

Some positive results on simultaneous stabilization

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Abstract.

A recent result of [1] on simultaneous stabilization of a family of plants is re-derived with some major simplifications. The result is that, under mild conditions, a finite number of LTI plants having unstable zeros of fixed multiplicities only at the origin and/or at infinity admit a common stabilizing controller.

1. Introduction

Although a solution to the problem of simultaneous stabilization of a finite number of LTI plants is highly desirable in algebraic control [2] and in robust control [3] alike, there are a number of strong negative results [4],[5],[6] on the topic and only a few positive ones [7],[8],[9],[10],[11].

In [1], any family of plants having unstable zeros of fixed multiplicities only at the origin and/or at infinity has been shown to admit a single stabilizing controller provided the high (and the low) frequency gains of the plants are “linked” by a positive definite matrix. Unlike similar results in the literature, [1] contains an explicit construction of a common controller together with a description of the set of all common controllers. Such a family is not obviously stabilizable by high-gain since its plants have unstable zeros. For instance, the set of all autoregressive (AR) transfer matrices of order n :

$$F_{ar} := \left\{ Q_i(s)^{-1} : i = 1, \dots, v, Q_i(s) = \sum_{j=0}^n Q_{ij} s^j, Q_m = I, Q_{ij} \text{ are real matrices} \right\}$$

is an admissible family. The family of “all-pole filters of a fixed order” may hence be stabilized by a single controller by the result of [1].

The purpose of this paper is to provide an alternative derivation of the main result of [1]. The derivation is based on a small-gain lemma of [11] and a “separation lemma”. The latter lemma may be of some independent interest. It splits the problem of construction of a stabilizing controller for a family of plants, each plant having a fixed and a varying numerator, into two easier sub-problems of constructing two common controllers for the fixed-numerator-plants and varying-numerator-plant, separately. The lemma of [11], on the other hand, streamlines the construction in [1] at the expense of using much more conservative controller pole and/or zero locations.

In other words, the small gains used in this paper are “smaller” than the gains used in [1] and, the large gains larger.

In order to make the main steps of the derivation as clear as possible, (i) we give the result under a more restrictive condition than that of [1], namely, we assume that high (and the low) frequency gains of the plants are all equal and (ii) we do not delve into the characterization of all controllers. Such extensions can be incorporated, [12].

2. Main Results

Let $\alpha(s), \beta(s)$ be two Hurwitz stable monic polynomials of degrees w and m , respectively.

Let

$$n_1(s) := \frac{1}{\alpha(s)}, \quad n_2(s) := \frac{s^m}{\beta(s)}$$

be two fixed elements of S denoting the set of proper, stable, rational functions of s with real coefficients. Let $D_i(s)$ for $i = 1, \dots, v$ be arbitrary nonsingular matrices over S . Consider that families of plants

$$\begin{aligned} F_1 &= \{D_i(s)^{-1}n_1(s) : i = 1, \dots, v\} \\ F_2 &= \{D_i(s)^{-1}n_2(s) : i = 1, \dots, v\} \\ F_3 &= \{D_i(s)^{-1}n_1(s)n_2(s) : i = 1, \dots, v\} \end{aligned}$$

where $v > 1$. Let us denote $D_i(\infty) := \lim_{s \rightarrow \infty} D_i(s)$. Thus, $D_i(s)$ is biproper if and only if $D_i(\infty)$ is nonsingular. We assume that the denominator matrices in F_1 are all biproper and the denomi-

nator matrices in F_2 are nonsingular at $s = 0$, i.e., $D_i(0)$ is nonsingular for all $i = 1, \dots, v$. In F_3 , the denominator matrices are assumed to be nonsingular at $s = 0$ if $m \geq 1$ and as $s \rightarrow \infty$ if $w \geq 1$. These assumptions ensure that the member plants are in coprime fractional representations over S .

Each member plant in F_1 has w zeros at infinity and no other unstable zeros. Each plant in F_2 has m zeros at $s = 0$ and no other unstable zeros. In both families, the plants can have varying (with i) stable zeros and varying stable as well as unstable poles. A member of the family F_3 , on the other hand, may zeros both at the origin and/or at infinity. A controller is said to stabilize a family if it stabilizes every plant in that family. In Proposition 1-3 below, we construct a stabilizing controller for each family.

