2005 American Control Conference
June 8-10, 2005. Portland, OR, USA



# PID Stabilizability Conditions and Controller Synthesis for MIMO Plants

A. N. Gündeş and Edgar C. Wai

***Abstract*— Stability using Proportional+Integral+Derivative (PID) controllers is investigated for linear multi-input multi-output (MIMO) plants. Necessary conditions are derived for existence of PID-controllers. Systematic PID-controller synthesis procedures are developed for several plant classes.**

## I. INTRODUCTION

Proportional+Integral+Derivative (PID) controllers are widely used and preferred due to their simplicity. Control texts treat PID-control is treated extensively (see e.g., [7]). In spite of the importance and wide-spread use of these low-order controllers, most PID design approaches lack systematic procedures and closed-loop stability proofs. Rigorous synthesis methods based on modern control theory are being studied in recent literature (see for example [5], [6], [8], [9]). The simplicity of PID-controllers is desirable from a tuning point-of-view, but it presents a major restriction: PID-controllers can control only certain classes of plants satisfactorily, while others cannot be stabilized. It is important to identify plant classes that admit PID-controllers, i.e., that can be stabilized using these simple controllers. Most results available show existence of PID-controllers only for low order plants. Explicit descriptions of high-order plant classes that admit PID-controllers and sufficient conditions for stabilizability of general multi-input multi-output (MIMO) or single-input single-output (SISO) unstable plants using PID-controllers are not available.

We consider closed-loop stabilization of linear, time-invariant (LTI), MIMO plants in the standard unity-feedback system configuration of Fig. 1. The order of the plants is not restricted. Although the continuous-time setting is used here, the results can be interpreted for discrete-time systems with minor modifications. Some of these results of existence of PID-controllers in the scalar plant case could be derived using root-locus arguments or via a generalization of the Hermite-Biehler Theorem (see [1], and [10], [4], [3]); similar derivations are not known for the MIMO case. Such existence proofs would not produce explicit synthesis procedures. The results here emphasize systematic designs with freedom in the design parameters. Our goal is to establish existence of stabilizing PID-controllers; we propose freedom in the design parameters that can be used towards satisfaction of performance criteria. We give simple illustrative examples, where we show only a few controllers out of the many that can be designed using the methods described here. It is possible to design various other controllers based on the given design specifications.

This work was supported by the NSF Grant ECS-9905729

The authors are with the Department of Electrical and Computer Engineering, University of California, Davis, CA 95616 gundes@ece.ucdavis.edu

The following notation is used: Let $\mathbb{C}$, $\mathbb{R}$ denote complex and real numbers. The extended closed right-half complex plane $\mathcal{U} = \{s \in \mathbb{C} \mid \mathcal{R}e(s) \geq 0\} \cup \{\infty\}$ is the unstable region; $\mathbf{R_p}$ denotes real proper rational functions of $s$; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in $\mathbf{S}$; $I_n$ is the $n \times n$ identity matrix. The $H_\infty$-norm of $M(s) \in \mathcal{M}(\mathbf{S})$ is $\|M\| := \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$. We drop $(s)$ in transfer matrices such as $G(s)$ when this causes no confusion. We use coprime factorizations over $\mathbf{S}$; i.e., $G = XY^{-1} \in \mathbf{R_p}^{n_y \times n_u}$ denotes a right-coprime-factorization (RCF), where $X \in \mathbf{S}^{n_y \times n_u}$, $Y \in \mathbf{S}^{n_u \times n_u}$, $\det Y(\infty) \neq 0$; $G = \widetilde{Y}^{-1}\widetilde{X}$ denotes a left-coprime-factorization (LCF), where $\widetilde{X} \in \mathbf{S}^{n_y \times n_u}$, $\widetilde{Y} \in \mathbf{S}^{n_y \times n_y}$, $\det \widetilde{Y}(\infty) \neq 0$.

## II. PROBLEM DESCRIPTION

Consider the LTI, MIMO unity-feedback system $Sys(G, C)$ in Fig. 1; $G \in \mathbf{R_p}^{n_y \times n_u}$ is the plant's, $C \in \mathbf{R_p}^{n_u \times n_y}$ is the controller's transfer-function. The system is well-posed, $G$ and $C$ have no unstable hidden-modes, and $G \in \mathbf{R_p}^{n_y \times n_u}$ is full (normal) row-rank.

*Definition 2.1: i)* The system $Sys(G, C)$ is said to be stable iff the closed-loop transfer-function from $(r, v)$ to $(y, w)$ is stable. *ii)* The controller $C$ is said to stabilize $G$ iff $C$ is proper and $Sys(G, C)$ is stable. ■

We consider a realizable form of proper PID-controllers,

$$C_{pid} = K_p + \frac{K_i}{s} + \frac{K_d\ s}{\tau_d s + 1}\ , \tag{1}$$

where $K_p$, $K_i$, $K_d \in \mathbb{R}^{n_u \times n_y}$ are called the proportional, integral, and derivative constants, and $\tau_d \in \mathbb{R}$ [2]. To implement the derivative term, a pole is added ($\tau_d > 0$) so that $C_{pid}$ in (1) is proper. The integral-action in $C_{pid}$ is present when $K_i \neq 0$. Subsets of PID-controllers obtained by setting one or two of the three constants equal to zero: When $K_d = 0$, $C_{pid}$ in (1) is in proportional+integral (PI) form $C_{pi} = K_p + \frac{K_i}{s}$. When $K_p = 0$, $C_{pid}$ is in integral+derivative (ID) form $C_{id} = \frac{K_i}{s} + \frac{K_d s}{\tau_d s + 1}$. When $K_i = 0$, $C_{pid}$ is in proportional+derivative (PD) form $C_{pd} = K_p + \frac{K_d s}{\tau_d s + 1}$. When two of the three constants are zero, $C_p$, $C_i$, $C_d$ correspond to pure proportional (P), integral (I), derivative (D) controllers, respectively.

0-7803-9098-9/05/$25.00 ©2005 AACC

*Definition 2.2:* A plant $G \in \mathbf{R_p}^{n_y \times n_u}$ is said to admit a PID-controller iff there exists a PID-controller $C_{pid}$ as in (1) such that the closed-loop system is stable. We say that $G$ is stabilizable by a PID-controller, and $C_{pid}$ is a stabilizing PID-controller. ■

*Lemma 2.1:* (*General existence conditions for stabilizing PID-controllers*): Let $G \in \mathbf{R_p}^{n_y \times n_u}$. Let (normal) rank$G(s) = n_y$. **a)** If $G$ admits a PID-controller such that the integral constant $K_i \in \mathbb{R}^{n_u \times n_y}$ is nonzero, then $G$ has no transmission-zeros at $s = 0$ and rank$K_i = n_y$. **b)** If $G$ admits a PID-controller such that any one of the three constants $K_p$, $K_d$, $K_i$ is nonzero, then $G$ admits a PID-controller such that any two of the three constants is nonzero, and $G$ admits a PID-controller such that all three constants is nonzero. If $G$ admits a PID-controller such that two of the three constants $K_p$, $K_d$, $K_i$ is nonzero, then $G$ admits a PID-controller such that all three constants is nonzero. In these statements, the integral constant is nonzero only if $G$ has no transmission-zeros at $s = 0$. ■

