ACKNOWLEDGMENT

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Bicoprime Factorizations of the Plant and Their Relation to Right- and Left-Coprime Factorizations

C. A. DESOER AND A. N. GÜNDEŞ

Abstract—**In a general algebraic framework, starting with a bicoprime factorization $P = N_{pr}D^{-1}N_{pl}$, we obtain a right-coprime factorization $N_pD_p^{-1}$, a left-coprime factorization $\tilde{D}_p^{-1}\tilde{N}_p$, and the generalized Bezout identities associated with the pairs (N_p, D_p) and $(\tilde{D}_p, \tilde{N}_p)$. We express the set of all H-stabilizing compensators for P in the unity-feedback configuration $S(P, C)$ in terms of (N_{pr}, D, N_{pl}) and the elements of the Bezout identity. The state-space representation $P = C(sI - A)^{-1}B$ is included as an example.**

INTRODUCTION

The set of all stabilizing compensators and achievable performance for a given plant P has been of great interest in the analysis and synthesis of linear time-invariant multiinput multioutput (MIMO) systems. Stabilizing compensators were first characterized in [11] for continuous-time and discrete-time lumped systems. An algebraic approach that included distributed as well as lumped continuous-time and discrete-time systems was given in [2]. Algebraic formulations were used by many researchers; for a detailed review of the factorization approach and related topics until 1985, see [9] and the references therein.

The well-known class of all stabilizing compensators is based on a right-coprime factorization ($P = N_pD_p^{-1}$) or a left-coprime factorization ($P = \tilde{D}_p^{-1}\tilde{N}_p$) of the plant P [3]–[5], [9], [10], [7]. It is useful to parametrize all stabilizing compensators starting with bicoprime factorizations ($P = N_{pr}D^{-1}N_{pl}$) as well, since a bicoprime-fraction representation (b.c.f.r.) is sometimes readily available (as in closed-loop input-output (I/O) maps of MIMO feedback systems). For example, in decentralized control it is more convenient to factorize an m-channel plant as

$$P = \begin{bmatrix} N_{pr1} \\ \vdots \\ N_{prm} \end{bmatrix} D^{-1}[N_{pl1} \cdots N_{plm}].$$

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In the b.c.f.r. (N_{pr}, D, N_{pl}), if $N_{pl} = I$ ($N_{pr} = I$), then the b.c.f.r. reduces to a right-coprime-fraction representation (r.c.f.r.) (a left-coprime-fraction representation (l.c.f.r.), respectively). Reducing a b.c.f.r. to a r.c.f.r. or a l.c.f.r. is a difficult problem. As a special case, stable rational coprime factorizations were obtained in [8] from a stabilizable and detectable *state-space realization* of $P\,(= C(sI - A)^{-1}B)$; in this case it is possible to use *constant* state-feedback and output-injection to obtain stable matrices $(sI - A + BK)^{-1}$ and $(sI - A + FC)^{-1}$. Note that B and C are constant matrices. In general, all of N_{pr}, D, and N_{pl} contain "dynamics," and we have a right-Bezout identity for (N_{pr}, D) and a left-Bezout identity for (D, N_{pl}); the purpose of this note is to use these Bezout identities appropriately to obtain the coprime factorizations.

In this note we use a completely general algebraic approach to obtain a r.c.f.r., a l.c.f.r., and the associated generalized Bezout identity from a b.c.f.r.: the main result is Proposition 2.5. In order to motivate the connection between coprime factorizations, we analyze the unity-feedback system $S(P, C)$, with P factorized as $N_pD_p^{-1}$, $\tilde{D}_p^{-1}\tilde{N}_p$, and $N_{pr}D^{-1}N_{pl}$. We write the set of all stabilizing compensators in terms of the b.c.f.r. of P in Theorem 2.6. Finally, in Example 2.8, we apply Proposition 2.5 to the state-space representation and show that we obtain the same coprime factorizations as in [8].

Due to the general algebraic setting, our results apply to lumped or distributed, continuous-time or discrete-time systems.

We use the following symbols and abbreviations: "I/O" input–output, "$a := b$" a is defined as b, "det A" the determinant of matrix A, "$m(H)$" the set of matrices with elements in H, "I_n" the $n \times n$ identity matrix.

I. ALGEBRAIC BACKGROUND

A. Notation [6], [9]

H is a principal ring (i.e., an entire commutative ring in which every ideal is principal).

$J \subset H$ is the group of units of H.

$I \subset H$ is a multiplicative subsystem, $0 \notin I$, $1 \in I$ (i.e., $x \in I, y \in I \Rightarrow xy \in I$).

$G = H/I := \{n/d : n \in H,\ d \in I\}$ is the ring of fractions of H associated with I.

G_s (Jacobson radical of the ring G) $:= \{x \in G : (1 + xy)^{-1} \in G$, for all $y \in G\}$.

Note that i) I = the set of units of G which are in H. ii) Let $A \in m(H)$, $B \in m(G)$, then a) $A^{-1} \in m(H)$ iff det $A \in J$ and b) $B^{-1} \in m(G)$ iff det $B \in I$. iii) Let $Y \in m(G_s)$, $X, Z \in m(G)$, then $XY, YZ \in m(G_s)$ and $(I + XY)^{-1}, (I + YZ)^{-1} \in m(G)$. iv) Let $a, b, \in H$, then $ab \in J$ iff a and $b \in J$. v) Let $c, d \in H$. Then $cd \in I$ iff c and $d \in I$ [5].