The following lemma is a result by Smith and Sondergeld stated in [11] for scalar case and for stable controllers. The extension given below is straightforward.

Lemma 1. If a strictly transfer matrix $G(s)$ is stabilized by a controller transfer matrix $G_c(s)$, then for any $q \geq 1$, there exists a small enough $\varepsilon_0 > 0$, possibly depending on q , such that $(\varepsilon s + 1)^{-q} G(s)$ is also stabilized by $G_c(s)$ for all $\varepsilon \in (0, \varepsilon_0]$

Proof. Let $G = D^{-1}N$ be a left coprime representation of G over S , where N is strictly proper as G is. Let $Z_c = N_c D_c^{-1}$ be a right coprime representation of G_c over S . Since G is stabilized by G_c , $U := DD_c + NN_c$ is unimodular over S . Let

$$\delta := 1/\|U(s)\|_\infty.$$

By Lemma 19 of [2], any matrix V over S that satisfies $\|V(s) - U(s)\|_\infty < 1/\|U(s)\|_\infty$ is unimodular, it is hence enough to show that there exists $\varepsilon > 0$ for which

$$\|[(\varepsilon s + 1)^{-q} - 1]N(s)N_c(s)\|_\infty < \delta. \quad (1)$$

By strict properness of $H := NN_c$, there exists w_0 for which $\sup_{w \geq w_0} \sigma(H(jw)) < \frac{\delta}{2}$, where $\sigma(H(jw))$ is the largest singular value of $H(jw)$. It follows that, for any finite ε ,

$$\sup_{w \geq w_0} \sigma([(\varepsilon jw + 1)^{-q} - 1]H(jw)) < \delta. \quad (2)$$

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On the other hand, in the interval $[0, w_0]$, we have that $\max_{w \in [0, w_0]} |(ejw + 1)^{-q} - 1| < kw_0^q \varepsilon$ for some k that depends on q . Let $\mu := \max_{w \in [0, w_0]} \sigma(H(jw))$ so that for every choice of ε less than $\delta / \mu kw_0^q$, it holds that

$$\sup_{w \in [0, w_0]} \overline{\sigma} \left(|(ejw + 1)^{-q} - 1| H(jw) \right) < \delta. \quad (3)$$

By (2) and (3), the norm inequality (1) follows.

Proposition 1. If

$$D_i(\infty) = D_i(\infty), \nabla i = 2, \dots, v, \quad (4)$$

then a controller $C(s)$ stabilizing the family F_1 exists. If $w \geq 1$, then one such controller is given by

$$C_1(s) = D_1(\infty) \frac{\alpha(s)}{(\varepsilon s + 1)^w - 1},$$

for some small enough $\varepsilon > 0$.

Proof. First observe that for each $i = 1, \dots, v$, the transfer matrix $D_i(s)^{-1} [D_1(\infty) - D_i(s)]$ is strictly proper, it is a left coprime representation over S , and it is stabilized by the controller $G_c(s) = I$.

By **Lemma 1**, there exists $\varepsilon_i > 0$ such that each

$$U_i(s) := D_i(s) + \frac{1}{(\varepsilon s + 1)^w} [D_1(\infty) - D_i(s)]$$

is unimodular over S for all $\varepsilon \in (0, \varepsilon_i]$. Let $\varepsilon_0 := \min_i \{\varepsilon_i\}$. It follows that for all $\varepsilon \in (0, \varepsilon_0]$,

$$U_i(s) := D_i(s) \left(1 - \frac{1}{(\varepsilon s + 1)^w} \right) + n_1(s) D_1(\infty) \frac{\alpha(s)}{(\varepsilon s + 1)^w}$$

are all unimodular for $i = 1, \dots, v$. The controller $C_1(s) = \left[D_1(\infty) \frac{\alpha(s)}{(\varepsilon s + 1)^w} \right] \left[1 - \frac{1}{(\varepsilon s + 1)^w} \right]^{-1}$ hence stabilizes every plant in the family F_1 .

Lemma 2. Let $U(s)$ be left (right) unimodular over S . Then, $\hat{U}(s) := U(s^{-1})$ is also left (right) unimodular over S .