*Lemma 2.2:* (*Integral controllers for stable systems*): Let $H \in \mathbf{S}^{n_y \times n_u}$, (normal) rank$H = n_y \le n_u$. *i)* There exists a stabilizing I-controller for $H$ if and only if rank$H(0) = n_y$. *ii)* Suppose rank$H(0) = n_y$. Let $H(0)^I \in \mathbb{R}^{n_u \times n_y}$ be any right-inverse of $H(0)$. Then for any $\rho \in \mathbb{R}$ satisfying

$$0 < \rho < \| \frac{H(s)\ H(0)^I - I}{s} \|^{-1}, \qquad (2)$$

$H$ is stabilized by the I-controller $C_i = \dfrac{\rho H(0)^I}{s}$. ■

*Proof*: *i)* Let $C_i = K_i/s$ stabilize $H$. For positive $a \in \mathbb{R}$, $M := \frac{s}{s+a} I_{n_y} + H \frac{K_i}{s+a} = (I_{n_y} + HC_i) \frac{s}{s+a} I_{n_y}$ is unimodular. Since $a \neq 0$, rank$M(0) = $ rank$(a^{-1} H(0) K_i) = n_y \le min\{$rank$H(0),$rank$K_i\} \le min\{n_y, n_u\} = n_y$ implies rank$H(0) = n_y$. Then there exists a right inverse $H(0)^I \in \mathbb{R}^{n_u \times n_y}$, i.e., $H(0)H(0)^I = I_{n_y}$. *ii)* Let $K_i = \rho H(0)^I$; define $\hat{M} := \frac{(s+a)}{(s+\rho)} M = \frac{s\,I}{s+\rho} + \frac{H C_i\, s}{s+\rho} = I_{n_y} + \frac{\rho s}{s+\rho} ( \frac{H(s)H(0)^I - I_{n_y}}{s} )$. Since $\| \frac{\rho s}{s+\rho} \| = \rho$, for any $\rho > 0$ as in (2), we have $\| \frac{\rho s}{s+\rho} ( \frac{H(s)H(0)^I - I_{n_y}}{s} ) \| < 1$. Therefore, $\hat{M}$ is unimodular; equivalently, $M$ is unimodular since $a, \rho > 0$; therefore $C_i = \rho H(0)^I / s$ stabilizes $H$. ■

*Proposition 2.1:* (*Two step PID-controller synthesis*): Let $G \in \mathbf{R_p}^{n_y \times n_u}$, (normal) rank$G(s) = n_y$, $G$ have no transmission-zeros at $s = 0$. Suppose $G$ admits a stabilizing PD-controller $C_{pd}$. Let $K_i/s$ be any I-controller that stabilizes $H := G(I + C_{pd} G)^{-1}$. Then

$$C_{pid} \;=\; C_{pd} + \frac{K_i}{s}\ , \qquad (3)$$

is a PID-controller that stabilizes $G$. ■

*Proof:* Let $G = XY^{-1}$ be an RCF, $G = \widetilde{Y}^{-1} \widetilde{X}$ be an LCF. Since $C_{pd}$ stabilizes $G$, $M_{pd} := Y + C_{pd} X$ and equivalently, $\widetilde{M}_{pd} = \widetilde{Y} + \widetilde{X} C_{pd}$ are unimodular, and $Y M_{pd}^{-1} + C_{pd} X M_{pd}^{-1} = I$. By assumption, $K_i/s$ stabilizes $H = X M_{pd}^{-1}$. For any $a \in \mathbb{R}$, $a > 0$, $M_g := \frac{s\,I}{s+a} + H \frac{K_i}{s+a} = \frac{s\,I}{s+a} + X M_{pd}^{-1} \frac{K_i}{s+a}$ is unimodular. An RCF $C_{pid} = N_c D_c^{-1}$ for $C_{pid} = C_{pd} + K_i/s$ is $N_c D_c^{-1} := (\frac{s}{s+a} C_{pd} + \frac{K_i}{s+a})(\frac{s}{s+a} I)^{-1} = (C_{pd}\, M_g + Y M_{pd}^{-1} \frac{K_i}{s+a})(\frac{s}{s+a} I)^{-1}$. Since $\widetilde{Y} X = \widetilde{X} Y$, we have $\widetilde{Y}\, D_c + \widetilde{X}\, N_c = \widetilde{Y}(\frac{s}{s+a} I + X M_{pd}^{-1} \frac{K_i}{s+a}) + \widetilde{X} C_{pd}\, M_g = (\widetilde{Y} + \widetilde{X} C_{pd}) M_g = \widetilde{M}_{pd} M_g$ is unimodular. Therefore, $C_{pid}$ in (3) stabilizes $G$. ■

In Proposition 2.1, the I-controller $K_i/s$ for the stable system $H := G(I + C_{pd} G)^{-1}$ can be chosen as in Lemma 2.2: Choose any $\rho \in \mathbb{R}$ satisfying

$$0 < \rho < \| \frac{(I + G C_{pd})^{-1} G(s) H(0)^I - I}{s} \|^{-1}, \qquad (4)$$

where $H(0) = G(0)(I + C_{pd}(0) G(0))^{-1} = G(0)(I + K_p G(0))^{-1}$. Condition (4) can be expressed as (5) and (6):

$$0 < \rho < \| (I + G C_{pd})^{-1} \frac{G(s)(H(0)^I - C_{pd}) - I}{s} \|^{-1}, \qquad (5)$$

$$0 < \rho < \| (I + G C_{pd})^{-1} \frac{G(s) G^I(0) - I}{s} - H(s) \frac{K_d}{\tau_d s + 1} \|^{-1}. \qquad (6)$$

Then $H$ is stabilized by the I-controller $C_i = \rho\ H(0)^I / s$.

Lemma 2.1 states that if a stabilizing $C_p$, $C_i$, $C_d$ exists for $G$, then the remaining constants can be selected to extend to stabilizing PI, ID, PD, PID-controllers. Lemma 2.1 does not explicitly describe the plants that admit P, I, or D-controllers. Specific plant classes are investigated next.