1.2. Example (Rational Functions in s): Let $u \supset \mathbb{C}_+$ be a closed subset of $\mathbb{C}$, symmetric about the real axis, and let $\mathbb{C} \setminus u$ be nonempty; let $\bar{u} := u \cup \{\infty\}$. The ring of proper scalar rational functions (with real coefficients) which are analytic in u is a principal ring; we denote it by $R_u(s)$. Let $H = R_u(s)$. By definition of J, $f \in J$ implies that f has neither poles nor zeros in $\bar{u}$. We choose I to be the multiplicative subset of $R_u(s)$ such that $f \in I$ implies that $f(\infty)$ is a nonzero constant in $\mathbb{R}$; equivalently, $I \subset R_u(s)$ is the set of proper, but not *strictly proper,* real rational functions which are analytic in u. Then $R_u(s)/I$ is the ring of *proper* rational functions $\mathbb{R}_p(s)$. The set of strictly proper rational functions $\mathbb{R}_{sp}(s)$ is the Jacobson radical of the ring $\mathbb{R}_p(s)$.

1.3. Definitions (Coprime Factorizations in H):

i) The pair (N_p, D_p), where $N_p, D_p \in m(H)$, is called *right-coprime* (r.c.) iff there exist U_p, $V_p \in m(H)$ such that

$$V_pD_p + U_pN_p = I; \tag{1.1}$$

ii) the pair (N_p, D_p) is called a *right-fraction representation* (r.f.r.) of $P \in m(G)$ iff

$$D_p \text{ is square, } \det D_p \in I \text{ and } P = N_pD_p^{-1}; \tag{1.2}$$

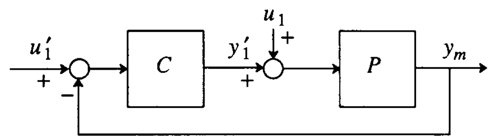

Fig. 1. The system $S(P, C)$.

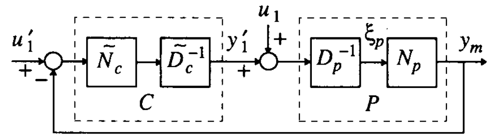

Fig. 2. $S(P, C)$ with $P = N_pD_p^{-1}$ and $C = \tilde{D}_c^{-1}\tilde{N}_c$.

iii) the pair (N_p, D_p) is called a *right-coprime-fraction representation* (r.c.f.r.) of

$$P \in m(G) \text{ iff } (N_p, D_p) \text{ is a r.f.r. of } P \text{ and } (N_p, D_p) \text{ is r.c.}$$

The definitions of *left-coprime* (l.c.), *left-fraction representation* (l.f.r.) and *left-coprime-fraction representation* (l.c.f.r.) are duals of i), ii), and iii), respectively [9], [7], [5].

iv) The triple (N_{pr}, D, N_{pl}), $N_{pr}, D, N_{pl} \in m(H)$ is called a *bicoprime-fraction representation* (b.c.f.r.) of $P \in m(G)$ iff the pair (N_{pr}, D) is *right-coprime,* the pair (D, N_{pl}) is *left-coprime,* $\det D \in I$ and $P = N_{pr}D^{-1}N_{pl}$. □

Note that every $P \in m(G)$ has a r.c.f.r. (N_p, D_p), a l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$, and a b.c.f.r. (N_{pr}, D, N_{pl}) in H because H is a principal ring [9].

II. Main Results

Consider the system $S(P, C)$ in Fig. 1.

2.1. Assumptions: A) $P \in G_s^{n_o \times n_i}$. Let (N_p, D_p) be a r.c.f.r., $(\tilde{D}_p, \tilde{N}_p)$ be a l.c.f.r., (N_{pr}, D, N_{pl}) be a b.c.f.r. of P, where $N_p \in H^{n_o \times n_i}$, $D_p \in H^{n_i \times n_i}$, $\tilde{D}_p \in H^{n_o \times n_o}$, $\tilde{N}_p \in H^{n_o \times n_i}$, $N_{pr} \in H^{n_o \times n}$, $D \in H^{n \times n}$, $N_{pl} \in H^{n \times n_i}$.

B) $C \in G^{n_i \times n_o}$. Let $(\tilde{D}_c, \tilde{N}_c)$ be a l.c.f.r. and (N_c, D_c) be a r.c.f.r. of C, where $\tilde{D}_c \in H^{n_i \times n_i}$, $\tilde{N}_c \in H^{n_i \times n_o}$, $N_c \in H^{n_i \times n_o}$, $D_c \in H^{n_o \times n_o}$.

If P satisfies assumption A) we have the following *generalized Bezout identities.*

1) For the r.c. pair (N_p, D_p) and the l.c. pair $(\tilde{D}_p, \tilde{N}_p)$, where $P = N_pD_p^{-1} = \tilde{D}_p^{-1}\tilde{N}_p$, there are matrices $V_p, U_p, \tilde{U}_p, \tilde{V}_p \in m(H)$ such that

$$\begin{bmatrix} V_p & U_p \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} \begin{bmatrix} D_p & -\tilde{U}_p \\ N_p & \tilde{V}_p \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{2.1}$$

$((N_p, D_p), (\tilde{D}_p, \tilde{N}_p))$ is called a *doubly-coprime factorization of P.*

2) For the b.c.f.r. (N_{pr}, D, N_{pl}) we have two *generalized Bezout identities*: for the r.c. pair (N_{pr}, D), there are matrices $V_{pr}, U_{pr}, \tilde{X}, \tilde{Y}, \tilde{U}, \tilde{V} \in m(H)$ such that

$$\begin{bmatrix} V_{pr} & U_{pr} \\ -\tilde{X} & \tilde{Y} \end{bmatrix} \begin{bmatrix} D & -\tilde{U} \\ N_{pr} & \tilde{V} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_o} \end{bmatrix}; \tag{2.2}$$

for the l.c. pair (D, N_{pl}) there are matrices $V_{pl}, U_{pl}, X, Y, U, V \in m(H)$ such that

$$\begin{bmatrix} D & -N_{pl} \\ U & V \end{bmatrix} \begin{bmatrix} V_{pl} & X \\ -U_{pl} & Y \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_i} \end{bmatrix}. \tag{2.3}$$

Let

$$y := \begin{bmatrix} y_m \\ y_1' \end{bmatrix}, \quad u := \begin{bmatrix} u_1 \\ u_1' \end{bmatrix}$$

the map $H_{yu}: u \mapsto y$ is called the I/O map.