Proof. A matrix $U(s)$ over S is and if all its poles are in the open left half complex plane, $U(\infty)$ has full row rank, i.e., $U(s)$ is proper, and $U(\lambda)$ has full row rank for all λ in the closed and extended right half plane. Since $\hat{U}(0) := U(\infty)$, $\hat{U}(0) := U(\infty)$, and since for every finite and nonzero complex number λ , we have $\text{Re } \lambda \geq 0$ if and only if $\text{Re } \lambda^{-1} \geq 0$, the result follows.

The following is a dual version of Lemma 1 in which zeros at infinity are replaced by zeros at the origin.

Lemma 3. Let $G(s)$ be a proper rational matrix such that $G(s^{-1})$ is strictly proper. If $G(s)$ is stabilized by a controller rational matrix $G_c(s)$ such that $G_c(s^{-1})$ is proper, then for any $q \geq 1$, there exists a small enough ε_0 , possibly depending on q , such that $\frac{s^q}{(s+\varepsilon)^q}G(s)$ is also stabilized by G_c for all $\varepsilon \in (0, \varepsilon_0]$

Proof. Let $G(s^{-1}) = \hat{D}(s)^{-1}\hat{N}(s)$ be a left coprime representation over S . Note that $\hat{N}(s)$ is strictly proper and $\hat{D}(s)$ is biproper. Let $D(s) := \hat{D}(s^{-1})$ and $N(s) := \hat{N}(s^{-1})$. Then, $G(s) = D(s)^{-1}N(s)$, where $(D(s), N(s))$ is left coprime over S by Lemma 2. Let $G_c(s) = N_c(s)D_c(s)^{-1}$ be a right coprime representation. Since, G_c stabilizes G , we have that $DD_c + NN_c := U$ is unimodular over S . Substituting s^{-1} for s , we obtain $\hat{D}(s)\hat{D}_c(s) + \hat{N}(s)\hat{N}_c(s) = \hat{U}(s)$, where $\hat{D}_c(s) := D_c(s^{-1})$, $\hat{N}_c(s) := N_c(s^{-1})$, and $\hat{U}(s) := U(s^{-1})$. Note that $\hat{N}_c(s)\hat{D}_c(s)^{-1} = G_c(s^{-1})$ is proper. Also, by Lemma 2, $\hat{U}(s)$ is unimodular over S . By Lemma 1, given any $q \geq 1$, there exists $\varepsilon > 0$ such that $\hat{D}(s)\hat{D}_c(s) + (\varepsilon s + 1)^{-q}\hat{N}(s)\hat{N}_c(s) = \hat{V}(s)$ is unimodular. Substituting s^{-1} for s , we obtain $D(s)D_c(s) + N(s)N_c(s)\frac{s^q}{(s+\varepsilon)^q} = V(s)$, where $V(s) = \hat{V}(s^{-1})$ is unimodular by Lemma 2. The result follows.

Proposition 2. If

$$D_i(0) = D_1(0), \forall i = 2, \dots, \nu, \quad (5)$$

then a controller $C(s)$ stabilizing the family F_2 exists. If $m \geq 1$, then one such controller is given by

$$C_2(s) = D_1(0) \frac{\beta(s)}{(s+1)[(s+\varepsilon)^m - s^m]},$$

for some small enough $\varepsilon > 0$.

Proof. First observe that for each $i = 1, \dots, \nu$, $G(s) := [(s+1)D_i(s)]^{-1} [D_1(0) - (s+1)D_i(s)]$ is such that $G(s^{-1})$ is strictly proper and the pair $(s+1)D_i(s), D_1(0) - (s+1)D_i(s)$ is left coprime over S by nonsingularity of $D_1(0)$. The controller $G_c(s) = I$ clearly stabilizes $G(s)$. By Lemma 3, there exists $\varepsilon_i > 0$ such that each

$$U_i(s) := (s+1)D_i(s) + \frac{s^m}{(s+\varepsilon)^m} [D_1(0) - (s+1)D_i(s)]$$

is unimodular over S for all $\varepsilon \in (0, \varepsilon_i]$. It follows that, with $\varepsilon_0 := \min_i \{\varepsilon_i\}$, for all $\varepsilon \in (0, \varepsilon_0]$,

$$U_i(s) = D_i(s) \frac{(s+1)[(s+\varepsilon)^m - s^m]}{(s+\varepsilon)^m} + n_2(s)D_1(0) \frac{\beta(s)}{(s+\varepsilon)^m}$$

are all unimodular for $i = 1, \dots, \nu$. The controller

$$C_2(s) = D_1(0) \frac{\beta(s)}{(s+\varepsilon)^m} \left[\frac{(s+1)[(s+\varepsilon)^m - s^m]}{(s+\varepsilon)^m} \right]^{-1} \text{ hence stabilizes every plant in the family } F_2.$$