## III. Main Results

We propose methods to explicitly synthesize $C_{pid}$ for five plant classes that admit PID-controllers. Let $G \in \mathbf{R_p}^{n_y \times n_u}$, (normal) rank$G(s) = n_y$. *1)* Stable plants: If $G \in \mathbf{S}^{n_y \times n_u}$, then there exist P, D, PD-controllers. If rank$G(0) = n_y$, then there also exist I, PI, ID, PID-controllers. *2)* Unstable (square) plants with no finite zeros in $\mathcal{U}$ and with relative degree equal to 0: If $G^{-1} \in \mathbf{S}^{n_y \times n_y}$, then there exist P, I, D, PD, PI, PID-controllers. *3)* Unstable (square) plants with no zeros in $\mathcal{U}$ and with relative degree equal to 1: If $\frac{1}{(s+a)} G^{-1} \in \mathbf{S}^{n_y \times n_y}$ (for any $a \in \mathbb{R}, a > 0$), then there exist P, PD, PI, PID-controllers. *4)* Unstable (square) plants with only one zero at $s = 0$, no other zeros in $\mathcal{U}$, and with relative degree equal to 0: If $\frac{s}{(s+a)} G^{-1} \in \mathbf{S}^{n_y \times n_y}$ (for any $a \in \mathbb{R}, a > 0$), then there exist P, D, PD-controllers. *5)* Unstable (square) plants with only one pole or only two poles at $s = 0$ and no other poles in $\mathcal{U}$: If either $\frac{s}{s+a} G \in \mathbf{S}^{n_y \times n_y}$, or $\frac{s^2}{(s+a)(s+b)} G \in \mathbf{S}^{n_y \times n_y}$ (for any $a, b \in \mathbb{R}, a, b > 0$), then there exist P, I, D, PI, PD, PID-controllers.

### A. Stable Plants

Let $G \in \mathbf{S}^{n_y \times n_u}$; rank$G = n_y$; $G$ has no poles in $\mathcal{U}$. By Lemma 2.1, PID-controllers with nonzero $K_i$ exist only if $G$ has no transmission-zeros at $s = 0$. Stable plants admit P, D, PD-controllers; $G$ admits I, PI, ID, PID-controllers if and only if rank$G(0) = n_y$. Proposition 3.1 (i) develops a PD-controller. The PID-controller in (ii) adds an integral term. An alternate PID-controller is in (iii), which is not based on adding an integral term onto a PD-controller.

*Proposition 3.1:* Let $G \in \mathbf{S}^{n_y \times n_u}$, rank$G(s) = n_y$. *i) PD-design*: Choose any $\hat{K}_p, \hat{K}_d \in \mathbb{R}^{n_y \times n_u}$, $\tau_d > 0$. Choose any $\alpha \in \mathbb{R}$ satisfying

$$0 < \alpha < \| \ ( \ \hat{K}_p + \frac{\hat{K}_d s}{\tau_d s + 1} \ ) \ G(s) \ \|^{-1} \ . \tag{7}$$

Let $K_p = \alpha \hat{K}_p$, $K_d = \alpha \hat{K}_d$; then a stabilizing PD-controller for $G$ is

$$C_{pd} = \alpha \hat{K}_p + \frac{\alpha \hat{K}_d s}{\tau_d s + 1} \ . \tag{8}$$

For $\hat{K}_d = 0$, (8) is a P-controller $C_p$; for $\hat{K}_p = 0$, (8) is a D-controller $C_d$. *ii) PID-design*: Let rank$G(0) = n_y$; let $G^I(0)$ be a right-inverse of $G(0)$. Let $C_{pd}$ be as in (8). Choose any $\rho \in \mathbb{R}$ satisfying (6). Let $K_i = \rho(\alpha \hat{K}_p + G(0)^I)$; then a stabilizing PID-controller for $G$ is

$$C_{pid} = \alpha \hat{K}_p + \frac{\rho(\alpha \hat{K}_p + G(0)^I)}{s} + \frac{\alpha \hat{K}_d s}{\tau_d s + 1} \ . \tag{9}$$

For $\hat{K}_d = 0$, (9) is a PI-controller $C_{pi}$; for $\hat{K}_p = 0$, (9) is an ID-controller $C_{id}$. *iii) Alternate PID-design*: Let rank$G(0) = n_y$; let $G^I(0)$ be a right-inverse of $G(0)$. Choose any $\hat{K}_p, \hat{K}_d \in \mathbb{R}^{n_y \times n_u}$, $\tau_d > 0$. Choose any $\gamma \in \mathbb{R}$ satisfying

$$0 < \gamma < \| G(s)(\hat{K}_p + \frac{\hat{K}_d s}{\tau_d s + 1}) + \frac{G(s)G^I(0) - I}{s} \|^{-1}. \tag{10}$$

Let $K_p = \gamma \hat{K}_p$, $K_d = \gamma \hat{K}_d$, $K_i = \gamma G^I(0)$; then

$$C_{pid} = \gamma \hat{K}_p + \frac{\gamma G^I(0)}{s} + \frac{\gamma \hat{K}_d s}{\tau_d s + 1} \tag{11}$$

is a PID-controller that stabilizes $G$. For $\hat{K}_d = 0$, (11) is a PI-controller $C_{pi}$; for $\hat{K}_p = 0$, (11) is an ID-controller $C_{id}$; for $\hat{K}_d = \hat{K}_p = 0$, (11) is an I-controller $C_i$. ■

*Proof*: *i)* Let $C_{pd}$ be as in (8) for $\alpha \in \mathbb{R}$ as in (7). Then $M_{pd} := I + C_{pd}G$ is unimodular and $C_{pd}$ stabilizes $G$. Since $\hat{K}_p, \hat{K}_d$ are arbitrary, they can be zero. *ii)* Since $C_{pd}$ in (8) is a stabilizing controller for $G$, $H = GM_{pd}^{-1} \in \mathbf{S}^{n_y \times n_u}$. Since $M_{pd}$ is unimodular, rank$H(0) =$ rank$G(0)M_{pd}(0)^{-1} =$ rank$G(0) = n_y$; $H(0) = G(0)(I + \alpha \hat{K}_p G(0))^{-1}$, $H(0)^I = \alpha \hat{K}_p + G^I(0)$. By Lemma 2.2, for any $\rho \in \mathbb{R}$ as in (6), $K_i/s = \rho H(0)^I/s = \rho(\alpha \hat{K}_p + G^I(0))$ is an I-controller stabilizing $H$. By Proposition 2.1, $C_{pid} = C_{pd} + K_i/s$ in (9) stabilizes $G$. *iii)* Let $C_{pid}$ be as in (11) for $\gamma \in \mathbb{R}$ as in (10). Then $M_{pid} := \frac{s\,I}{s+\gamma} + G\frac{s}{s+\gamma}C_{pid} = I + \frac{\gamma\, s}{s+\gamma}[G\,(\hat{K}_p + \frac{\hat{K}_d s}{\tau_d s+1}) + \frac{GG^I(0)-I}{s}]$ is unimodular and $C_{pid}$ stabilizes $G$. ■

*Example 3.1:* Let $G = \begin{bmatrix} \frac{1}{s+1} & 3 \\ \frac{3}{s+2} & \frac{s+1}{s+3} \end{bmatrix}$; rank$G(0) = 2$. Let $\hat{K}_p = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $\hat{K}_d = \begin{bmatrix} 1 & 0 \\ 6 & 0 \end{bmatrix}$, $\tau_d = 0.1$. Then $\alpha = 0.0026 < 0.0053$ satisfies (7). The PD-controller in (8) is $C_{pd} = \begin{bmatrix} \frac{0.0286s+0.026}{s+10} & 0.0052 \\ \frac{0.156s}{s+10} & 0.0052 \end{bmatrix}$. Choose $\rho = 0.5420 < 1.0840$ satisfying (6). The PID-controller in (9) is $C_{pid} = C_{pd} + \frac{1}{s}\begin{bmatrix} -0.0419 & 0.3931 \\ 0.1951 & -0.1272 \end{bmatrix}$. The closed-loop poles are $\{-6.6929, -2.8749, -1.1724 \pm j0.5087, -0.6096\}$. ■