2.2. Definition (H-Stability): The system $S(P, C)$ is said to be *H-stable* iff $H_{yu} \in m(H)$.

2.3. Definition (H-Stabilizing Compensator): 1) C is called an *H-stabilizing compensator for P* iff $C \in G^{n_i \times n_o}$ *satisfies assumption B) and* the system $S(P, C)$ is H-stable. 2) The set

$$S(P) := \{C : C \ H\text{-stabilizes } P\} \tag{2.4}$$

is called the *set of all H*-stabilizing compensators for P.

We analyze the system $S(P, C)$ by factorizing P and C as in the four cases below; the first two analyses give the well-known set $S(P)$ of all H-stabilizing compensators in terms of familiar r.c.f.r. and l.c.f.r. of P [9], [10], [3].

2.4. Analysis: Case 1: Let $P = N_pD_p^{-1}$ and let $C = \tilde{D}_c^{-1}\tilde{N}_c$, where (N_p, D_p) is r.c. and $(\tilde{D}_c, \tilde{N}_c)$ is l.c. (see Fig. 2). $S(P, C)$ is then described by (2.5), (2.6)

$$[\tilde{D}_cD_p + \tilde{N}_cN_p]\xi_p = [\tilde{D}_c \ \vdots \ \tilde{N}_c] \begin{bmatrix} u_1 \\ u_1' \end{bmatrix}, \tag{2.5}$$

$$\begin{bmatrix} N_p \\ D_p \end{bmatrix} \xi_p = \begin{bmatrix} y_m \\ y_1' \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I_{n_i} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_1' \end{bmatrix}. \tag{2.6}$$

$S(P, C)$ is H-stable if and only if $[\tilde{D}_cD_p + \tilde{N}_cN_p] \in m(H)$ is H-unimodular [9], [10], [5]. It is well known (see, for example, [9], [3], [5], [7]) that the set $S(P)$ of all H-stabilizing compensators is given by

$$S(P) = \{(V_p - Q\tilde{N}_p)^{-1}(U_p + Q\tilde{D}_p) : Q \in H^{n_i \times n_o}\} \tag{2.7}$$

where $V_p, U_p, \tilde{N}_p, \tilde{D}_p$ are as in (2.1).

Case 2: Now let $P = \tilde{D}_p^{-1}\tilde{N}_p$, $C = N_cD_c^{-1}$, where $(\tilde{D}_p, \tilde{N}_p)$ is l.c. and (N_c, D_c) is r.c. (see Fig. 3). $S(P, C)$ is then described by (2.8), (2.9)

$$[\tilde{D}_pD_c + \tilde{N}_pN_c]\xi_c = [\tilde{N}_p \ \vdots \ \tilde{D}_p] \begin{bmatrix} u_1 \\ u_1' \end{bmatrix}, \tag{2.8}$$

$$\begin{bmatrix} -D_c \\ N_c \end{bmatrix} \xi_c = \begin{bmatrix} y_m \\ y_1' \end{bmatrix} + \begin{bmatrix} 0 & -I_{n_o} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_1' \end{bmatrix}. \tag{2.9}$$

$S(P, C)$ is H-stable if and only if $[\tilde{D}_pD_c + \tilde{N}_pN_c] \in m(H)$ is H-unimodular (which is equivalent to $[\tilde{D}_cD_p + \tilde{N}_cN_p] \in m(H)$ is H-unimodular). The set $S(P)$ of all H-stabilizing compensators is given by

$$S(P) = \{(\tilde{U}_p + D_pQ)(\tilde{V}_p - N_pQ)^{-1} : Q \in H^{n_i \times n_o}\} \tag{2.10}$$

where $\tilde{U}, \tilde{V}, N_p, D_p$ are as in (2.1).

Case 3: Now let $P = N_{pr}D^{-1}N_{pl}$ and let $C = \tilde{D}_c^{-1}\tilde{N}_c$, where (N_{pr}, D, N_{pl}) is a b.c.f.r. and $(\tilde{D}_c, \tilde{N}_c)$ is l.c. (see Fig. 4). $S(P, C)$ is then described by (2.11), (2.12).

$$\begin{bmatrix} D & \vdots & -N_{pl} \\ \cdots & & \cdots \\ \tilde{N}_cN_{pr} & \vdots & \tilde{D}_c \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ y_1' \end{bmatrix} = \begin{bmatrix} N_{pl} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & \tilde{N}_c \end{bmatrix} \begin{bmatrix} u_1 \\ \cdots \\ u_1' \end{bmatrix}, \tag{2.11}$$

$$\begin{bmatrix} N_{pr} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I_{n_i} \end{bmatrix} \begin{bmatrix} \xi_x \\ \cdots \\ y_1' \end{bmatrix} = \begin{bmatrix} y_m \\ \cdots \\ y_1' \end{bmatrix}. \tag{2.12}$$

Equations (2.11), (2.12) are of the form $D_H\xi = N_Lu$, $N_R\xi = y$, where (N_R, D_H) is a r.c. pair and (D_H, N_L) is a l.c. pair, $N_R, D_H, N_L \in m(H)$. The system $S(P, C)$ is H-stable if and only if $D_H^{-1} \in m(H)$; equivalently, $S(P, C)$ is H-stable if and only if

$$D_H = \begin{bmatrix} D & -N_{pl} \\ \tilde{N}_cN_{pr} & \tilde{D}_c \end{bmatrix} \text{ is } H\text{-unimodular.} \tag{2.13}$$

