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Brief Paper

Simultaneously stabilizing controller design for a class of MIMO systems☆

A.N. Gündeş*, M.G. Kabuli

Department of Electrical and Computer Engineering, University of California, Davis, CA 95616, USA

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Abstract

It is shown that a class of linear, time-invariant, multi-input multi-output plants can be simultaneously stabilized. This class of plants all have the same number of zeros at infinity, at zero, or both, but no other zeros in the unstable region. If they have zeros at zero or infinity, then their gain matrices at zero and infinity also satisfy a positive-definiteness condition. There is no restriction on the poles of the plants considered in this class. An explicit design procedure is proposed to achieve simultaneously stabilizing controllers. All simultaneously stabilizing controllers for this class of plants are also characterized in terms of a parameter matrix that satisfies a unimodularity condition.

Keywords: Simultaneous stabilization

1. Introduction

Simultaneous stabilization of a set of linear time-invariant (LTI), single-input single-output (SISO) or multi-input multi-output (MIMO) plants is a challenging control problem. Controller design for a number of different plants is encountered in many applications, such as when considering a class of systems generated by a given nominal plant in different modes of operation, or when the actual plant is only known to belong to a finite set of plants, or when partial failures of sensors or actuators change the original plant description so drastically that the resulting systems cannot be described as small perturbations of the nominal plant.

In the case of two plants, the well-known parametrization of all stabilizing controllers leads to explicit necessary and sufficient conditions for existence of simultaneously stabilizing controllers (Vidyasagar, 1985). These simple conditions require that a pseudo-plant associated with the two given plants has the parity-interlacing-property (PIP), i.e., it has an even number of poles between consecutive pairs of real-axis zeros in the region of instability. However, there are no known necessary and sufficient conditions to check for existence of simultaneously stabilizing controllers for a completely general class of three or more plants. Although it is necessary for the plants to satisfy the PIP pairwise, this is not sufficient in the case of three or more plants. In fact, conditions restricted to checking the real-axis pole/zero locations are not sufficient to guarantee that a single controller can stabilize all of the plants simultaneously (Blondel, Gevers, Mortini, & Rupp, 1994) and the simultaneous stabilization problem for three or more plants is in general rationally undecidable (Blondel, 1994). Since necessary and sufficient conditions applicable to a completely general class of three or more different plants are not available, it is important to identify classes for which simultaneous stabilization is possible to achieve. Identifying such classes has proven to be a very difficult problem even in the SISO case. Some important sufficient conditions for simultaneous stabilizability have been considered in the literature. For example, a special class of uncertain SISO plants was considered in Barmish and Wei (1986), where it was shown that a class of SISO minimum-phase, strictly proper plants that have the same high-frequency gain sign are simultaneously

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* Corresponding author. Tel.: + 15307524989; fax: + 15307528428.
E-mail addresses: gundes@ece.ucdavis.edu (A.N. Gündeş), kabuli@ece.ucdavis.edu (M.G. Kabuli).

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stabilizable by a stable and strictly proper controller. This result was extended to discrete-time systems in Wei and Barmish (1988), and modified for MIMO systems in Wei (1993). Various special cases of sufficient conditions presented in the literature for the simultaneous stabilization of SISO plants were generalized in Bredemann (1995) (see also the references therein), where the simultaneous stabilizability conditions for two plants were formulated in terms of the difference of the plants, and conditions for simultaneous stabilizability were derived for SISO plants with zeros either at zero or at infinity (but not both). A sufficient condition for strong simultaneous stabilization of SISO systems that have the same relative degree and the same unstable zeros was presented in Abdallah, Dorato, and Bredemann (1997). It was shown in Gündeş and Kabuli (1999) that a class of MIMO plants that have no other unstable poles except at zero are simultaneously stabilizable and that the controller can be chosen stable and strictly proper. While conditions guaranteeing simultaneous stabilizability have been studied extensively, most of the previous results are applicable to SISO systems. Furthermore, most of these results derive existence conditions and do not tackle explicit controller design, which is a difficult problem even in the case of two plants.

In this paper, we consider the class $\mathscr{P} = \{P_0, P_1, \ldots, P_n\}$ of $n+1$ LTI, MIMO plants that have no other zeros in the region of instability except at infinity and/or zero. These plants all have w blocking-zeros at infinity and m blocking-zeros at zero, where w and m are non-negative integers. In addition, when $w \neq 0$, or $m \neq 0$, the high frequency gains or the low frequency gains of the plants in this class are related by positive-definite matrices (see Assumptions 2.1 for a complete description of the class). There are no restrictions on the poles of the plants considered here. In the special case of SISO plants ($\eta = 1$), Assumption 2.1(iii) is equivalent to $(P_0/P_j)(\infty) > 0$; Assumption 2.1(iv) is equivalent to $(P_0/P_j)(0) > 0$, for all $j \in \{0, 1, \ldots, n\}$. Note that more restricted classes of plants with zeros either at zero or at infinity (but not both) were considered in previous literature and controller design for these classes has proven to be a challenging task even in the SISO case. Although the class considered here includes finitely many MIMO plants as "centers", and proposes simultaneously stabilizing controller design that guarantees stabilization of these centers, "small" perturbations around these centers are also stabilized using the same controller as in standard robustness results. However, the finitely many plants in the class are not related to one another by any small-gain restrictions.

The main result of this paper (Proposition 2.3) gives a simple design procedure of constructing a simultaneously stabilizing controller C_0, called the nominal controller. This nominal controller C_0 is biproper; in fact, it has a stable inverse. All simultaneously stabilizing controllers are obtained from this nominal controller in terms of a stable controller-parameter that satisfies an additional unimodularity condition. Following the main result, we apply the design method of Proposition 2.3 to a class of MIMO plants in Example 2.5. We provide a proof of Proposition 2.3 in the appendix.

Due to the algebraic framework described in the following notation, the results apply to continuous-time as well as discrete-time systems. A continuous-time setting was assumed throughout for simplicity; in the discrete-time case, all evaluations and discussions involving poles and zeros at $s = 0$ should be interpreted at $z = 1$.

Notation. Let $\mathscr{U}$ be the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). The sets of real numbers, rational functions (with real coefficients), proper and strictly proper rational functions, proper rational functions that have no poles in the region of instability $\mathscr{U}$ are denoted by $\mathbb{R}, R, R_p, R_s, \mathscr{R}$. The set of matrices whose entries are in $\mathscr{R}$ is denoted by $\mathscr{M}(\mathscr{R})$; M is called stable iff $M \in \mathscr{M}(\mathscr{R})$; a square $M \in \mathscr{M}(\mathscr{R})$ is called unimodular iff $M^{-1} \in \mathscr{M}(\mathscr{R})$. For $M \in \mathscr{M}(\mathscr{R})$, the norm $\|\cdot\|$ is defined as $\|M\| = \sup_{s \in \partial\mathscr{U}} \bar{\sigma}(M(s))$; $\bar{\sigma}$ denotes the maximum singular value and $\partial\mathscr{U}$ denotes the boundary of $\mathscr{U}$. The product notation used with matrices $M_i \in \mathscr{M}(\mathscr{R})$ assumes an ascending order in the index, i.e., $\prod_{i=1}^{m} M_i = M_1 M_2 \cdots M_m$.