*B. Unstable plants with no $\mathcal{U}$-zeros:*

Let $G \in \mathbf{R_p}^{n_y \times n_y}$ be square, rank$G = n_y$; $G$ has no transmission-zeros in $\mathcal{U}$ (including infinity). Therefore, $G$ has an RCF $G = XY^{-1} = I_{n_y}(G^{-1})^{-1}$. The necessary condition in Lemma 2.1 for existence of PID-controllers with nonzero $K_i$ is satisfied since $G$ has no transmission-zeros at $s = 0 \in \mathcal{U}$. Then $G$ admits P, I, PI, PD, PID-controllers; $G$ admits D-controllers if and only if $G$ has no poles at $s = 0$. Proposition 3.2 (i) develops a PD-controller synthesis; it gives a P-controller for $K_d = 0$. For $G$ with no poles at $s = 0$, a D-controller is in (ii). The PID-controller in (iii) adds an integral term; it gives a PI-controller for $K_d = 0$. An ID-controller is in (iv) for plants with no poles at $s = 0$ based on the D-controller in (ii). An alternate PID-controller is in (v), which is not based on adding an integral term.

*Proposition 3.2:* Let $G \in \mathbf{R_p}^{n_y \times n_y}$, rank$G(s) = n_y$. Let $G$ have no transmission-zeros in $\mathcal{U}$. *i) PD-design*: Choose any $K_d \in \mathbb{R}^{n_y \times n_y}$, $\tau_d > 0$; choose any nonsingular $\hat{K}_p \in \mathbb{R}^{n_y \times n_y}$. Choose any $\alpha \in \mathbb{R}$ satisfying

$$\alpha > \| \ \hat{K}_p^{-1} \ ( \ G^{-1}(s) + \frac{K_d s}{\tau_d s + 1} \ ) \ \| \ . \tag{12}$$

Let $K_p = \alpha \hat{K}_p$; then a stabilizing PD-controller for $G$ is

$$C_{pd} = \alpha \, \hat{K}_p + \frac{K_d s}{\tau_d s + 1} \ . \tag{13}$$

For $K_d = 0$, (13) is a P-controller. *ii) PID-design*: Let $C_{pd}$ be as in (13). With $G^{-1}(0)$ replacing $G^I(0)$ in (6), choose any positive $\rho \in \mathbb{R}$ satisfying (6). Let $K_i = \rho(\alpha \hat{K}_p + G^{-1}(0))$; then a stabilizing PID-controller for $G$ is

$$C_{pid} = \alpha \hat{K}_p + \frac{\rho(\alpha \hat{K}_p + G^{-1}(0))}{s} + \frac{K_d s}{\tau_d s + 1} \ . \tag{14}$$

If $K_d = 0$, (14) is a PI-controller. *iii) D-design*: Let rank$G^{-1}(s)|_{s=0} = n_y$. Choose any $\tau_d > 0$. Choose any $\delta \in \mathbb{R}$ satisfying

$$\delta > \| \ \frac{(\tau_d s + 1) \ G^{-1}(s) \ G(0) - I}{s} \ \| \ . \tag{15}$$

Let $K_p = 0$, $K_d = \delta \, G^{-1}(0)$; then a stabilizing D-controller for $G$ is

$$C_d = \frac{\delta \, G^{-1}(0) \, s}{\tau_d s + 1} \ . \tag{16}$$

*iv) ID-design*: Let rank$G^{-1}(s)|_{s=0} = n_y$. Let $K_p = 0$; let $C_d$ be as in (16). Choose any positive $\beta \in \mathbb{R}$ satisfying

$$0 < \beta < \| \ \frac{(I + GC_d)^{-1} G(s) G^{-1}(0) - I}{s} \ \|^{-1} \ . \tag{17}$$

Let $K_i = \beta G^{-1}(0)$; then a stabilizing ID-controller for $G$ is

$$C_{id} = \frac{\beta G^{-1}(0)}{s} + \frac{\delta \, G^{-1}(0) \, s}{\tau_d s + 1} \ . \tag{18}$$

*v) Alternate PID-design*: Choose $K_p, K_d \in \mathbb{R}^{n_y \times n_y}$, $\tau_d > 0$ such that $\det[I + (K_p + \tau_d^{-1} K_d) G(\infty)] \neq 0$. Let $W_{pd} := G^{-1}(s) + K_p + \dfrac{K_d s}{\tau_d s + 1}$. Choose any $\gamma \in \mathbb{R}$ satisfying

$$\gamma > \| s \, [ \, W_{pd} \, (G(\infty)^{-1} + K_p + \tau_d^{-1} K_d \,)^{-1} - I \, ] \, \| . \quad (19)$$

Let $K_i = \gamma \, ( \, G(\infty)^{-1} + K_p + \tau_d^{-1} K_d \, )$; then

$$C_{pid} = K_p + \frac{\gamma [G(\infty)^{-1} + K_p + \tau_d^{-1} K_d]}{s} + \frac{K_d s}{\tau_d s + 1} \quad (20)$$

is a stabilizing PID-controller for $G$. If $K_d = 0$, (20) is a PI-controller; if $K_p = 0$, (20) is an ID-controller; if $K_p = K_d = 0$, (20) is an I-controller. ∎