Let

$$R := \begin{bmatrix} V_{pl} & X \\ -U_{pl} & Y \end{bmatrix}$$

by (2.3), $R \in m(H)$ is H-unimodular. Postmultiply D_H by R

$$D_HR = \begin{bmatrix} I_n & 0 \\ \tilde{N}_cN_{pr}V_{pl} - \tilde{D}_cU_{pl} & \tilde{N}_cN_{pr}X + \tilde{D}_cY \end{bmatrix}; \tag{2.14}$$

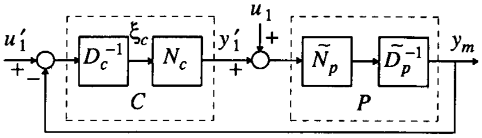

Fig. 3. $S(P, C)$ with $P = \tilde{D}_p^{-1}\tilde{N}_p$ and $C = N_c D_c^{-1}$.

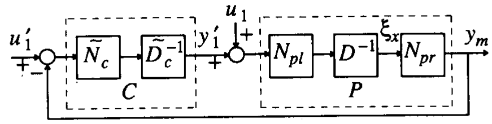

Fig. 4. $S(P, C)$ with $P = N_{pr}D^{-1}N_{pl}$ and $C = \tilde{D}_c^{-1}\tilde{N}_c$.

D_H is H-unimodular if and only if $D_H R$ is H-unimodular; hence, (2.13) holds if and only if

$$\tilde{N}_c N_{pr} X + \tilde{D}_c Y =: D_{HR} \text{ is } H\text{-unimodular}. \tag{2.15}$$

The set $S(P)$ of all H-stabilizing compensators is then the set of all $\tilde{D}_c^{-1}\tilde{N}_c$ such that (2.15) is satisfied.

Case 4: Finally, let $P = N_{pr}D^{-1}N_{pl}$ and let $C = N_c D_c^{-1}$, where (N_{pr}, D, N_{pl}) is a b.c.f.r. and (N_c, D_c) is r.c. (see Fig. 5). $S(P, C)$ is then described by (2.16), (2.17)

$$\begin{bmatrix} D & -N_{pl}N_c \\ N_{pr} & D_c \end{bmatrix}\begin{bmatrix} \xi_x \\ \xi_c \end{bmatrix} = \begin{bmatrix} N_{pl} & 0 \\ 0 & I_{n_o} \end{bmatrix}\begin{bmatrix} u_1 \\ u_1' \end{bmatrix}, \tag{2.16}$$

$$\begin{bmatrix} N_{pr} & 0 \\ 0 & N_c \end{bmatrix}\begin{bmatrix} \xi_x \\ \xi_c \end{bmatrix} = \begin{bmatrix} y_m \\ y_1' \end{bmatrix}. \tag{2.17}$$

Following similar steps as in Case 3 of the analysis, we conclude that $S(P, C)$ is H-stable if and only if

$$\hat{D}_H := \begin{bmatrix} D & -N_{pl}N_c \\ N_{pr} & D_c \end{bmatrix} \text{ is } H\text{-unimodular}. \tag{2.18}$$

Let

$$L := \begin{bmatrix} V_{pr} & U_{pr} \\ -\tilde{X} & \tilde{Y} \end{bmatrix}$$

by (2.2), $L \in m(H)$ is H-unimodular; and hence, $\hat{D}_H$ is H-unimodular if and only if $L\hat{D}_H$ is H-unimodular. The set $S(P)$ of all H-stabilizing compensators is then the set of all $N_c D_c^{-1}$ such that

$$\tilde{X}N_{pl}N_c + \tilde{Y}D_c =: D_{HL} \text{ is } H\text{-unimodular}. \tag{2.19}$$

□

We obtain a r.c.f.r. (N_p, D_p) and a l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ for P from the given b.c.f.r. (N_{pr}, D, N_{pl}) in Proposition 2.5 below; using the relationship between these coprime-factorizations, the set of all H-stabilizing compensators is given by (2.7) and equivalently, by (2.10).

2.5 Proposition: Let $P \in m(G_s)$. Let (N_{pr}, D, N_{pl}) be a b.c.f.r. of P; hence, (2.2), (2.3) hold. Under these conditions,

$$(N_p, D_p) := (N_{pr}X, Y) \text{ is a r.c.f.r. of } P, \tag{2.20}$$

$$(\tilde{D}_p, \tilde{N}_p) := (\tilde{Y}, \tilde{X}N_{pl}) \text{ is a l.c.f.r. of } P, \tag{2.21}$$

where $X, Y, \tilde{X}, \tilde{Y} \in m(H)$ are defined in (2.2), (2.3).

Comments: 1) Using (2.2), (2.3) we obtain a generalized Bezout identity for the doubly-coprime pair $((N_{pr}X, Y), (\tilde{Y}, \tilde{X}N_{pl}))$

$$\begin{bmatrix} V + UV_{pr}N_{pl} & UU_{pr} \\ -\tilde{X}N_{pl} & \tilde{Y} \end{bmatrix}\begin{bmatrix} Y & -U_{pl}\tilde{U} \\ N_{pr}X & \tilde{V} + N_{pr}V_{pl}\tilde{U} \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{2.22}$$