2. Main results

Consider the standard LTI, MIMO, unity-feedback system $\mathscr{S}(P_j, C)$, where $P_j : e_P \mapsto y$, $C : e \mapsto y_C$, $e = r - y$, $e_P = y_C + u$; $P_j \in R_p^{\eta \times \eta}$ and $C \in R_p^{\eta \times \eta}$ represent the transfer-functions of the plant and the controller, respectively. It is assumed that P_j and C have no hidden modes corresponding to eigenvalues in the region of instability $\mathscr{U}$.

2.1. Assumptions

The plant $P_j \in R_p^{\eta \times \eta}$ belongs to the class $\mathscr{P} := \{P_0, P_1, \ldots, P_n\}$. For $j \in \{0, 1, \ldots, n\}$, $P_j \in \mathscr{P}$ satisfies the following assumptions: (i) (normal) $\operatorname{rank} P_j = \eta$; (ii) let w and m be non-negative integers; P_j has w blocking-zeros at ∞ and m blocking-zeros at zero (i.e., $s^{w-1} P_j(\infty) = 0$, $s^w P_j(\infty) \neq 0$, $s^{-(m-1)} P_j(0) = 0$, $s^{-m} P_j(0) \neq 0$) but it has no other transmission-zeros in $\mathscr{U}$; (iii) when $w \neq 0$, $(s^w P_j)(\infty) \Delta_j = (s^w P_0)(\infty)$, for some symmetric positive-definite matrix $\Delta_j \in \mathbb{R}^{\eta \times \eta}$; (iv) when $m \neq 0$, $(s^{-m} P_j)(0) \Theta_j = (s^{-m} P_0)(0)$, for some symmetric positive-definite matrix $\Theta_j \in \mathbb{R}^{\eta \times \eta}$.

In the special case of SISO plants ($\eta = 1$), Assumption 2.1(iii) is equivalent to $(P_0/P_j)(\infty) > 0$; Assumption 2.1(iv) is equivalent to $(P_0/P_j)(0) > 0$, for all $j \in \{0, 1, \ldots, n\}$.

By Assumption 2.1, each $P_j \in R_p^{\eta \times \eta}$ in the class $\mathscr{P}$ has a left-coprime-factorization (LCF) $P_j = D_j^{-1}N$ given by

$$P_j = D_j^{-1}N =: D_j^{-1}\frac{s^m}{\prod_{i=1}^{w}(s+\alpha_i)\prod_{i=1}^{m}(s+\beta_i)}I, \tag{1}$$

where $D_j \in \mathscr{R}^{\eta\times\eta}$, $\det D_j(\infty) \neq 0$, $-\alpha_i \in \mathbb{R}\backslash\mathscr{U}$ for $i = 1,\ldots,w$, $-\beta_i \in \mathbb{R}\backslash\mathscr{U}$ for $i = 1,\ldots,m$ (i.e., for continuous-time systems, $\alpha_i > 0$, $\beta_i > 0$). Note that (1) the numerator (matrix) N is the same for every plant $P_j \in \mathscr{P}$ since the plants P_j all have the same blocking-zeros and no other transmission-zeros in $\mathscr{U}$; (2) $P_j = ND_j^{-1}$ is a right-coprime-factorization (RCF) since N commutes with D_j^{-1}; (3) if $m \neq 0$, then $\det D_j(0) \neq 0$, i.e., since P_j has blocking-zeros at $s = 0$, it does not have any poles at $s = 0$. Furthermore, by (1), the assumption $(s^w P_j)(\infty)\Delta_j = (s^w P_0)(\infty)$ is equivalent to $D_j(\infty)D_0^{-1}(\infty) = \Delta_j$, and the assumption $(s^{-m}P_j)(0)\Theta_j = (s^{-m}P_0)(0)$ is equivalent to $D_j(0)D_0^{-1}(0) = \Theta_j$.

2.2. Definitions

The system $\mathscr{S}(P_j, C)$ is said to be stable iff the transfer-function from (r,u) to (y, y_C) is stable. The controller C is said to be a stabilizing controller for the plant P_j (or C stabilizes P_j) iff $C \in \mathscr{M}(R_p)$ and the system $\mathscr{S}(P_j, C)$ is stable; C is said to simultaneously stabilize all $P_j \in \mathscr{P}$ iff the system $\mathscr{S}(P_j, C)$ is stable for all $P_j \in \mathscr{P}$.

Let $P = \tilde{D}^{-1}\tilde{N}$ be any LCF of $P \in \mathscr{M}(R_p)$ ($\tilde{D}$, $\tilde{N} \in \mathscr{M}(\mathscr{R})$, $\det \tilde{D}(\infty) \neq 0$). It is well-known that the controller $C \in \mathscr{M}(R_p)$ stabilizes the plant $P \in \mathscr{M}(R_p)$ if and only if $(\tilde{D}D_c + \tilde{N}N_c)$ is unimodular for any RCF $C = N_c D_c^{-1}$ (with D_c, $N_c \in \mathscr{M}(\mathscr{R})$, $\det D_c(\infty) \neq 0$) (Vidyasagar, 1985). In the problem studied here, for the particular RCF $P_j = D_j^{-1}N$ given in (1), the controller $C = N_c D_c^{-1}$ simultaneously stabilizes all $P_j \in \mathscr{P}$ if and only if

$$D_j D_c + NN_c \quad \text{is unimodular} \tag{2}$$

for all $j \in \{0,1,\ldots,n\}$. Let one of the plants $P_0 = D_0^{-1}N \in \mathscr{P}$ be called the nominal plant. For the left-coprime pair (D_0, N), there exist $\tilde{V}, \tilde{U} \in \mathscr{M}(\mathscr{R})$ such that $D_0\tilde{V} + N\tilde{U} = I$. All stabilizing controllers for P_0 are $C = (\tilde{U} + D_0 Q)(\tilde{V} - NQ)^{-1}$, where $Q \in \mathscr{M}(\mathscr{R})$ is such that $\det(\tilde{V} - NQ)(\infty) \neq 0$ (this condition guarantees that C is proper and it holds for all $Q \in \mathscr{M}(\mathscr{R})$ when $w \neq 0$, i.e., $P_j \in \mathscr{M}(R_s)$). In the case of two plants, i.e., $\mathscr{P} = \{P_0, P_1\}$, by (2), there exists a controller that simultaneously stabilizes P_0 and P_1 if and only if there exists a $Q \in \mathscr{M}(\mathscr{R})$ such that $D_1(\tilde{V} - NQ) + N(\tilde{U} + D_0 Q) = I + (D_1 - D_0)\tilde{V} + (D_0 - D_1)NQ$ is unimodular. From the parity-interlacing-property (PIP), such $Q \in \mathscr{M}(\mathscr{R})$ exists if and only if $\det[I + (D_1 - D_0)\tilde{V}(s_0)]$ has the same sign at all blocking-zeros $s_0 \in \mathscr{U}$ of $(D_0 - D_1)N$. In the case of three or more plants, i.e., $\mathscr{P} = \{P_0, P_1, \ldots, P_n\}$, $n > 1$, PIP between any pairs of plants is a necessary condition for existence of controllers that simultaneously stabilize all $P_j \in \mathscr{P}$; however, it is not sufficient. We now verify that P_0 and P_j for any $P_j \in \mathscr{P}$ actually satisfy the PIP under Assumptions 2.1: For $s_0 \in \mathscr{U}$ such that $(D_0 - D_j)(s_0) = 0$, we have $\det[I + (D_j - D_0)\tilde{V}(s_0)] = 1 > 0$. For $s_0 \in \mathscr{U}$ such that $N(s_0) = 0$ (i.e., $s_0 = \infty$ or $s_0 = 0$), we have $(D_0\tilde{V} + N\tilde{U})(s_0) = I = D_0(s_0)\tilde{V}(s_0)$ implies $\det[I + (D_j - D_0)\tilde{V}(s_0)] = \det(D_j\tilde{V})(s_0) = \det D_j(s_0)D_0^{-1}(s_0)$. By Assumption 2.1, since Δ_j and Θ_j are positive-definite, $\det D_j(\infty)D_0^{-1}(\infty) = \det \Delta_j > 0$, $\det D_j(0)D_0^{-1}(0) = \det \Theta_j > 0$, and hence, PIP holds.