*Proof*: By assumption, $G = XY^{-1}$, $X = I_{n_y}$, $Y = G^{-1}$. *i)* Let $C_{pd}$ be as in (13) for $\alpha \in \mathbb{R}$ as in (12). Then $C_{pd}$ stabilizes $G$ since $M_{pd} := Y + C_{pd} X = G^{-1} + C_{pd} = K_p (I + \frac{1}{\alpha} \hat{K}_p^{-1} (G^{-1} + \frac{K_d s}{\tau_d s + 1}))$ is unimodular. Since $K_d$ is arbitrary, it can be zero. *ii)* Since $C_{pd}$ in (13) stabilizes $G$, $H := G(I + C_{pd} G)^{-1}$ is stable. Since $M_{pd}$ is unimodular, $H = X M_{pd}^{-1} = M_{pd}^{-1}$ is unimodular; hence $\text{rank} H(0) = n_y$, $H(0)^{-1} = M_{pd}(0) = \alpha \hat{K}_p + G^{-1}(0)$. By Lemma 2.2, for $\rho \in \mathbb{R}$ as in (6), $K_i / s = \rho H(0)^{-1} / s$ stabilizes $H$. By Proposition 2.1, $C_{pid} = C_{pd} + K_i / s$ in (14) stabilizes $G$. *iii)* Let $C_d$ be as in (16) for $\delta \in \mathbb{R}$ as in (15). Then $M_d := Y + C_d X = G^{-1} + \frac{\delta G(0)^{-1} s}{\tau_d s + 1}$ is unimodular since $\delta, \tau_d > 0$ and $M_d G(0) \frac{(\tau_d s + 1)}{(\delta s + 1)} = I + \frac{s}{(\delta s + 1)} [\frac{(\tau_d s + 1) G^{-1} G(0) - I}{s}]$ is unimodular. Therefore, $C_d$ stabilizes $G$. *iv)* Since $C_d$ in (16) stabilizes $G$, $H$ is stable. Since $M_d$ is unimodular, $H = X M_d^{-1} = M_d^{-1}$ is unimodular; hence $\text{rank} H(0) = n_y$, $H(0)^{-1} = M_d(0) = G(0)^{-1}$. By Lemma 2.2, for $\beta \in \mathbb{R}$ as in (17), $K_i / s = \beta H(0)^{-1} / s$ is an I-controller stabilizing $H$. By Proposition 2.1, $C_{id} = C_d + K_i / s$ in (18) stabilizes $G$. *v)* Let $C_{pid}$ be as in (20) for $\gamma \in \mathbb{R}$ as in (19). Since $K_p, K_d, \tau_d$ are such that $W_{pd}(\infty)$ is nonsingular, $W_{pd}^{-1}(\infty)$ exists. Then $M_{pid} := \frac{s \, G^{-1}}{s + \gamma} + \frac{s \, C_{pid}}{s + \gamma} = \frac{s}{s + \gamma} W_{pd} + \frac{\gamma}{s + \gamma} W_{pd}(\infty) = [I + \frac{1}{s + \gamma} \, s \, (W_{pd}(s) W_{pd}^{-1}(\infty) - I)] W_{pd}(\infty)$ is unimodular. Therefore, $C_{pid}$ stabilizes $G$. Since $\hat{K}_p, \hat{K}_d$ are arbitrary, they can be zero. ∎

*Example 3.2:* Given $G = \frac{(s+2)(s+3)(s+4)}{5s(s-1)(s-5)}$; $G^{-1}$ is stable. Let $K_d = -1.5$, $\tau_d = 0.01$, $\hat{K}_p = 2$. Choosing $\alpha = 200$ as in (12), the PD-controller in (13) is $C_{pd} = 400 - 1.5s/(0.01s + 1)$. The closed-loop poles are $\{-158.71, -3.50 \pm j0.87, -1.83\}$. Following Proposition 3.2-(v), let $K_p = -10$, $K_d = 0.5$, $\tau_d = 0.01$. Let $K_i = 45\gamma$ with $\gamma = 120$ satisfying (19). Then the PID-controller in (20) is $C_{pid} = -10 + 5400/s + 0.5s/(0.01s + 1)$. The closed-loop poles are $\{-2.0442, -2.7801, -4.2818, -53.5580 \pm j94.693\}$. ∎

*Example 3.3:* Let $G = \begin{bmatrix} \frac{s+1}{s-1} & \frac{s+1}{s-1} \\ 0 & 2 \end{bmatrix}$; $G^{-1}$ is stable; $\text{rank} G^{-1}(s)|_{s=0} = 2$. Follow Proposition 3.2-(iii): Choosing $\tau_d = 0.1$, (15) is satisfied for $\delta > 2.7596$. With $\delta = 3$, the D-controller in (16) is $C_d = \begin{bmatrix} \frac{-30s}{s+10} & \frac{-15s}{s+10} \\ 0 & \frac{15s}{s+10} \end{bmatrix}$. The closed-loop poles are $\{-0.3621 \pm j0.4623, -0.3226\}$. ∎

### C. Unstable plants with only one zero at infinity:

Let $G \in \mathbf{R_p}^{n_y \times n_y}$ be square, have full (normal) row-rank and be strictly-proper. Other than a blocking-zero at infinity (of multiplicity one), let $G$ have no transmission-zeros in $\mathcal{U}$. For SISO systems, the unstable plant has relative degree one. Therefore, $G$ has an RCF $G = XY^{-1} = \dfrac{1}{s+a} I_{n_y} (\dfrac{1}{s+a} G^{-1})^{-1}$, $a \in \mathbb{R}$, $a > 0$. The necessary condition in Lemma 2.1 for existence of PID-controllers with nonzero $K_i$ is satisfied since $G$ has no transmission-zeros at $s = 0 \in \mathcal{U}$. Plants in this class admit P, PI, PD, PID-controllers; but some (for example, $G = \frac{1}{s-p}$, $p > 0$) do not admit stabilizing D-controllers $C_d = K_d s / (\tau_d s + 1)$ for any $\tau_d > 0$ or I-controllers $C_i = K_i / s$. Proposition 3.3 (i) develops a PD-controller synthesis; the PID-design in (ii) adds an integral term to the PD-controller in (i).

*Proposition 3.3:* Let $G \in \mathbf{R_p}^{n_y \times n_y}$, $\text{rank} G(s) = n_y$. Let $\dfrac{1}{s+a} G^{-1} \in \mathcal{M}(\mathbf{S})$, for $a \in \mathbb{R}$, $a > 0$. *i) PD-design*: Choose any $K_d \in \mathbb{R}^{n_y \times n_y}$, $\tau_d > 0$. With $Y(\infty)^{-1} = (s + a) G(s)|_{s \to \infty}$, choose any positive $\alpha \in \mathbb{R}$ satisfying

$$\alpha > \| \; Y(\infty)^{-1} \; ( \; G^{-1}(s) + \frac{K_d s}{\tau_d s + 1} \; ) \; - sI \; \| \, . \quad (21)$$

Let $K_p = \alpha Y(\infty)$; then a stabilizing PD-controller for $G$ is

$$C_{pd} = \alpha \; Y(\infty) + \frac{K_d s}{\tau_d s + 1} \; . \quad (22)$$

If $K_d = 0$, (22) is a P-controller.