Note the similarity between (2.1) and (2.22). We refer to the matrices on the left-hand side of (2.22) as M and M^{-1}, respectively, (2.22) then reads

$$MM^{-1} = I_{n_o+n_i}. \tag{2.23}$$

2) If, instead of $N_{pr}D^{-1}N_{pl}$, the plant is given by $P = N_{pr}D^{-1}N_{pl} + E$, where $E \in m(H)$, then a r.c.f.r. and a l.c.f.r. are given by

$$(N_p, D_p) := (N_{pr}X + EY, Y), \ (\tilde{D}_p, \tilde{N}_p) := (\tilde{Y}, \tilde{X}N_{pl} + \tilde{Y}E),$$

$$V_p := V + UV_{pr}N_{pl} - UU_{pr}E, \ U_p := UU_{pr},$$

$$\tilde{U}_p := -U_{pl}\tilde{U}, \ \tilde{V}_p := \tilde{V} + N_{pr}V_{pl}\tilde{U} - EU_{pl}\tilde{U}.$$

Proof of Proposition 2.5: By assumption, $P = N_{pr}D^{-1}N_{pl}$, and (2.2), (2.3) hold. Clearly $N_{pr}X, Y, \tilde{Y}, \tilde{X}N_{pl} \in m(H)$. We must show that $(N_{pr}X, Y)$ is a r.c. pair with $\det Y \in I$ and that $(\tilde{Y}, \tilde{X}N_{pl})$ is a l.c. pair with $\det \tilde{Y} \in I$.

By (2.22), $(N_{pr}X, Y)$ is a r.c. pair and $(\tilde{Y}, \tilde{X}N_{pl})$ is a l.c. pair; more specifically, if $(N_{pr}X, Y) := (N_p, D_p)$ and $(\tilde{Y}, \tilde{X}N_{pl}) := (\tilde{D}_p, \tilde{N}_p)$, then

$$V_pD_p + U_pN_p = I_{n_i}, \ \tilde{N}_p\tilde{U}_p + \tilde{D}_p\tilde{V}_p = I_{n_o}, \tag{2.24}$$

where

$$V_p := V + UV_{pr}N_{pl}, \ U_p := UU_{pr}, \ \tilde{U}_p := U_{pl}\tilde{U}, \ \tilde{V}_p = \tilde{V} + N_{pr}V_{pl}\tilde{U}. \tag{2.25}$$

Now $P \in m(G_s)$. Postmultiplying P by Y and using $N_{pl}Y = DX$ from the Bezout equation (2.3), we obtain

$$PY = N_{pr}D^{-1}N_{pl}Y = N_{pr}X \in m(G_S). \tag{2.26}$$

Premultiplying P by $\tilde{Y}$ and using $\tilde{Y}N_{pr} = \tilde{X}D$ from the Bezout equation (2.2), we obtain

$$\tilde{Y}P = \tilde{Y}N_{pr}D^{-1}N_{pl} = \tilde{X}N_{pl} \in m(G_S). \tag{2.27}$$

By (2.26), $N_p := N_{pr}X \in m(G_s)$ and hence, $U_pN_p := UU_{pr}N_{pr}X \in m(G_s)$; then (2.24) implies that $\det(V_pD_p) = \det(I_{n_i} - U_pN_p) \in I$ and hence, $\det V_p \in I$ and $\det D_p := \det Y \in I$. From (2.22)–(2.24), since $\det M \in J$ and $\det Y \in I$, we obtain $\det Y \det M = \det \left[\begin{smallmatrix} Y & 0 \\ 0 & I_{n_o} \end{smallmatrix}\right] M = \det \tilde{Y} \in I$.

At this point we know that $Y^{-1} \in m(G)$ and $\tilde{Y}^{-1} \in m(G)$. By (2.26),

$$P = N_{pr}XY^{-1} \tag{2.28}$$

and similarly, by (2.27),

$$P = \tilde{Y}^{-1}\tilde{X}N_{pl}. \tag{2.29}$$

Finally, since (2.28) and (2.24) hold and since $\det Y \in I$, with $N_p, D_p \in m(H)$, $(N_{pr}X, Y) =: (N_p, D_p)$, is a r.c.f.r. of P. Similarly, from (2.29), (2.24), and $\det \tilde{Y} \in I$, with $\tilde{D}_p, \tilde{N}_p \in m(H)$, $(\tilde{Y}, \tilde{X}N_{pl}) =: (\tilde{D}_p, \tilde{N}_p)$, is a l.c.f.r. of P. □

Comment: If $P \in m(G)$ but not $m(G_s)$, (2.20), (2.21) still give a r.c.f.r. and a l.c.f.r. of P, respectively. The only difference in this case is in showing that $\det Y \in I$ and $\det \tilde{Y} \in I$: Consider the Bezout equation (2.2); since $P \in m(G)$, $\det V_{pr}$ is not necessarily $\in I$. Choose $T \in m(H)$ such that $\det(V_{pr} - T\tilde{X}) \in I$ [9]; then by (2.2),

$$\begin{bmatrix} V_{pr} - T\tilde{X} & U_{pr} + T\tilde{Y} \\ -\tilde{X} & \tilde{Y} \end{bmatrix}\begin{bmatrix} D & -\tilde{U} - DT \\ N_{pr} & \tilde{V} - N_{pr}T \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{2.30}$$

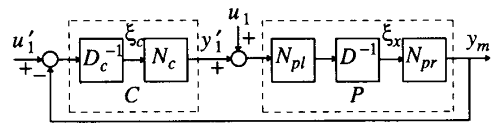

Fig. 5. $S(P, C)$ with $P = N_{pr}D^{-1}N_{pl}$ and $C = N_c D_c^{-1}$.

Since det $D \in I$, from (2.30) we get det $((V_{pr} - T\tilde{X})D)$ = det $(I_n - (U_{pr} + T\tilde{Y})N_{pr})$ = det $(I_{n_o} - N_{pr}(U_{pr} + T\tilde{Y}))$ = det $((\tilde{V} - TN_{pr})\tilde{Y}) \in I$; equivalently, det $(\tilde{V} - TN_{pr}) \in I$ and det $\tilde{Y} \in I$. So by (2.22), since det $M \in J$, we obtain det Y = det $\tilde{Y}$ det $M^{-1} \in I$.