Checking that the PIP holds pairwise for the plants $P_j \in \mathscr{P}$ only confirms that the necessary condition is satisfied. It does not guarantee existence of simultaneously stabilizing controllers. Furthermore, this check is still a long way from explicit construction of simultaneously stabilizing controllers. In Proposition 2.3, an explicit design procedure is given that achieves a simultaneously stabilizing controller for the class $\mathscr{P}$. In addition to defining one such controller explicitly, all controllers can also be characterized based on the nominal plant P_0. The selection of the nominal plant $P_0 \in \mathscr{P}$ is completely arbitrary. Four possible cases are considered in Proposition 2.3: (a) P_j has blocking-zeros at infinity and zero ($w > 0$, $m > 0$), (b) P_j has blocking-zeros only at infinity ($w > 0$, $m = 0$), (c) P_j has blocking-zeros only at zero ($w = 0$, $m > 0$), (d) P_j has no blocking-zeros at infinity and at zero ($w = 0$, $m = 0$). Since these plants have no other transmission-zeros in the region of instability $\mathscr{U}$, the last case corresponds to the class of minimum-phase plants. Existence of simultaneously stabilizing controllers for case (d) is rather obvious and is included here only for completeness.

2.3. Proposition

Let $P_j \in R_p^{\eta\times\eta}$, $P_j \in \mathscr{P}$ *satisfy Assumptions* 2.1, $j \in \{0,1,\ldots,n\}$, *i.e.*, $P_j = D_j^{-1}N$ *as in* (1).

(a) If $w > 0$, $m > 0$, let $k_1 \in \mathbb{R}$ satisfy (3); if $w > 1$, let $k_2 \in \mathbb{R}$ satisfy (4); if $w > 2$, let $k_v \in \mathbb{R}$, $v = 3,\ldots,w$, satisfy (5):

$$k_1 > \max_{j\in\{0,\ldots,n\}} \|s(\Delta_j - D_j D_0^{-1}(\infty))\|, \tag{3}$$

$$k_2 > \max_{j\in\{0,\ldots,n\}} \left\| s\left[I + D_j D_0^{-1}(\infty)\frac{s}{k_1}\right]^{-1}\right\|, \tag{4}$$

$$k_v > \max_{j\in\{0,\ldots,n\}} \left\| s\left(I + D_j D_0^{-1}(\infty)\sum_{i=1}^{v-1} s^i \prod_{\ell=1}^{i}\frac{1}{k_\ell}\right)^{-1} \times\left(I + D_j D_0^{-1}(\infty)\sum_{i=1}^{v-2} s^i \prod_{\ell=1}^{i}\frac{1}{k_\ell}\right)\right\|. \tag{5}$$

Define $K \in \mathbb{R}^{\eta \times \eta}$ as

$$K := D_0^{-1}(\infty) \sum_{i=1}^{w} s^i \prod_{\ell=1}^{i} \frac{1}{k_\ell}. \tag{6}$$

Let $f_1 \in \mathbb{R}, f_1 > 0$ satisfy (7); if $m > 1$, let $f_2 \in \mathbb{R}, f_2 > 0$ satisfy (8); if $m > 2$, let $f_v \in \mathbb{R}, f_v > 0$, $v = 3, \ldots, m$, satisfy (9):

$$f_1 < \min_{j \in \{0,\ldots,n\}} \left\| s^{-1} \left[\Theta_j \prod_{i=1}^{w} \frac{\alpha_i}{k_i} - (I + D_j K)^{-1} D_j D_0^{-1}(0) \times \prod_{i=1}^{w} \frac{(s+\alpha_i)}{k_i} \right] \right\|^{-1}, \tag{7}$$

$$f_2 < \min_{j \in \{0,\ldots,n\}} \left\| s^{-1} \left[I + D_j \left(K + D_0^{-1}(0) s^{-1} f_1 \times \prod_{i=1}^{w} \frac{(s+\alpha_i)}{k_i} \right) \right]^{-1} (I + D_j K) \right\|^{-1}, \tag{8}$$

$$f_v < \min_{j \in \{0,\ldots,n\}} \left\| s^{-1} \left[I + D_j \left(K + D_0^{-1}(0) \sum_{i=1}^{v-1} \frac{1}{s^i} \prod_{\ell=1}^{i} f_\ell \times \prod_{i=1}^{w} \frac{(s+\alpha_i)}{k_i} \right) \right]^{-1} \left[I + D_j \left(K + D_0^{-1}(0) \sum_{i=1}^{v-2} \frac{1}{s^i} \times \prod_{\ell=1}^{i} f_\ell \prod_{i=1}^{w} \frac{(s+\alpha_i)}{k_i} \right) \right] \right\|^{-1}. \tag{9}$$

A controller $C_0 \in R_p^{\eta \times \eta}$ that simultaneously stabilizes all $P_j \in \mathscr{P}$ is

$$C_0 = \left(\frac{s^m}{\prod_{i=1}^{w}(s+\alpha_i) \prod_{i=1}^{m}(s+\beta_i)} K + \frac{s^m}{\prod_{i=1}^{m}(s+\beta_i)} D_0^{-1}(0) \sum_{i=1}^{m} \frac{1}{s^i} \prod_{\ell=1}^{i} f_\ell \prod_{i=1}^{w} \frac{1}{k_i} \right)^{-1}. \tag{10}$$

Furthermore, all simultaneously stabilizing controllers C are

$$C = \left(C_0^{-1} - \frac{s^m}{\prod_{i=1}^{w}(s+\alpha_i) \prod_{i=1}^{m}(s+\beta_i)} Q \right)^{-1} (I + Q D_0), \tag{11}$$

for $j \in \{1, \ldots, n\}$, $Q \in \mathscr{R}^{\eta \times \eta}$ is such that G_j is unimodular:

$$G_j := I + (D_0 - D_j) \times \left(I + C_0^{-1} D_j \frac{\prod_{i=1}^{w}(s+\alpha_i) \prod_{i=1}^{m}(s+\beta_i)}{s^m} \right)^{-1} Q. \tag{12}$$