*ii) PID-design*: Let $C_{pd}$ be as in (22). With $K_d$ arbitrary, and $G^{-1}(0)$ replacing $G^I(0)$ in condition (6), choose any positive $\rho \in \mathbb{R}$ satisfying (6). Let $K_i = \rho(G^{-1}(0) + \alpha Y(\infty))$; then a stabilizing PID-controller for $G$ is

$$C_{pid} = \alpha Y(\infty) + \frac{\rho(\alpha Y(\infty) + G^{-1}(0))}{s} + \frac{K_d s}{\tau_d s + 1} \; . \quad (23)$$

If $K_d = 0$, (23) is a PI-controller. ∎

*Proof*: By assumption, $G = XY^{-1}$; $X = (s + a)^{-1} I_{n_y}$, $Y = (s + a)^{-1} G^{-1}$. *i)* Let $C_{pd}$ be as in (22) for $\alpha \in \mathbb{R}$ satisfying (21). Then $M_{pd} := Y + C_{pd} X = \frac{1}{s+a} [G^{-1} + (K_p + \frac{K_d s}{\tau_d s + 1})]$ is unimodular since $a, \; \alpha > 0$ and $\frac{(s+a)}{(s+\alpha)} Y(\infty)^{-1} M_{pd} = I + \frac{1}{s+\alpha} [\, Y(\infty)^{-1} (\, G^{-1} + \frac{K_d s}{\tau_d s + 1} \,) - sI]$ is unimodular. Since $K_d$ is arbitrary, it can be zero. *ii)* Since $C_{pd}$ in (22) stabilizes $G$, $H = X M_{pd}^{-1}$ is stable. Since $M_{pd}$ is unimodular, $H = (s + a)^{-1} M_{pd}^{-1}$ implies $\text{rank} H(0) = n_y$, $H(0)^{-1} = a M_{pd}(0) = K_p + G^{-1}(0) = \alpha Y(\infty) + G^{-1}(0)$. By Lemma 2.2, for $\rho \in \mathbb{R}$ as in (6), $K_i / s = \rho H(0)^{-1} / s$ stabilizes $H$. By Proposition 2.1, $C_{pid} = C_{pd} + K_i / s$ in (23) stabilizes $G$. ∎

*Example 3.4:* Given $G = \frac{-2(s+2)(s+3)}{(s-4)[(s-2)^2+9]}$, let $K_d = -0.1$, $\tau_d = 0.004$. Choose $\alpha = 38$ satisfying (21). With $Y(\infty)^{-1} = -2$, the PD-controller in (22) is $C_{pd} = -19 - 0.1s/(0.004s + 1)$. The closed-loop poles are $\{-0.91, -304.42, -12.34 \pm j2.5\}$. If $K_d = 0$, choosing $\alpha = 80$ satisfying (21), the P-controller $C_p = -40$. The closed-loop poles are $\{-1.2591, -5.1854, -65.556\}$. For a

PID-design, start with $C_{pd}$, i.e., $K_p = -19$, $K_d = -0.1$, $\tau_d = 0.004$. Choosing $\rho = 0.39$ satisfying (6), $C_{pid} = -19 - 5.72/s - 0.1s/(0.004s+1)$ as in (23). The closed-loop poles are $\{-304.41, -12.23 \pm j3.58, -0.56 \pm j0.17\}$.

*Example 3.5:* Given $G = \frac{1}{s-1}\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, let $\tau_d = 0.1$, $K_d = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. With $Y(\infty)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, choose $\alpha = 22 > 21.9507$ satisfying (21). The PD-controller in (22) is $C_{pd} = \begin{bmatrix} \frac{5.6s-44}{s+10} & 8.8 \\ 13.2 & -4.4 \end{bmatrix}$. The closed-loop poles are $\{-35, -21, -6\}$. With this $C_{pd}$, $\rho = 2.8360 < 5.6719$ satisfies (6). The PID-controller in (23) is $C_{pid} = C_{pd} + \frac{1}{s}\begin{bmatrix} -11.9112 & 23.8224 \\ 35.7336 & -11.9112 \end{bmatrix}$. The closed-loop poles are $\{-33.4801, -17.6200, -3.7599 \pm j1.9108, -3.38\}$. ∎

*D. Unstable plants with only one zero at the origin:*

Let $G \in \mathbf{R_p}^{n_y \times n_y}$ be square, have full (normal) row-rank. Other than a blocking-zero at $s = 0$ (of multiplicity one), let $G$ have no transmission-zeros in $\mathcal{U}$ (including infinity); i.e., the $G$ has no poles at $s = 0$. Therefore, $G$ has an RCF $G = XY^{-1} = \frac{s}{as+1} I_{n_y} (\frac{s}{as+1} G^{-1})^{-1}$, $a \in \mathbb{R}$, $a > 0$. The necessary condition in Lemma 2.1 for existence of PID-controllers with nonzero $K_i$ is not satisfied. Plants in this class admit P-controllers and PD-controllers. Proposition 3.4 develops a PD-controller synthesis, which converts to a P-design. These plants may admit D-controllers, but the synthesis in Proposition 3.4 does not address this case.

*Proposition 3.4:* Let $G \in \mathbf{R_p}^{n_y \times n_y}$, $\text{rank}G(s) = n_y$. Let $\frac{s}{as+1} G^{-1} \in \mathcal{M}(\mathbf{S})$, for $a \in \mathbb{R}$, $a > 0$. Choose any $K_d \in \mathbb{R}^{n_y \times n_y}$, $\tau_d > 0$. With $Y(0)^{-1} = s^{-1} G(s)|_{s=0}$, choose any positive $\alpha \in \mathbb{R}$ satisfying

$$\alpha > \| \frac{s\, Y(0)^{-1}\, G^{-1}(s) - I}{s} + Y(0)^{-1} \frac{K_d s}{\tau_d s + 1} \|. \quad (24)$$

Let $K_p = \alpha Y(0)$; then a stabilizing PD-controller for $G$ is

$$C_{pd} = \alpha\, Y(0) + \frac{K_d s}{\tau_d s + 1}\ . \quad (25)$$

If $K_d = 0$, (25) is a P-controller. ∎

*Proof*: By assumption, $G = XY^{-1}$, $X = s(as+1)^{-1} I_{n_y}$, $Y = s(as+1)^{-1} G^{-1}$ implies $Y(0)^{-1} sG|_{s=0} = I$. Let $C_{pd}$ be as in (25) for $\alpha \in \mathbb{R}$ as in (24). Then $M_{pd} := Y + C_{pd} X = \frac{s}{as+1}\,[\, G^{-1} + K_p + \frac{K_d s}{\tau_d s+1}\,] = Y(0)[\frac{\alpha s}{as+1} I + \frac{s}{as+1} Y(0)^{-1} (G^{-1} + \frac{K_d s}{\tau_d s+1})]$ is unimodular since $a, \alpha > 0$ and $\frac{(as+1)}{(\alpha s+1)} Y(0)^{-1} M_{pd} = I + \frac{s}{\alpha s+1} [\frac{Y(0)^{-1} s G^{-1} - I}{s} + Y(0)^{-1} \frac{K_d s}{\tau_d s+1}\,]$ is unimodular. Therefore, $C_{pd}$ stabilizes $G$. Since $K_d$ is arbitrary, it can be zero. ∎

*Example 3.6:* Given $G = \frac{s}{s-1}\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, let $\tau_d = 1$, $K_d = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}$. With $Y(0)^{-1} = \begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}$, choose $\alpha = 13 > 12.9020$ satisfying (24). The PD-controller in (25) is $C_{pd} = \begin{bmatrix} 26 & \frac{-10s-13}{s+1} \\ \frac{-38s-39}{s+1} & \frac{14s+13}{s+1} \end{bmatrix}$. The closed-loop poles are $\{-50.8161, -1.0237, -0.0832, -0.0770\}$. ∎

*E. Plants with poles in the stable region and at the origin:*

The plants in this class are square and have poles at $s = 0$ of multiplicity one or two, but no other $\mathcal{U}$-poles. The poles at $s = 0$ appear in some or all entries of $G$. If the poles at $s = 0$ appear in some but not all entries of $G$, we further assume that $G$ has no transmission-zeros at $s = 0$. When every entry of $G$ has poles at $s = 0$, the assumption of no transmission-zeros at $s = 0$ is obviously satisfied.