Note that the family $F_3 = \{D_i(s)^{-1}n_1(s)n_2(s) : i = 1, \dots, \nu\}$ is related to F_1 and F_2 in an obvious way. It would simplify things a great deal if we could employ the constructions of Propositions 1 and 2 in order to find a stabilizing controller for F_3 . The following "separation lemma" provides a mechanism for doing exactly this. Note that the stated result, in which one numerator may also vary and be non-scalar, is a bit more than what is needed.

Lemma 4. Given a matrix M over S and a family of plants $\{D_j^{-1}N_j : j = 1, \dots, \nu\}$ suppose there exist matrices N_{c1}, D_{c1} over S such that $D_j D_{c1} + N_j N_{c1} =: U_j$ are unimodular matrices for $j = 1, \dots, \nu$. Also suppose there exist matrices N_{c2}, D_{c2} over S such that $U_j^{-1}D_j D_{c2} + M N_{c2}$ are unimodular matrices for $j = 1, \dots, \nu$. If M and N_{c1} commute and $D_{c1} M N_{c2} + D_{c2}$ is nonsingular, then a stabilizing (possibly non-proper) controller for the family of plants of $\{D_j^{-1}N_j M : j = 1, \dots, \nu\}$ is given by $N_{c1} N_{c2} (D_{c1} M N_{c2} + D_{c2})^{-1}$.

Proof. Let $V_j := U_j^{-1}D_jD_{c2} + MN_{c2}$. Then, $(D_jD_{c1} + N_jN_{c1})MN_{c2} = U_jMN_{c2} = U_j(V_j - U_j^{-1}D_jD_{c2})$ so that $D_j(D_{c1}MN_{c2} + D_{c2}) + N_jN_{c1}MN_{c2} = U_jV_j$. Since $N_{c1}M = MN_{c1}$, we finally have $D_j(D_{c1}MN_{c2} + D_{c2}) + N_jMN_{c1}N_{c2} = U_jV_j$ proving the claim.

Observe in Lemma 4 that if MN_{c2} is strictly proper, then properness of the controllers stabilizing the “separated families” is enough for the properness of the resulting controller for the “united family”.

Proposition 3. If

$$D_i(\infty) = D_i(0), \quad D_i(0) = D_i(0), \quad \forall i = 2, \dots, v, \quad (6)$$

then a controller $C(s)$ stabilizing the family F_3 exists. If $w \geq 1$ and $m \geq 1$, then one such controller is given by

$$C_3(s) = \alpha(s)\beta(s) \left\{ s^m \left[(\varepsilon s + 1)^w - 1 \right] D_1(\infty)^{-1} + \left[(s + \varepsilon)^m - s^m \right] (\varepsilon s + 1)^w D_1(0)^{-1} \right\}^{-1}$$

for some small enough $\varepsilon > 0$.

Proof. Let

$$N_{c1}(s) = D_1(\infty) \frac{\alpha(s)}{(\varepsilon s + 1)^w}, \quad D_{c1}(s) = \frac{(\varepsilon s + 1)^w - 1}{(\varepsilon s + 1)^w} I,$$

so that, by Proposition 1, $N_{c1}D_{c1}^{-1}$ is a stabilizing controller for the family F_1 , for all ε sufficiently small, achieving unimodular $U_i := D_iD_{c1} + n_iN_{c1}$ for $i = 1, \dots, v$. Consider the family of plants

$$\bar{F}_2 = \left\{ D_i(s)^{-1}U_i(s)n_2(s) : i = 1, \dots, v \right\}$$

Note that $\bar{D}_i(s) := U_i(s)^{-1}D_i(s)$ is such that

$$\bar{D}_i(0) = U_i(0)^{-1}D_i(0) = [n_1(0)N_{c1}(0)]^{-1}D_i(0) = D_1(\infty)^{-1}D_i(0)$$