(b) If $w > 0$, $m = 0$, let k_v, $v = 1, \ldots, w$, satisfy (3), (4), (5). A controller $C_0 \in R_p^{\eta \times \eta}$ that simultaneously stabilizes all $P_j \in \mathscr{P}$ is

$$C_0 = \left(\left(\prod_{i=1}^{w} (s+\alpha_i) \right)^{-1} K \right)^{-1} = D_0(\infty) \left(\sum_{i=1}^{w} s^i \prod_{\ell=1}^{i} \frac{1}{k_\ell} \right)^{-1} \prod_{i=1}^{w} (s+\alpha_i). \tag{13}$$

Furthermore, all simultaneously stabilizing controllers C are

$$C = \left(C_0^{-1} - \frac{1}{\prod_{i=1}^{w} (s+\alpha_i)} Q \right)^{-1} (I + Q D_0), \tag{14}$$

for $j \in \{1, \ldots, n\}$, $Q \in \mathscr{R}^{\eta \times \eta}$ is such that $\tilde{G}_j$ is unimodular:

$$\tilde{G}_j := I + (D_0 - D_j) \left(I + C_0^{-1} D_j \prod_{i=1}^{w} (s+\alpha_i) \right)^{-1} Q. \tag{15}$$

(c) If $w = 0$, $m > 0$, choose $X \in \mathscr{R}^{\eta \times \eta}$ such that $\det X(\infty) \neq 0$. Let $f_1 \in \mathbb{R}, f_1 > 0$ satisfy (16); if $m > 1$, let $f_2 \in \mathbb{R}, f_2 > 0$ satisfy (17); if $m > 2$, let $f_v \in \mathbb{R}, f_v > 0$, $v = 3, \ldots, m$, satisfy (18):

$$f_1 < \min_{j \in \{0,\ldots,n\}} \| s^{-1} [\Theta_j - D_j D_0^{-1}(0)] - D_j D_0^{-1}(0) X \|^{-1}, \tag{16}$$

$$f_2 < \min_{j \in \{0,\ldots,n\}} \| s^{-1} [I + D_j D_0^{-1}(0)(I + sX) s^{-1} f_1]^{-1} \|^{-1}, \tag{17}$$

$$f_v < \min_{j \in \{0,\ldots,n\}} \left\| s^{-1} \left[I + D_j D_0^{-1}(0)(I + sX) \sum_{i=1}^{v-1} \frac{1}{s^i} \prod_{\ell=1}^{i} f_\ell \right]^{-1} \times \left[I + D_j D_0^{-1}(0)(I + sX) \sum_{i=1}^{v-2} \frac{1}{s^i} \prod_{\ell=1}^{i} f_\ell \right] \right\|^{-1}. \tag{18}$$

A controller $C_0 \in R_p^{\eta \times \eta}$ that simultaneously stabilizes all $P_j \in \mathscr{P}$ is

$$C_0 = \left(\frac{s^m}{\prod_{i=1}^{m}(s+\beta_i)} D_0^{-1}(0)(I + sX) \sum_{i=1}^{m} \frac{1}{s^i} \prod_{\ell=1}^{i} f_\ell \right)^{-1}. \tag{19}$$

Furthermore, all simultaneously stabilizing controllers C are

$$C = \left(C_0^{-1} - \frac{s^m}{\prod_{i=1}^{m}(s+\beta_i)} Q \right)^{-1} (I + Q D_0), \tag{20}$$

$Q \in \mathscr{R}^{\eta \times \eta}$ is such that $\det(f_1 D_0^{-1}(0) X - Q)(\infty) \neq 0$ and for $j \in \{1, \ldots, n\}$, $\hat{G}_j$ is unimodular:

$$\hat{G}_j := I + (D_0 - D_j) \left(I + C_0^{-1} D_j \frac{\prod_{i=1}^{m}(s+\beta_i)}{s^m} \right)^{-1} Q. \tag{21}$$

(d) If $m = 0$ and $w = 0$, let $A \in \mathscr{R}^{\eta \times \eta}$ be such that $\|A\| < \min_{j \in \{0,\ldots,n\}} \|D_j\|^{-1}$ and $\det A(\infty) \neq 0$. A controller $C_0 \in R_p^{\eta \times \eta}$ that simultaneously stabilizes all $P_j \in \mathscr{P}$ is $C_0 = A^{-1}$. Furthermore, all simultaneously stabilizing controllers are $C = (A - Q)^{-1}(I + Q D_0)$; $Q \in \mathscr{R}^{\eta \times \eta}$ satisfies $\det(A - Q)(\infty) \neq 0$ and $I + (D_0 - D_j)(I + A D_j)^{-1} Q$ is unimodular for $j \in \{1, \ldots, n\}$.

2.4. Comments

(i) In Proposition 2.3, the choices of $-\alpha_i \in \mathbb{R} \backslash \mathscr{U}$ for $i=1,\ldots,w$, and $-\beta_i \in \mathbb{R} \backslash \mathscr{U}$ for $i=1,\ldots,m$, are completely arbitrary (in the continuous-time case, $\alpha_i > 0$ and $\beta_i > 0$). However, these choices will determine some of the zeros of the designed controllers: When $w>0$, $m>0$, C_0 has zeros at $-\alpha_i$ and $-\beta_i$; when $w>0$, $m=0$, C_0 has zeros at $-\alpha_i$; when $w=0$, $m>0$, C_0 has zeros at $-\beta_i$.

(ii) In all of the four cases considered in Proposition 2.3, the explicitly defined nominal controller C_0 has a stable inverse although C_0 itself is not necessarily stable. The nominal controller is biproper since C_0 is proper and C_0^{-1} is stable. In case (d), the controller C_0 is stable if and only if $A \in \mathscr{M}(\mathscr{R})$ satisfying $\|A\| < \min_j \|D_j\|^{-1}$ is chosen unimodular. In case (b), where the plants have $w>0$ blocking-zeros at infinity but no zeros at zero, C_0 has a pole at $s=0$ (and possibly other $\mathscr{U}$-poles depending on the constants $k_1,\ldots,k_w$). In particular, when $w=1, C_0 = k_1 D_0(\infty) + (\alpha_1 k_1/s) D_0(\infty)$ is in the form of a proportional-plus-integral controller. This design with a pole at $s=0$ in case (b) provides integral-action in the closed-loop system. In case (c), where the plants have $m>0$ blocking-zeros at zero but no zeros at infinity, C_0 may have $\mathscr{U}$-poles depending on the choice of $X \in \mathscr{M}(\mathscr{R})$ and the constants $f_1,\ldots,f_m$. When $w=1$, if X is chosen constant nonsingular, where the eigenvalues of X^{-1} have positive real parts, then the corresponding $C_0 = f_1^{-1}(s+\beta_1)(sI + X^{-1})^{-1} X^{-1} D_0(0)$ is stable.