2.6. Theorem (Set of All H-Stabilizing Compensators): Let $P \in m(G_s)$ and let (N_{pr}, D, N_{pl}) be a b.c.f.r. of P, hence (2.2) and (2.3) hold. Then

$$S(P)=\{(V+UV_{pr}N_{pl}-Q\tilde{X}N_{pl})^{-1}(UU_{pr}+Q\tilde{Y}) : Q \in m(H)\}; \tag{2.32}$$

equivalently,

$$S(P)=\{(U_{pl}\tilde{U}+YQ)(\tilde{V}+N_{pr}V_{pl}\tilde{U}-N_{pr}XQ)^{-1} : Q \in m(H)\}; \tag{2.33}$$

where the matrices in (2.32), (2.33) are as in the generalized Bezout equation (2.22).

Comment: By Proposition 2.5 we know how to obtain a r.c.f.r. (N_p, D_p) and a l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$ from a b.c.f.r. (N_{pr}, D, N_{pl}) of $P \in m(G_s)$: with (N_p, D_p) as in (2.20), $(\tilde{D}_p, \tilde{N}_p)$ as in (2.21), and V_p, U_p, $\tilde{V}_p$, $\tilde{U}_p$ as in (2.25), the generalized Bezout equation (2.22) is the same as the Bezout equation (2.1). Furthermore, observe that (2.20) substituted into (2.15) implies that $D_{HR} = \tilde{N}_c N_p + \tilde{D}_c D_p$, and hence, H-stability using Analysis 2.4—Case 3 is equivalent to establishing H-stability using Case 1. Therefore, it is no surprise that $S(P)$ in (2.32) is the same as $S(P)$ in (2.7), with (2.20) and (2.25) in mind. Similarly, (2.21) substituted into (2.19) implies that $D_{HL} = \tilde{N}_p N_c + \tilde{D}_p D_c$, and hence, H-stability using Analysis 2.4—Case 4 is equivalent to Case 2. Therefore, $S(P)$ in (2.33) is the same as $S(P)$ in (2.10), with (2.21) and (2.25) in mind.

Although the discussion above justifies Theorem 2.6, we now give a formal proof.

Proof of Theorem 2.6: We only prove that the set $S(P)$ in (2.32) is the set of all H-stabilizing compensators; the proof of equation (2.33) is entirely similar.

If C is defined by the expression in (2.32) then C H-stabilizes P. Let

$$C=\tilde{D}_c^{-1}\tilde{N}_c,\ \tilde{D}_c=V+UV_{pr}N_{pl}-Q\tilde{X}N_{pl},\ \tilde{N}_c=UU_{pr}+Q\tilde{Y}. \tag{2.34}$$

We must show that i) C satisfies assumption B), i.e., $\tilde{D}_c$, $\tilde{N}_c \in m(H)$ with det $\tilde{D}_c \in I$ and the pair $(\tilde{D}_c, \tilde{N}_c)$ is l.c., and ii) $S(P, C)$ is H-stable, i.e., (2.15) holds.

i) From (2.34), $\tilde{D}_c$, $\tilde{N}_c \in m(H)$. By the generalized Bezout equation (2.22),

$$D_{HR}=\tilde{N}_c N_{pr}X+\tilde{D}_c Y=(UU_{pr}+Q\tilde{Y})N_{pr}X + (V+UV_{pr}N_{pl}-Q\tilde{X}N_{pl})Y=I_{n_i}. \tag{2.35}$$

By (2.35), $(\tilde{D}_c, \tilde{N}_c)$ is a l.c. pair. In the proof of Proposition 2.5 we showed that $N_{pr}X \in m(G_s)$ [see (2.26)], and hence $\tilde{N}_c N_{pr} X \in m(G_s)$. We conclude from (2.35) that det $(\tilde{D}_c Y)$ = det $(I_{n_i} - \tilde{N}_c N_{pr} X) \in I$, therefore det $\tilde{D}_c \in I$; consequently, $(\tilde{D}_c, \tilde{N}_c)$ is a l.c.f.r. of C.

ii) From (2.35), $D_{HR} = I_{n_i}$; hence $S(P, C)$ is H-stable since (2.15) holds.

Any C that H-stabilizes P is an element of the set $S(P)$ defined by (2.32). Let $C \in m(G)$ H-stabilize P. Let $(\tilde{D}_c, \tilde{N}_c)$ be a l.c.f.r. of C. By assumption, $S(P, C)$ is H-stable; equivalently, by normalizing (2.15), $D_{HR} = I_{n_i}$. Then

$$[\tilde{D}_c \vdots \tilde{N}_c] \begin{bmatrix} Y & -U_{pl}\tilde{U} \\ N_{pr}X & \tilde{V}+N_{pr}V_{pl}\tilde{U} \end{bmatrix} =: [I_{n_i} \vdots Q] \tag{2.36}$$

where $Q := -\tilde{D}_c U_{pl}\tilde{U} + \tilde{N}_c(\tilde{V} + N_{pr}V_{pl}\tilde{U}) \in H^{n_i \times n_o}$. Postmultiply both sides of (2.36) by the H-unimodular matrix M defined in (2.22), (2.23)

$$[\tilde{D}_c \vdots \tilde{N}_c]=[I_{n_i} \vdots Q] \begin{bmatrix} V+UV_{pr}N_{pl} & UU_{pr} \\ -\tilde{X}N_{pl} & \tilde{Y} \end{bmatrix} \tag{2.37}$$

from (2.37), $C = \tilde{D}_c^{-1}\tilde{N}_c$ is in the set $S(P)$ in (2.32) for some $Q \in H^{n_i \times n_o}$ (in fact, there is a unique Q for each C; we prove this in Corollary 2.7). □

2.7. Corollary: Let C_1, $C_2 \in S(P)$; then $C_1 = C_2$ if and only if $Q_1 = Q_2$. Equivalently, the map $Q \to C$, $Q \in m(H)$, $C \in S(P)$, is one-to-one.