*1) Plants with poles in the stable region and only one pole at the origin:* Let $Y := \frac{s}{as+1} I$, $a \in \mathbb{R}, a > 0$. Let $G \in \mathbf{R_p}^{n_y \times n_y}$ have full normal rank. Let $X := GY = \frac{s}{as+1} G \in \mathbf{S}^{n_y \times n_y}$ for $a \in \mathbb{R}, a > 0$, and let $\text{rank}X(0) = n_y$. Some or all entries of $G(s)$ have a pole (of multiplicity one) at $s = 0$; $G$ has no other poles in $\mathcal{U}$. Furthermore, $X(0) = GY(s)|_{s=0} = sG(s)|_{s=0}$ nonsingular implies $G$ has no transmission-zeros at $s = 0$; hence, the necessary condition in Lemma 2.1 for existence of PID-controllers with nonzero $K_i$ is satisfied. Then $G$ admits P, PD, PI, PID-controllers. $G$ does not admit D-controllers $C_d = K_d s/(\tau_d s + 1)$ since the plant pole at $s = 0$ would then cancel the controller's zero. Some plants (for example, $G = \frac{1}{s(s+p)}$, $p \geq 0$) do not admit I-controllers $C_i = K_i/s$. Proposition 3.5 (i) develops a PD-controller synthesis; the PID-controller in (ii) adds an integral term.

*Proposition 3.5:* Let $G \in \mathbf{R_p}^{n_y \times n_y}$, $\text{rank}G(s) = n_y$. Let $X := \frac{s}{as+1} G \in \mathcal{M}(\mathbf{S})$, for $a \in \mathbb{R}$, $a > 0$. Let $X(0) = s\, G(s)|_{s=0}$ be nonsingular. *i) PD-design*: Choose any $\hat{K}_d \in \mathbb{R}^{n_y \times n_y}$, $\tau_d > 0$. Choose any $\alpha \in \mathbb{R}$ satisfying

$$0 < \alpha < \| \frac{X(0)^{-1} s\ G(s) - I}{s} + \frac{\hat{K}_d s}{\tau_d s + 1}\ G(s)\ \|^{-1}\ . \quad (26)$$

Let $K_p = \alpha X(0)^{-1}$, $K_d = \alpha \hat{K}_d$; then a stabilizing PD-controller for $G$ is

$$C_{pd} = \alpha X(0)^{-1} + \frac{\alpha \hat{K}_d s}{\tau_d s + 1}\ . \quad (27)$$

If $\hat{K}_d = 0$, (27) is a P-controller.

*ii) PID design*: Let $C_{pd}$ be as in (27). With $K_d = \alpha \hat{K}_d$, and $G^{-1}(0)$ replacing $G^I(0)$ in (6), choose any $\rho \in \mathbb{R}$ satisfying (6). Let $K_i = \rho\alpha X(0)^{-1}$; then a stabilizing PID-controller for $G$ is

$$C_{pid} = \alpha X(0)^{-1} + \frac{\rho\alpha X(0)^{-1}}{s} + \frac{\alpha \hat{K}_d s}{\tau_d s + 1}\ . \quad (28)$$

If $K_d = 0$, (28) is a PI-controller. ∎

*Proof*: By assumption, $G = XY^{-1}$, where $X = (as+1)^{-1} s G$, $Y = (as+1)^{-1} sI$. *i)* Let $C_{pd}$ be as in (27) for $\alpha \in \mathbb{R}$ satisfying (26). Then $M_{pd} := Y + C_{pd}\, X = \frac{s\,I}{as+1} + C_{pd} \frac{s\,G}{as+1} = \frac{(s+\alpha)}{(as+1)} [I + \frac{\alpha\, s}{s+\alpha} (\frac{X(0)^{-1} s\, G - I}{s} + \frac{K_d\, s}{\tau_d s+1} G)]$ is unimodular since $a, \alpha > 0$. Therefore, $C_{pd}$ stabilizes $G$. Since $K_d$ is arbitrary, it can be chosen as zero. *ii)* Since $C_{pd}$ in (27) is a stabilizing controller for $G$, $H = X M_{pd}^{-1}$ is stable. Since $\text{rank}X(0) = n_y$, $\text{rank}H(0) = n_y$, $H(0)^{-1} = K_p = \alpha X(0)^{-1}$. By Lemma 2.2, for any $\rho \in \mathbb{R}$ as in

(6), $K_i/s = \rho H(0)^{-1}/s$ stabilizes $H$. By Proposition 2.1, $C_{pid} = C_{pd} + K_i/s$ in (28) stabilizes $G$. ∎

*Example 3.7:* Given $G = \begin{bmatrix} \frac{s+2}{s} & \frac{1}{s} \\ \frac{s+1}{s} & 1 \end{bmatrix}$, let $\tau_d = 0.1$, $\hat{K}_d = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$. With $X(0) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, choose $\alpha = 0.0116 < 0.0231$ satisfying (26). The PD-controller in (27) is $C_{pd} = \begin{bmatrix} \frac{0.116s}{s+10} & 0.0116 \\ \frac{0.2436s+0.116}{s+10} & \frac{0.3248s-0.232}{s+10} \end{bmatrix}$. ∎

*2) Plants with poles in the stable region and two poles at the origin:* Define $Y := \dfrac{s^2}{(as+1)(bs+1)} I$, $a, b \in \mathbb{R}, a, b > 0$. Let $G \in \mathbf{R_p}^{n_y \times n_y}$ have full normal rank. Let $X := GY = \frac{s^2}{(as+1)(bs+1)} G \in \mathbf{S}^{n_y \times n_y}$, for $a, b \in \mathbb{R}, a, b > 0$ and let $\text{rank} X(0) = n_y$. Some or all entries of $G(s)$ have a pole of multiplicity two at $s = 0$; other entries may have a pole of multiplicity one or no poles at $s = 0$; $G$ has no other poles in $\mathcal{U}$. Furthermore, $X(0) = GY(s)|_{s=0} = s^2 G(s)|_{s=0}$ nonsingular implies $G$ has no transmission-zeros at $s = 0$; hence, the necessary condition in Lemma 2.1 for existence of PID-controllers with nonzero $K_i$ is satisfied. Then $G$ admits PD, PID-controllers; it does not admit D-controllers $C_d = K_d s/(\tau_d s + 1)$ since the plant pole at $s = 0$ would cancel the controller's zero. Some plants (for example, $G = \frac{1}{s^2(s+p)}$, $p \geq 0$) do not admit P-controllers $C_p = K_p$ or I-controllers $C_i = K_i/s$. Proposition 3.6 (i) develops a PD-controller synthesis; the PID-controller in (ii) adds an integral term.