(iii) The choice of $Q=0$ obviously satisfies the unimodularity conditions on (12), (15), (21) and $I + (D_0 - D_j)(I + AD_j)^{-1} Q$ in Proposition 2.3(a)–(d). With $Q=0$, the controller C becomes the nominal controller C_0. Other simple choices satisfying these unimodularity conditions include choices for $Q \in \mathscr{M}(\mathscr{R})$ based on the small-gain approach as follows: A sufficient condition for G_j in (12), $\tilde{G}_j$ in (15), $\hat{G}_j$ in (15) unimodular is if $Q \in \mathscr{M}(\mathscr{R})$ satisfies

$$\|Q\| < \left\| (D_0 - D_j) \left(I + C_0^{-1} D_j \frac{\prod_{i=1}^{w}(s+\alpha_i)\prod_{i=1}^{m}(s+\beta_i)}{s^m} \right)^{-1} \right\|^{-1},$$

or

$$\|Q\| < \left\| (D_0 - D_j) \left(I + C_0^{-1} D_j \prod_{i=1}^{w} (s+\alpha_i) \right)^{-1} \right\|^{-1}$$

or

$$\|Q\| < \left\| (D_0 - D_j) \left(I + C_0^{-1} D_j \frac{\prod_{i=1}^{m}(s+\beta_i)}{s^m} \right)^{-1} \right\|^{-1},$$

respectively. A sufficient condition for $I + (D_0 - D_j)(I + AD_j)^{-1} Q$ unimodular is if $Q \in \mathscr{M}(\mathscr{R})$ satisfies $\|Q\| < \|(D_0 - D_j)(I + AD_j)^{-1}\|^{-1}$.

(iv) In Proposition 2.3(c), $X \in \mathscr{M}(\mathscr{R})$ is chosen biproper in order to guarantee that the designed nominal controller C_0 in (19) is proper. The additional condition $\det(f_1 D_0^{-1}(0) X - Q)(\infty) \neq 0$ on $Q \in \mathscr{M}(\mathscr{R})$ ensures that the controller C in (20) is proper. A sufficient condition to satisfy this constraint is to choose $Q \in \mathscr{M}(\mathscr{R})$ strictly proper.

2.5. Example

Consider the class $\mathscr{P} = \{P_0, P_1, P_0 H, P_1 F, F P_1\}$ of five 2×2 plants:

$$P_0 = \begin{bmatrix} \dfrac{s}{s^2+64} & \dfrac{-s^2}{(s^2-1)(s-3)} \\ \dfrac{s}{(s-2)(s+10)} & \dfrac{s}{(s+1)(s-2)} \end{bmatrix},$$

$$H = \begin{bmatrix} \dfrac{(s+4)(s+16)}{s^2+16} & \dfrac{s}{(s+1)(s+4)} \\ 0 & \dfrac{40(s-1)(s-3)}{(40s-5)(s-6)} \end{bmatrix},$$

$$P_1 = \begin{bmatrix} \dfrac{13s}{(s-2)(s-8)} & \dfrac{-s(s+4)}{(s^2-4)(s-8)} \\ \dfrac{s(17s+7)}{((s+2)^2+1)(s+1)} & \dfrac{s(5s+16)}{(s+1)(s-2)(s+5)} \end{bmatrix},$$

$$F = \mathrm{diag}\left[\frac{3(s-2)}{s-3}, \ \frac{3s+14}{s+7} \right].$$

For $j \in \{0,1,\ldots,4\}$, each $P_j \in \mathscr{P}$ satisfies Assumptions 2.1, with rank $P_j = 2$, $w=1$, $m=1$. The symmetric positive-definite matrices $\Delta_j = P_j^{-1}(\infty) P_0(\infty) \in \mathbb{R}^{2\times 2}$, $\Theta_j = P_j^{-1}(0) P_0(0) \in \mathbb{R}^{2\times 2}$ are:

$$\Delta_0 = I,$$

$$\Delta_1 = \frac{1}{41}\begin{bmatrix} 3 & -2 \\ -2 & 15 \end{bmatrix},$$

$$\Delta_2 = I, \quad \Delta_3 = \frac{1}{3}\Delta_1, \quad \Delta_4 = \Delta_3, \quad \Theta_0 = I,$$

$$\Theta_1 = \frac{1}{36}\begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix},$$

$$\Theta_2 = \frac{1}{4} I, \quad \Theta_3 = \frac{1}{2}\Theta_1, \quad \Theta_4 = \Theta_3.$$

Choosing $\alpha_1 = 5$, $\beta_1 = 10$, write each P_j in the form of (1); then $N = s/(s+5)(s+10)$. Since $w=1$, $m=1$, we only need to compute the constants k_1 and f_1 satisfying (3) and (7). Compute $\max_{j\in\{0,\ldots,4\}} \|s(\Delta_j - D_j D_0^{-1}(\infty))\| = 31.4075$; then choose $k_1 = 70$ satisfying (3). Compute $\min_{j\in\{0,\ldots,4\}} \| s^{-1}(\Theta_j 5/70 - (I + D_j D_0^{-1}(\infty) s/70)^{-1} D_j D_0^{-1}(0)(s+5)/70)\|^{-1} = 0.374301$; then choose $f_1 = 0.37$

satisfying (7). By (10), the nominal controller is

$$C_0=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix} =\left(\frac{s^2}{k_1(s+\alpha_1)(s+\beta_1)}D_0^{-1}(\infty)+\frac{f_1}{k_1(s+\beta_1)}D_0^{-1}(0)\right)^{-1},$$

where

$$C_{11}=\frac{70(s+5)(s+10)(s^2-9.25s-46.25)}{2s^4-9.8859s^3-52.1035s^2-26.7823s-66.8457},$$

$$C_{12}=\frac{70(s+5)(s+10)s^2}{2s^4-9.8859s^3-52.1035s^2-26.7823s-66.8457},$$

$$C_{21}=\frac{-70(s+5)(s+10)(s^2-0.925s-4.625)}{2s^4-9.8859s^3-52.1035s^2-26.7823s-66.8457},$$

$$C_{22}=\frac{70(s+5)(s+10)(s^2+0.2891s+1.4453)}{2s^4-9.8859s^3-52.1035s^2-26.7823s-66.8457}.$$

Since $C_0^{-1}\in\mathscr{M}(\mathscr{R})$, an RCF of C_0 is $C_0=I(C_0^{-1})^{-1}$. Using (2), it can be shown that C_0 simultaneously stabilizes all $P_j\in\mathscr{P}$ since $D_jC_0^{-1}+N=:\Psi_j$ is unimodular for $j\in\{0,1,\ldots,4\}$. By (11), all simultaneously stabilizing controllers are

$$C=\frac{(s+5)(s+10)}{s}\times\left(\frac{s}{70}D_0^{-1}(\infty)+\frac{0.37(s+5)}{70s}D_0^{-1}(0)-Q\right)^{-1}(I+QD_0),$$

where $Q\in\mathscr{R}^{2\times 2}$ satisfies G_j.

$$G_j=I+(D_0-D_j)\times\left[I+\left(\frac{s}{70}D_0^{-1}(\infty)+\frac{0.37(s+5)}{70s}D_0^{-1}(0)\right)D_j\right]^{-1}Q$$

is unimodular, $j\in\{1,\ldots,4\}$.