Proof: Let $S(P)$ be given as in (2.32); the proof for (2.33) is entirely similar.

Let $C_1 = \tilde{D}_{c1}^{-1}\tilde{N}_{c1}$, $C_2 = \tilde{D}_{c2}^{-1}\tilde{N}_{c2}$; by (2.36)

$$[\tilde{D}_{c1} \vdots \tilde{N}_{c1}]M^{-1}=[I_{n_i} \vdots Q_1]=\tilde{D}_{c1}[I_{n_i} \vdots C_1]M^{-1}, \tag{2.38}$$

$$[\tilde{D}_{c2} \vdots \tilde{N}_{c2}]M^{-1}=[I_{n_i} \vdots Q_2]=\tilde{D}_{c2}[I_{n_i} \vdots C_2]M^{-1}. \tag{2.39}$$

But $C_1 = C_2$ in (2.38), (2.39) implies $[I_{n_i} \vdots C_1]M^{-1} = \tilde{D}_{c1}^{-1}[I_{n_i} \vdots Q_1] = \tilde{D}_{c2}^{-1}[I_{n_i} \vdots Q_2]$ and hence, $\tilde{D}_{c1} = \tilde{D}_{c2}$; consequently, $Q_1 = Q_2$.

Now suppose C_1 is given by a l.c.f.r. $(\tilde{D}_{c1}, \tilde{N}_{c1})$ but C_2 is given by a r.c.f.r. (N_{c2}, D_{c2}); then by (2.33) and (2.22),

$$M \begin{bmatrix} -N_{c2} \\ D_{c2} \end{bmatrix} = \begin{bmatrix} -Q_2 \\ I_{n_o} \end{bmatrix}. \tag{2.40}$$

By (2.40), (2.38), and (2.23) we obtain

$$[\tilde{D}_{c1} : \tilde{N}_{c1}]M^{-1}M \begin{bmatrix} -N_{c2} \\ D_{c2} \end{bmatrix} = [I_{n_i} \vdots Q_1] \begin{bmatrix} -Q_2 \\ I_{n_o} \end{bmatrix}. \tag{2.41}$$

But $C_1 = C_2$ implies $\tilde{N}_{c1}D_{c2} = \tilde{D}_{c1}N_{c2}$; therefore by (2.41), $(-\tilde{D}_{c1}N_{c2} + \tilde{N}_{c1}D_{c2}) = Q_1 - Q_2 = 0$. We conclude that for each $C \in S(P)$, there is a unique $Q \in m(H)$. □

2.8. Example: Let $H = R_u(s)$ as in Example 1.2. Let $P = \mathbb{R}_{sp}(s)^{n_o \times n_i}$ be represented by its state-space representation $\dot{x} = Ax + Bu$, $y = Cx$, where (C, A, B) is stabilizable and detectable in $\bar{u}$. Then $P = (s + a)^{-1}C[(s + a)^{-1}(sI - A)]^{-1}B$, where $a \in \mathbb{R}$, $-a \in \mathbb{C} \setminus \bar{u}$. The pair $((s + a)^{-1}C, (s + a)^{-1}(sI - A))$ is r.c. in $R_u(s)$, the pair $((s + a)^{-1}(sI - A), B)$ is l.c. in $R_u(s)$, and det $[(s + a)^{-1}(sI - A)] \in I$. Therefore, $(N_{pr}, D, N_{pl}) = ((s + a)^{-1}C, (s + a)^{-1}(sI - A), B)$ is a b.c.f.r. of P. Choose $K \in \mathbb{R}^{n_i \times n}$ and $F \in \mathbb{R}^{n \times n_o}$ such that $(A - BK)$ and $(A - FC)$ have all eigenvalues in $\mathbb{C} \setminus u$; Let $G_K := (sI_n - A + BK)^{-1}$ and let $G_F := (sI_n - A + FC)^{-1}$; then $(s + a)(sI_n - A + BK)^{-1} = (s + a)G_K$ and $(s + a)(sI_n - A + FC)^{-1} = (s + a)G_F$ are $\in m(R_u(s))$. For this special b.c.f.r., (2.2), (2.3) and (2.22) become

$$\begin{bmatrix} (s+a)G_F & (s+a)G_F F \\ -CG_F & I_{n_o}-CG_F F \end{bmatrix} \begin{bmatrix} (s+a)^{-1}(sI_n-A) & -F \\ (s+a)^{-1}C & I_{n_o} \end{bmatrix} = I_{n+n_o}; \tag{2.2S}$$

$$\begin{bmatrix} (s+a)^{-1}(sI_n-A) & -B \\ (s+a)^{-1}K & I_{n_i} \end{bmatrix} \begin{bmatrix} (s+a)G_K & (s+a)G_K B \\ -KG_K & I_{n_i}-KG_K B \end{bmatrix} = I_{n+n_i}; \tag{2.3S}$$

$$\begin{bmatrix} I_{n_i}+KG_F B & KG_F F \\ -CG_F B & I_{n_o}-CG_F F \end{bmatrix} \begin{bmatrix} I_{n_i}-KG_K B & -KG_K F \\ CF_K B & I_{n_o}+CG_K F \end{bmatrix} = I_{n_i+n_o}. \tag{2.22S}$$

Note that (2.22S) gives the coprime factorizations obtained in [8, eq. (1)–(4)].