*Proposition 3.6:* Let $G \in \mathbf{R_p}^{n_y \times n_y}$, $\text{rank} G(s) = n_y$. Let $X := \dfrac{s^2}{(as+1)(bs+1)} G \in \mathcal{M}(\mathbf{S})$, for $a \in \mathbb{R}$, $a > 0$. Let $X(0) = s^2 G(s)|_{s=0}$ be nonsingular. *i) PD-design*: Choose any $\delta \in \mathbb{R}$ satisfying

$$0 < \delta < \| \frac{\frac{1}{(\tau_d s+1)} X(0)^{-1} s^2 \; G(s) \; - I}{s} \|^{-1} . \qquad (29)$$

Let $K_d = \alpha X(0)^{-1}$, $\tau_d > 0$. Choose any $\alpha \in \mathbb{R}$ satisfying

$$0 < \alpha < \| \frac{\delta X(0)^{-1} (I + G \frac{K_d s}{(\tau_d s+1)})^{-1} s G(s) - I}{s} \|^{-1} . \qquad (30)$$

Let $K_p = \alpha \delta X(0)^{-1}$; then

$$C_{pd} = \alpha \delta X(0)^{-1} + \frac{\delta X(0) s}{\tau_d s + 1} \qquad (31)$$

is a stabilizing PD-controller for $G$.
*ii) PID design*: Let $C_{pd}$ be as in (31). With $K_d = \delta\, X(0)^{-1}$, and $G^{-1}(0)$ replacing $G^I(0)$ in (6), choose any $\rho \in \mathbb{R}$ satisfying (6). Let $K_i = \rho \alpha \delta X(0)^{-1}$; then

$$C_{pid} = \delta \alpha X(0)^{-1} + \frac{\delta \alpha \rho \; X(0)^{-1}}{s} + \frac{\delta X(0)^{-1} s}{\tau_d s + 1} \qquad (32)$$

is a stabilizing PID-controller for $G$. ∎
*Proof*: By assumption, $G = XY^{-1}$, $X = (as+1)^{-1}(bs+1)^{-1} s^2\, G$, $Y = (as+1)^{-1}(bs+1)^{-1} s^2 I$. *i)* For $\delta \in \mathbb{R}$ as in (29), $W_d := \frac{s\, I}{as+1} + \frac{K_d\, s}{(\tau_d s+1)} (bs+1) X = \frac{(s+\delta)}{(as+1)} [\; I + \frac{\delta\, s}{s+\delta} ( \frac{\frac{1}{(\tau_d s+1)} X(0)^{-1} s^2 G - I}{s} ) ]$ is unimodular. Let $C_{pd}$ be as in (31) for $\alpha \in \mathbb{R}$ as in (29). Define $H_d := (bs+1)\; X\; W_d^{-1} = s(I + G \frac{K_d\, s}{(\tau_d s+1)})^{-1} G$; then $H_d(0)^{-1} = \delta X(0)^{-1}$. Then $M_{pd} := Y + C_{pd}\, X \frac{s}{bs+1} [ \frac{s}{as+1} I + \frac{K_d\, s}{\tau_d s+1} (bs + 1) X ] + K_p X = \frac{(s+\alpha)}{(bs+1)} [ I + \frac{\alpha\, s}{s+\alpha} ( \frac{\delta\, X(0)^{-1} H_d - I}{s} ) ] W_d$ is unimodular since $b, \alpha > 0$ and $W_d$ is unimodular. Therefore, $C_{pd}$ stabilizes $G$. *ii)* Since $C_{pd}$ in (31) stabilizes $G$, $H = X M_{pd}^{-1} = X W_d^{-1} (bs+1) s^{-1} (I + K_p X W_d^{-1} (bs + 1) s^{-1})^{-1} = s^{-1} H_d (I + K_d s^{-1} H_d)^{-1}$ is stable. Since $\text{rank} X(0) = n_y$, $\text{rank} H(0) = n_y$, $H(0)^{-1} = K_p = \alpha \delta X(0)^{-1}$. By Lemma 2.2, for any $\rho \in \mathbb{R}$ satisfying (6), $K_i/s = \rho H(0)^{-1}/s$ is an I-controller that stabilizes $H$. By Proposition 2.1, $C_{pid} = C_{pd} + K_i/s$ in (32) stabilizes $G$. ∎

## IV. Conclusions

We showed existence of stabilizing PID-controllers for several LTI, MIMO plant classes. We proposed systematic PID-controller synthesis procedures that guarantee robust closed-loop stability. We achieved stabilizing PID-controller designs with freedom in the design parameters that can be used towards satisfaction of performance criteria. Future goals of this study include consideration of performance issues. While PID-controllers can be designed using various other methods, the systematic procedures proposed here are straightforward and offer great design flexibility.

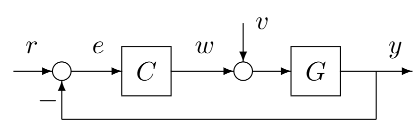


Fig. 1. Unity-Feedback System $Sys(G, C)$.

## References


[1] F. R. Gantmacher, *Matrix Theory*, Vol. II, New York: Chelsea, 1990.
[2] G. C. Goodwin, S. F. Graebe, and M. E. Salgado, *Control System Design*, Prentice Hall, 2001.
[3] Ming-Tzu Ho, A. Datta, S. P. Bhattacharyya, "An extension of the generalized Hermite-Biehler theorem: relaxation of earlier assumptions," *Proc. 1998 American Contr. Conf.*, pp. 3206-3209, 1998.
[4] M. G. Kabuli, A. N. Gündeş, N. At, M. Koca, "Pole placement using constant output feedback," *Proc. 1998 American Control Conference*, pp. 1151-1152, 1998.
[5] L. H. Keel, J. I. Rego, S. P. Bhattacharyya, "A new approach to digital PID controller design," *IEEE Trans. Automatic Control*, AC-48: 4, pp. 687-692, 2003.
[6] Ching-An Lin, A. N. Gündeş, "Multi-input multi-output PI controller design," *Proc. 39th IEEE Conf. on Decision and Control*, pp. 3702 -3707, 2000.
[7] K. Ogata, *Modern Control Engineering, 3rd edition*, Prentice Hall, New Jersey, 1997.
[8] G. J. Silva, A. Datta, S. P. Bhattacharyya, "New results on the synthesis of PID controllers," *IEEE Trans. Automatic Control*, AC-47: 2, pp. 241-252, 2002.
[9] G. J. Silva, A. Datta, S. P. Bhattacharyya, "Robust control design using the PID controller," *Proc. 41st IEEE Conf. on Decision and Control*, pp. 1313-1318, 2002.
[10] M. Yousefpor, A. N. Gündeş, A. Palazoğlu, "A test for stabilizability by constant output-feedback," *Proc. 32nd IEEE Conf. on Decision and Control*, 1826-1827, 1993; and *International Journal of Control*, vol. 62:5, pp. 1239-1246, 1995.