3. Conclusions

We considered simultaneous stabilization of a class of LTI, square MIMO plants that all have the same number of blocking-zeros at infinity and/or at zero; these plants have no other zeros in the region of instability, and satisfy positive-definiteness assumptions on the high frequency and the low frequency gain matrices as described in Assumptions 2.1. We proved that this class of plants are simultaneously stabilizable and proposed an explicit design method to find a nominal controller. We characterized all simultaneously stabilizing controllers based on this nominal controller by choosing a controller-parameter satisfying additional unimodularity conditions. The design method was illustrated by an example of a class of 2×2 MIMO plants that all have one blocking-zero at infinity and one blocking-zero at zero. As these results show, although necessary and sufficient conditions are not available to check simultaneous stabilizability of a completely general class of more than two plants, it may be possible to identify classes of three or more plants for which simultaneously stabilizing controllers exist and to design controllers explicitly for such classes.

Appendix

Proof of Proposition 2.3. By assumption, $P_j=D_j^{-1}N$ is an RCF of P_j, where

$$N:=\frac{s^m}{\prod_{i=1}^{w}(s+\alpha_i)\prod_{i=1}^{m}(s+\beta_i)}I.$$

Furthermore, $D_j(\infty)D_0^{-1}(\infty)=\Delta_j$, and $D_j(0)D_0^{-1}(0)=\Theta_j$.

(a) Choose any $k_1\in\mathbb{R}$ satisfying (3). Define

$$\begin{aligned}W_{1j}&:=\frac{k_1}{(s+\alpha_1)}I+D_jD_0^{-1}(\infty)\frac{s}{(s+\alpha_1)}\\&=(k_1I+D_jD_0^{-1}(\infty)s)(s\Delta_j+k_1I)^{-1}\frac{(s\Delta_j+k_1I)}{(s+\alpha_1)}\\&=[I-s(\Delta_j-D_jD_0^{-1}(\infty))(s\Delta_j+k_1I)^{-1}]\frac{(s\Delta_j+k_1I)}{(s+\alpha_1)},\end{aligned}\tag{22}$$

for $j\in\{0,\ldots,n\}$, $W_{1j}\in\mathscr{M}(\mathscr{R})$ is unimodular since k_1 satisfies (3), $\alpha_1>0$, Δ_j is symmetric, positive definite and $D_j(\infty)D_0^{-1}(\infty)=\Delta_j$ implies $s(\Delta_j-D_jD_0^{-1}(\infty))\in\mathscr{M}(\mathscr{R})$. If $w>1$, choose any $k_2\in\mathbb{R}$ satisfying (4), i.e., for $j\in\{0,\ldots,n\}$, let $k_2>\max_j\|s(I-W_{1j}^{-1}D_jD_0^{-1}(\infty)s/(s+\alpha_1))\|$ (note that by (22), $[I-W_{1j}^{-1}D_jD_0^{-1}(\infty)s/(s+\alpha_1)]=W_{1j}^{-1}[W_{1j}-D_jD_0^{-1}(\infty)s/(s+\alpha_1)]=W_{1j}^{-1}k_1/(s+\alpha_1)=[I+D_jD_0^{-1}(\infty)s/k_1]^{-1}$). Define

$$\begin{aligned}W_{2j}&:=\frac{k_2}{(s+\alpha_2)}I+W_{1j}^{-1}D_jD_0^{-1}(\infty)\frac{s}{(s+\alpha_1)}\frac{s}{(s+\alpha_2)}\\&=\left[I-s(I-W_{1j}^{-1}D_j\ D_0^{-1}(\infty)\frac{s}{(s+\alpha_1)})\frac{1}{(s+k_2)}\right]\\&\quad\times\frac{(s+k_2)}{(s+\alpha_2)}\end{aligned}\tag{23}$$

for $j\in\{0,\ldots,n\}$, $W_{2j}\in\mathscr{M}(\mathscr{R})$ is unimodular since k_2 satisfies (4), $\alpha_2>0$, and $[W_{1j}^{-1}D_jD_0^{-1}(\infty)s(s+\alpha_1)^{-1}](\infty)=I$ implies $s[I-W_{1j}^{-1}D_jD_0^{-1}(\infty)s(s+\alpha_1)^{-1}]\in\mathscr{M}(\mathscr{R})$. If $w>2$, continue similarly for $v=3,\ldots,w$ with k_v satisfying (5), i.e., when $v=w$, for $j\in\{0,\ldots,n\}$, k_w satisfies $k_w>\max_j\|s(I+D_jD_0^{-1}(\infty)\sum_{i=1}^{w-1}s^i\prod_{\ell=1}^{i}1/k_\ell)^{-1}(I+D_jD_0^{-1}(\infty)\sum_{i=1}^{w-2}s^i\prod_{\ell=1}^{i}1/k_\ell)\|=\max_j\|s(I-(\prod_{i=1}^{w-1}W_{ij})^{-1}D_jD_0^{-1}(\infty)s^{w-1}\prod_{i=1}^{w-1}(s+\alpha_i)^{-1})\|=\max_j\|W_{(w-1)j}^{-1}k_{w-1}$

$s(s+\alpha_{w-1})^{-1}\|$. Define

$$W_{wj} := \frac{k_w}{(s+\alpha_w)} I + \left(\prod_{i=1}^{w-1} W_{ij}\right)^{-1} D_j D_0^{-1}(\infty) s^w \prod_{i=1}^{w} (s+\alpha_i)^{-1}$$
$$= \left[I - s\left(I - \left(\prod_{i=1}^{w-1} W_{ij}\right)^{-1} D_j D_0^{-1}(\infty) s^{w-1} \prod_{i=1}^{w-1} (s+\alpha_i)^{-1}\right)(s+k_w)^{-1}\right] \frac{(s+k_w)}{(s+\alpha_w)}, \tag{24}$$

for $j \in \{0,\ldots,n\}$, $W_{wj} \in \mathscr{M}(\mathscr{R})$ is unimodular since k_w satisfies (5), $\alpha_w > 0$, and $[(\prod_{i=1}^{w-1} W_{ij})^{-1} D_j D_0^{-1}(\infty) s^{w-1} \prod_{i=1}^{w-1}(s+\alpha_i)^{-1}](\infty) = I$ implies $s[I - (\prod_{i=1}^{w-1} W_{ij})^{-1} D_j D_0^{-1}(\infty) s^{w-1} \prod_{i=1}^{w-1}(s+\alpha_i)^{-1}] \in \mathscr{M}(\mathscr{R})$. Since W_{1j}, W_{2j} are unimodular, $(W_{1j}W_{2j}) = [k_1 k_2 I + D_j D_0^{-1}(\infty)(sk_2 + s^2)](s+\alpha_1)^{-1}(s+\alpha_2)^{-1}$ is unimodular. Similarly, W_j is unimodular for $j \in \{0,\ldots,n\}$:

$$W_j := \prod_{i=1}^{w} W_{ij} = (I + D_j K) \prod_{i=1}^{w} \frac{k_i}{(s+\alpha_i)}. \tag{25}$$