III. Conclusions

From a given b.c.f.r. (N_{pr}, D, N_{pl}) for $P \in m(G_s)$, we obtain a r.c.f.r. (N_p, D_p), a l.c.f.r. $(\tilde{D}_p, \tilde{N}_p)$, and the associated generalized Bezout identities. We find the class of all H-stabilizing compensators for P based on (N_{pr}, D, N_{pl}): with V, U, V_{pr}, U_{pr}, $\tilde{X}$, $\tilde{Y}$ as in (2.22), $C \in G^{n_i \times n_o}$, given by

$$C=\tilde{D}_c^{-1}\tilde{N}_c=(V+UV_{pr}N_{pl}-Q\tilde{X}N_{pl})^{-1}(UU_{pr}+Q\tilde{Y}) \tag{3.1}$$

H-stabilizes P, where $Q \in m(H)$ is a free parameter. If we design a two-degrees-of-freedom compensator $C := [C_{21} \vdots C_{22}]$ (as in [3], [4], [9], for example), then $C = \tilde{D}_c^{-1}[Q_{21} \vdots \tilde{N}_c]$, where $Q_{21} \in m(H)$, and $\tilde{D}_c^{-1}\tilde{N}_c$ is given by (3.1); in this case there are two free parameters.

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A Sufficient Condition for Output Feedback Stabilization of Uncertain Dynamical Systems

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Abstract—**A different proof than the one given in [6] is given for the existence of an output feedback controller which stabilizes an uncertain single-input single-output dynamical system with a linear nominal part and matched uncertainties. Yet there is no need to assume that the nominal system is stable. A simple expression is obtained for the feedback gain which is necessary for the closed-loop nominal system to become strictly positive real.**

INTRODUCTION

Recently [1], a condition was given for the existence of a static output feedback stabilizing controller for an uncertain dynamical system with a linear nominal part and matched [2]–[5] uncertainties with known bounds. For a single-input single-output system, the condition requires that the input–output transfer function $g(s)$ of the nominal linear system be strictly positive real (SPR) [1]. Consider the output feedback stabilization of the following single-input, single-output system:

$$\dot{x}(t) = Ax(t) + b[u(t) + e(t, x)];\ x(t) \in R^n,\ b \in R^n,\ u(t) \in R$$
$$y(t) = C^T x(t);\ C \in R^n \tag{1}$$

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where the known triple (C^T, A, b) defines a nominal linear system and $e{:}R \times R^n \rightarrow R$ models all uncertainty which is assumed matched. The input–output transfer function $g(s)$ for the nominal linear system in (1) is given by

$$g(s) \triangleq C^T(sI - A)^{-1}b. \tag{2}$$

In [1] it is established that if $g(s)$ is SPR, then stability (in the sense defined in [1]) can be achieved, in the presence of arbitrary admissible uncertainty $e(t, x)$, by linear static output feedback

$$u(t) = -\gamma y(t) \tag{3}$$

with sufficiently large gain $\gamma > 0$.

A less restrictive sufficient condition for the existence of a stabilizing controller was given in [6]. In [6] it is assumed that the nominal system is stable and that its transfer function $g(s)$:

a) has a positive leading coefficient,

b) satisfies $g(s) \neq 0$ for Re $[s] \geq 0$,

and

c) $g(s)$ has relative degree one, i.e., if its denominator polynomial is of order n, then its numerator polynomial is of order $(n - 1)$.

It was shown in [6] that a nominal system satisfying conditions a)–c) can always become SPR by applying negative constant gain output feedback $u = -ky$. Once the nominal system becomes SPR, additional gain is necessary to stabilize system (1) with uncertainties.

In this note we provide a different proof for the lemma in [6], yet there is no need to assume that the nominal system is stable, as assumed in the Introduction and in the example in [6]. Furthermore, the proof we give allows us to obtain a simple expression for the feedback gain K which is necessary for the closed-loop nominal system to become SPR. Thus, it is not necessary to solve Lyapunov's equation as was done in [6].

FREQUENCY DOMAIN CONDITIONS FOR SPR FUNCTIONS [7]

A transfer function $\phi(s)$ for a linear single-input, single-output system, with relative degree $m = 1$, is SPR if and only if

a) $\phi(s)$ is analytic in Re $[s] \geq 0$,

b) Re $[\phi(j\omega)] > 0\ \forall\ \omega \in (-\infty, \infty)$,

and

c) $\lim_{\omega^2 \rightarrow \infty} \omega^2$ Re $[\phi(j\omega)] > 0$.

Allowing A in (1) to be unstable, we prove the following lemma and derive an expression for the gain K that is necessary to make the closed-loop nominal system SPR.

Lemma: Given a transfer function

$$g(s) = \frac{n(s)}{d(s)} = \frac{s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}. \tag{3}$$

Assume the roots of $n(s) = 0$ are in Re $[s] < 0$. Then, for a sufficiently large gain $K \geq \bar{K}$, the closed-loop transfer function

$$h(s) = \frac{g(s)}{1 + Kg(s)} \tag{4}$$

is SPR.

Proof: We have to show that the SPR conditions are satisfied.

Consider the function

$$\frac{1}{h(s)} = K + \frac{1}{g(s)} = K + \frac{d(s)}{n(s)}. \tag{5}$$

By assumption, $n(s)$ has no roots on the imaginary axis, thus, $n(j\omega) \neq 0$ and Re $d(j\omega)/n(j\omega)$ is finite for all finite ω. Furthermore, it can be easily verified that $\lim_{\omega \rightarrow \infty}$ Re $d(j\omega)/n(j\omega) = a_{n-1} - b_{n-2}$.

Hence, the following is well defined:

$$K_0 = \inf_{s=j\omega} \left[\text{Re}\ \frac{1}{g(s)} \right] \triangleq -\bar{K}. \tag{6}$$

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