is independent of i . Following the construction in Proposition 2, where $(s+1)D_i(s)$ is replaced by $\bar{D}_i(s)$, for all ε sufficiently small,

$$N_{c2}(s) = D_1(\infty)^{-1}D_1(0) \frac{\beta(s)}{(s + \varepsilon)^m}, \quad D_{c2}(s) = \frac{(s + \varepsilon)^m - s^m}{(s + \varepsilon)^m} I$$

give a (non-proper) stabilizing controller $N_{c_2} D_{c_2}^{-1}$ for the family \bar{F}_2 . By Proposition 3,

$$N_{c_3}(s) = \frac{\alpha(s)}{(\varepsilon s + 1)^\omega} \frac{\beta(s)}{(s + \varepsilon)^m} D_1(0),$$

$$D_{c_3}(s) = \frac{(s + \varepsilon)^m - s^m}{(s + \varepsilon)^m} I + \frac{(\varepsilon s + 1)^\omega - 1}{(\varepsilon s + 1)^\omega} \frac{s^m}{(s + \varepsilon)^m} D_1(\infty)^{-1} D_1(0)$$

is such that $C_3(s) = N_{c_3}(s) D_{c_3}(s)^{-1}$ is proper stabilizing controller for F_3 .

3. Discussion

All there common controllers of Propositions 1-3 are high-gain controllers since for each finite s

$$\lim_{\varepsilon \rightarrow 0} C_i(s) \rightarrow \infty, i = 1, 2, 3.$$

Consider the case $D_i(\infty) = I$ and $D_i(0) = I$ in which the controllers have the simple expressions

$$C_1(s) = \frac{\alpha(s)}{(\varepsilon s + 1)^\omega - 1} I,$$

$$C_2(s) = \frac{\beta(s)}{(s + \varepsilon)^m [(s + \varepsilon)^m - s^m]} I,$$

$$C_3(s) = \frac{\alpha(s) \beta(s)}{s^m [(\varepsilon s + 1)^\omega - 1] + [(s + \varepsilon)^m - s^m] (\varepsilon s + 1)^\omega} I.$$

The controllers are all diagonal with the same diagonal entries. Moreover, (i) for any $\varepsilon > 0$, $C_1(s)$ is minimum-phase and stable except for a single pole at the origin. Its left hand plane set of poles is

$$\left\{ \frac{1}{\varepsilon} \left(-1 + e^{j \frac{2\pi k}{\omega}} \right) : k = 1, \dots, \omega - 1 \right\}.$$

(ii) For any $\varepsilon > 0$, $C_2(s)$ is stable and minimum phase, i.e., it is a unit of S. The set of poles of $C_2(s)$, other than one pole at -1 , is

$$\left\{ -\frac{\varepsilon}{2} \left(1 + e^{-j \frac{2\pi k}{\omega}} \right) : k = 1, \dots, \omega - 1 \right\}.$$

(iii) For sufficiently small $\varepsilon > 0$ $C_3(s)$ is also and minimum phase, [12].

Some further results are derived at no cost. Consider a new family consisting of the inverse plants of F_2 :

$$\hat{F}_2 := \{n_2(s)^{-1} D_i(s)^{-1} U_i(s) : i = 1, \dots, v\}$$

Member plants have all m fixed poles at the origin, no other unstable poles, and varying numerator matrices. Since the stabilizing controller $C_2(s)$ of F_2 is biproper, its inverse

$$\hat{C}_2(s) = D_1(0)^{-1} \frac{(s+1) \left[(s+\varepsilon)^m - s^m \right]}{\beta(s)}$$

is a stabilizing controller for the family \hat{F}_2 . This is of course a low-gain controller.

The restrictive assumptions of Propositions 1-3 on the frequency gains at infinity and at zero, i.e., $D_1(\infty)D_1(\infty)^{-1} = I$, $D_1(0)D_1(0)^{-1} = I$, are not necessary. (See the discussion in [3] on necessary conditions.) While these can be relaxed to

$$A_i := D_1(\infty)D_1(\infty)^{-1}, B_i := D_1(0)D_1(0)^{-1}, \quad \forall i = 2, \dots, v$$

are such that $-A_i - B_i$ are Hurwitz stable matrices, [12], any further relaxation seems to be difficult.

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