For simplicity, define $\alpha := \prod_{i=1}^{w} \alpha_i / k_i$. If $m > 0$, choose any $f_1 \in \mathbb{R}$ satisfying (7). Define

$$M_{1j} := \frac{s}{(s+\beta_1)} I + W_j^{-1} D_j D_0^{-1}(0) \frac{f_1}{(s+\beta_1)}$$
$$= (sI + W_j^{-1} D_j D_0^{-1}(0) f_1)(sI + \alpha f_1 \Theta_j)^{-1} \frac{(sI + \alpha f_1 \Theta_j)}{(s+\beta_1)}$$
$$= [I - s^{-1}(\alpha \Theta_j - W_j^{-1} D_j D_0^{-1}(0)) f_1 s (sI + \alpha f_1 \Theta_j)^{-1}] \frac{(sI + \alpha f_1 \Theta_j)}{(s+\beta_1)}, \tag{26}$$

for $j \in \{0,\ldots,n\}$, $M_{1j} \in \mathscr{M}(\mathscr{R})$ is unimodular since f_1 satisfies (7), $\beta_1 > 0$, Θ_j is symmetric, positive definite, and $D_j(0) D_0^{-1}(0) = \Theta_j$ implies $s^{-1}(\Theta_j \alpha - W_j^{-1} D_j D_0^{-1}(0)) \in \mathscr{M}(\mathscr{R})$. If $m > 1$, choose any $f_2 \in \mathbb{R}$ satisfying (8), i.e., for $j \in \{0,\ldots,n\}$, let $f_2 < \min_j \| s^{-1}[I - M_{1j}^{-1} W_j^{-1} D_j D_0^{-1}(0) f_1 (s+\beta_1)^{-1}]\| = \min_j \| M_{1j}^{-1}\ (s+\beta_1)^{-1}\|$. Define

$$M_{2j} := \frac{s}{(s+\beta_2)} I + M_{1j}^{-1} W_j^{-1} D_j D_0^{-1}(0) \frac{f_1}{(s+\beta_1)} \frac{f_2}{(s+\beta_2)}$$
$$= \left[I - s^{-1}(I - M_{1j}^{-1} W_j^{-1} D_j\ D_0^{-1}(0) f_1 (s+\beta_1)^{-1}) \frac{f_2\ s}{(s+f_2)}\right] \frac{(s+f_2)}{(s+\beta_2)} \tag{27}$$

for $j \in \{0,\ldots,n\}$, $M_{2j} \in \mathscr{M}(\mathscr{R})$ is unimodular since f_2 satisfies (8), $\beta_2 > 0$, and $[M_{1j}^{-1} W_j^{-1} D_j D_0^{-1}(0) f_1 (s+\beta_1)^{-1}](0) = I$ implies $s^{-1}[I - M_{1j}^{-1} W_j^{-1} D_j D_0^{-1}(0) f_1 (s+\beta_1)^{-1}] \in \mathscr{M}(\mathscr{R})$. If $m > 2$, continue similarly for $v = 3,\ldots,m$, with f_v satisfying (9). Define

$$M_{mj} := \frac{s}{(s+\beta_m)} I + \left(\prod_{i=1}^{m-1} M_{ij}\right)^{-1} \times W_j^{-1} D_j D_0^{-1}(0) \prod_{i=1}^{m} \frac{f_i}{(s+\beta_i)}$$
$$= \left[I - s^{-1}\left(I - \left(\prod_{i=1}^{m-1} M_{ij}\right)^{-1} \times W_j^{-1} D_j D_0^{-1}(0) \prod_{i=1}^{m-1} \frac{f_i}{(s+\beta_i)}\right) \times \frac{f_m s}{(s+f_m)}\right] \frac{(s+f_m)}{(s+\beta_m)} \tag{28}$$

for $j \in \{0,\ldots,n\}$, $M_{mj} \in \mathscr{M}(\mathscr{R})$ is unimodular since f_m satisfies (9), $\beta_m > 0$, and $[(\prod_{i=1}^{m-1} M_{ij})^{-1} W_j^{-1} D_j D_0^{-1}(0) \prod_{i=1}^{m-1} f_i (s+\beta_i)^{-1}](0) = I$ implies $s^{-1}[I - (\prod_{i=1}^{m-1} M_{ij})^{-1} W_j^{-1} D_j D_0^{-1}(0) \prod_{i=1}^{m-1} f_i (s+\beta_i)^{-1}) \in \mathscr{M}(\mathscr{R})$. Since M_{1j}, M_{2j} are unimodular, $(M_{1j} M_{2j}) = (s^2 I + W_j^{-1} D_j D_0^{-1}(0)(f_1 s + f_1 f_2))(s+\beta_1)^{-1}(s+\beta_2)^{-1}$ is unimodular. Similarly, M_j is unimodular for $j \in \{0,\ldots,n\}$:

$$M_j := \prod_{i=1}^{m} M_{ij} = \left(I + W_j^{-1} D_j D_0^{-1}(0) \sum_{i=1}^{m} \frac{1}{s^i} \prod_{\ell=1}^{i} f_\ell \right) \frac{s^m}{\prod_{i=1}^{m}(s+\beta_i)}. \tag{29}$$

Define C_0 as in (10); $C_0^{-1} \in \mathscr{M}(\mathscr{R})$ since $(1/\prod_{i=1}^{w}(s+\alpha_i)) \sum_{i=1}^{w} s^i \in \mathscr{R}$ and $s^m \sum_{i=1}^{m} 1/s^i \in \mathscr{R}$. Note that an RCF $C_0 = N_{c0} D_{c0}^{-1}$ and an LCF $C_0 = \tilde{D}_{c0}^{-1} \tilde{N}_{c0}$ are $N_{c0} = \tilde{N}_{c0} = I$, $D_{c0} = C_0^{-1} = \tilde{D}_{c0}$ since $C_0^{-1} \in \mathscr{M}(\mathscr{R})$. Since W_j, M_j are unimodular, and $k_i \neq 0$, $i = 1,\ldots,w$, the product $(W_j M_j)$ and $\Psi_j, \tilde{\Psi}_j$ are also unimodular:

$$W_j M_j = (I + D_j K) \frac{s^m}{\prod_{i=1}^{w}(s+\alpha_i) \prod_{i=1}^{m}(s+\beta_i)} \prod_{i=1}^{w} k_i + \frac{s^m}{\prod_{i=1}^{m}(s+\beta_i)} D_j D_0^{-1}(0) \sum_{i=1}^{w} \frac{1}{s^i} \prod_{\ell=1}^{i} f_\ell$$
$$= (D_j C_0^{-1} + N) \prod_{i=1}^{w} k_i =: \Psi_j \prod_{i=1}^{w} k_i, \tag{30}$$

$$\Psi_j := D_j C_0^{-1} + N, \quad \tilde{\Psi}_j := C_0^{-1} D_j + N. \tag{31}$$

Note that $C_0^{-1} \Psi_j^{-1} = \tilde{\Psi}_j^{-1} C_0^{-1}$, $\Psi_j^{-1} D_0 = D_0 \tilde{\Psi}_j^{-1}$. By (31), we have the following important identity:

$$\begin{bmatrix} C_0^{-1} & I \\ -N & D_0 \end{bmatrix} \begin{bmatrix} D_0 \tilde{\Psi}_0^{-1} & -\Psi_0^{-1} \\ N \tilde{\Psi}_0^{-1} & C_0^{-1} \Psi_0^{-1} \end{bmatrix} = I. \tag{32}$$

