{Y}$ as in (2.22), $C \in H^{\rho + 4}$, given by

\[
C = D_2^{-1} \tilde{N}_2 = (V + U_\rho N_\rho - Q \tilde{X} \tilde{N}_2)^{-1}(U_\rho + Q \tilde{Y}) \quad \text{(3.1)}
\]

$H$-stabilizes $P$, where $Q \in m(H)$ is a free parameter. If we design a two-degrees-of-freedom compensator $C = \tilde{C}_2 \tilde{C}_1$ as in [3], [4], [9], for
example), then \( C = D^{-1} [Q_1 : N_1] \), where \( Q_1 \in M(H) \), and \( D^{-1} N_1 \) is given by (3.1); in this case there are two free parameters.

REFERENCES


A Sufficient Condition for Output Feedback Stabilization of Uncertain Dynamical Systems

AVRAHAM STEINBERG

Abstract—A different proof than the one given in [6] is given for the existence of an output feedback controller which stabilizes an uncertain single-input single-output dynamical system with a linear nominal part and matched uncertainties. Yet there is no need to assume that the nominal system is stable. A simple expression is obtained for the feedback gain which is necessary for the closed-loop nominal system to become strictly positive real.

INTRODUCTION

Recently [1], a condition was given for the existence of a static output feedback stabilizing controller for an uncertain dynamical system with a linear nominal part and matched uncertainties. Even then it is necessary for the closed-loop nominal system to be strictly positive real (SPR) [11]. Consider the output feedback stabilization of the following single-input, single-output system:

\[
\begin{align*}
   x(t) &= Ax(t) + b[u(t) + e(t, x)]; \\
   x(t) &
\end{align*}
\]

where the known triple \((C', A, b)\) defines a nominal linear system and \(e: R^p \times R^n \rightarrow R\) models all uncertainty which is assumed matched. The input-output transfer function \(g(s)\) for the nominal linear system in (1) is given by

\[
g(s) = C'[sI - A]^{-1}b. \tag{2}
\]

In [1] it is established that if \(g(s)\) is SPR, then stability (in the sense defined in [1]) can be achieved, in the presence of arbitrary admissible uncertainty \(e(t,x)\), by linear static output feedback

\[
u(t) = -\gamma e(t) \tag{3}
\]

with sufficiently large gain \(\gamma > 0\).

A less restrictive sufficient condition for the existence of a stabilizing controller was given in [6]. In [6] it is assumed that the nominal system is stable and that its transfer function \(g(s)\):

a) has a positive leading coefficient,

b) satisfies \(g(s) \neq 0\) for \(\text{Re } s \geq 0\), and

c) \(g(s)\) has relative degree one, i.e., if its denominator polynomial is of order \(n\), then its numerator polynomial is of order \((n - 1)\).

It was shown in [6] that a nominal system satisfying conditions a)–c) can always become SPR by applying negative constant gain output feedback \(u = -\gamma y\). Once the nominal system becomes SPR, additional gain is necessary to stabilize system (1) with uncertainties.

In this note we provide a different proof for the lemma in [6], yet there is no need to assume that the nominal system is stable, as assumed in the Introduction and in the example in [6]. Furthermore, the proof we give allows us to obtain a simple expression for the feedback gain \(K\) which is necessary for the closed-loop nominal system to become SPR. Thus, it is not necessary to solve Lyapunov's equation as was done in [6].

FREQUENCY DOMAIN CONDITIONS FOR SPR FUNCTIONS [7]

A transfer function \(g(s)\) for a linear single-input, single-output system, with relative degree \(m = 1\), is SPR if and only if

a) \(g(s)\) is analytic in \(\text{Re } s \geq 0\),

b) \(\text{Re } [g(j\omega)] > 0 \ \forall \ \omega \in (-\infty, \infty)\), and

c) \(\lim_{\omega \rightarrow \infty} \omega^2 \text{Re } [g(j\omega)] > 0\).

Allowing \(A\) in (1) to be unstable, we prove the following lemma and derive an expression for the gain \(K\) that is necessary to make the closed-loop nominal system SPR.

Lemma: Given a transfer function

\[
g(s) = \frac{n(s)}{d(s)} = \frac{s^{n-1} + b_1 s^{n-2} + \cdots + b_n s + b_0}{s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}. \tag{3}
\]

Assume the roots of \(n(s) = 0\) are in \(\text{Re } s > 0\). Then, for a sufficiently large gain \(K \geq K\), the closed-loop transfer function

\[
h(s) = \frac{g(s)}{1 + Kg(s)} \tag{4}
\]

is SPR.

Proof: We have to show that the SPR conditions are satisfied. Consider the function

\[
\frac{1}{h(s)} = K + \frac{1}{g(s)} = K + \frac{d(s)}{n(s)}. \tag{5}
\]

By assumption, \(n(s)\) has no roots on the imaginary axis, thus, \(\text{Re } [d(j\omega)]/\text{Re } [n(j\omega)]\) is finite for all finite \(\omega\). Furthermore, it can be easily verified that \(\lim_{\omega \rightarrow \infty} \text{Re } [d(j\omega)/n(j\omega)] = a_{m-1} - b_{n-1}\).

Hence, the following is well defined:

\[
K_e = \inf_{s \rightarrow a} \left[ \text{Re } \frac{1}{g(s)} \right] = K, \tag{6}
\]