In (32), $P_0 = (N \tilde{\Psi}_0^{-1})(D_0 \tilde{\Psi}_0^{-1})^{-1} = (I - C_0^{-1} \Psi_0^{-1} D_0)(\Psi_j^{-1} D_0)^{-1}$ is another RCF of $P_0 = N D_0^{-1}$. By (32), the controller C stabilizes P_0 if and only if C is given by (11), where $Q \in \mathscr{M}(\mathscr{R})$; C becomes C_0 given in (10) when $Q = 0$. An RCF for C can also be obtained from (32) as

$C =: N_c D_c^{-1} = (\Psi_0^{-1}(I + D_0 Q))(\tilde{\Psi}_0^{-1}(C_0^{-1} - NQ))^{-1}$. The controller C is proper if and only if $\det(C_0^{-1} - QN)(\infty) = \det C_0^{-1}(\infty) \neq 0$, which follows by construction since $\det C_0(\infty) = \det D_0(\infty) \prod_{i=1}^{w} k_i \neq 0$. The controller C stabilizes all $P_j = D_j^{-1} N \in \mathscr{P}$ in addition to $P_0 = D_0^{-1} N$ if and only if $Q \in \mathscr{M}(\mathscr{R})$ is such that $D_j D_c + N N_c = I + (D_j - D_0) D_c = I + (D_j - D_0)[C_0^{-1}\Psi_0^{-1} - N\tilde{\Psi}_0^{-1} Q]$ is unimodular. But since N is diagonal, $N\tilde{\Psi}_0^{-1} = \tilde{\Psi}_0^{-1} N$; since $C_0^{-1}\Psi_0^{-1} = \tilde{\Psi}_0^{-1} C_0^{-1}$, $D_0 \tilde{\Psi}_0^{-1} = \Psi_0^{-1} D_0$, using the definitions of Ψ_0, Ψ_j, $\tilde{\Psi}_0$, $\tilde{\Psi}_j$ given by (31), it follows that this last unimodularity is equivalent to $I + (D_j - D_0)[\tilde{\Psi}_0^{-1}(C_0^{-1} - NQ)] = \Psi_j \Psi_0^{-1}[I + D_0 Q - \Psi_0 \Psi_j^{-1} D_j Q] = \Psi_j \Psi_0^{-1}[I + D_0(I - C_0^{-1} D_j \tilde{\Psi}_j^{-1})Q - N D_j \tilde{\Psi}_j^{-1} Q] = \Psi_j \Psi_0^{-1}[I + (D_0 - D_j) N \tilde{\Psi}_j^{-1} Q] = \Psi_j \Psi_0^{-1} G_j$ is unimodular, equivalently, G_j in (12) is unimodular.

(b) If $w > 0$ but $m = 0$, then $N = (\prod_{i=1}^{w}(s + \alpha_i))^{-1} I$. The proof is as in (a) above, with $M_j = I$.

(c) If $w = 0$ and $m > 0$, then $N = s^m(\prod_{i=1}^{m}(s + \beta_i))^{-1} I$. Choose any biproper $X \in \mathscr{M}(\mathscr{R})$ and choose any $f_1 \in \mathbb{R}$ satisfying (7). Define

$$M_{1j} := \frac{s}{(s+\beta_1)} I + D_j D_0^{-1}(0)(I + sX)\frac{f_1}{(s+\beta_1)}$$
$$= [I - (s^{-1}[\Theta_j - D_j D_0^{-1}(0)] - D_j D_0^{-1}(0) X)\, f_1 s (sI + f_1 \Theta_j)^{-1}]\frac{(sI + f_1\Theta_j)}{(s+\beta_1)}, \tag{33}$$

for $j \in \{0, \ldots, n\}$, $M_{1j} \in \mathscr{M}(\mathscr{R})$ is unimodular since f_1 satisfies (16), $\beta_1 > 0$, Θ_j is symmetric, positive definite, and $s^{-1}(\Theta_j - D_j D_0^{-1}(0)) \in \mathscr{M}(\mathscr{R})$. If $m > 1$, choose any $f_2 \in \mathbb{R}$ satisfying (17), i.e., for $j \in \{0, \ldots, n\}$, let $f_2 < \min_j \| s^{-1}[I - M_{1j}^{-1} D_j D_0^{-1}(0)(I + sX) f_1 (s+\beta_1)^{-1}] \| = \min_j \| M_{1j}^{-1}(s+\beta_1)^{-1} \|$. Define

$$M_{2j} := \frac{s}{(s+\beta_2)} I + M_{1j}^{-1} D_j D_0^{-1}(0)(I + sX)\frac{f_1}{(s+\beta_1)}\frac{f_2}{(s+\beta_2)}$$
$$= \left[I - s^{-1}(I - M_{1j}^{-1} D_j D_0^{-1}(0)(I + sX) f_1 (s+\beta_1)^{-1}) \frac{f_2 s}{(s+f_2)} \right] \frac{(s+f_2)}{(s+\beta_2)}, \tag{34}$$

for $j \in \{0, \ldots, n\}$, $M_{2j} \in \mathscr{M}(\mathscr{R})$ is unimodular since f_2 satisfies (17), $\beta_2 > 0$, and $[M_{1j}^{-1} D_0^{-1}(0)(I + sX) f_1 (s+\beta_1)^{-1}](0) = I$ implies $s^{-1}[I - M_{1j}^{-1} D_j D_0^{-1}(0)(I + sX) f_1 (s+\beta_1)^{-1}] \in \mathscr{M}(\mathscr{R})$. If $m > 2$, continue similarly for $v = 3, \ldots, m$, with f_v satisfying (9). Define

$$M_{mj} := \frac{s}{(s+\beta_m)} I + \left(\prod_{i=1}^{m-1} M_{ij}\right)^{-1} D_j D_0^{-1}(0)(I + sX)\prod_{i=1}^{m}\frac{f_i}{(s+\beta_i)}$$
$$= \left[I - s^{-1}\left(I - \left(\prod_{i=1}^{m-1} M_{ij}\right)^{-1} D_j D_0^{-1}(0)(I + sX) \times \prod_{i=1}^{m-1}\frac{f_i}{(s+\beta_i)}\right)\frac{f_m s}{(s+f_m)}\right]\frac{(s+f_m)}{(s+\beta_m)}, \tag{35}$$

for $j \in \{0, \ldots, n\}$, $M_{mj} \in \mathscr{M}(\mathscr{R})$ is unimodular since f_m satisfies (18), $\beta_m > 0$, and $[(\prod_{i=1}^{m-1} M_{ij})^{-1} D_j D_0^{-1}(0)(I + sX)\prod_{i=1}^{m-1} f_i (s+\beta_i)^{-1}](0) = I$ implies $s^{-1}[I - (\prod_{i=1}^{m-1} M_{ij})^{-1} D_j D_0^{-1}(0)(I + sX)\prod_{i=1}^{m-1} f_i (s+\beta_i)^{-1}] \in \mathscr{M}(\mathscr{R})$. Since M_{1j}, M_{2j} are unimodular, $(M_{1j} M_{2j}) = [s^2 I + D_j D_0^{-1}(0)(I + sX)(f_1 s + f_1 f_2)](s+\beta_1)^{-1}(s+\beta_2)^{-1}$ is unimodular. Similarly, $M_j := \prod_{i=1}^{m} M_{ij} = [I + D_j D_0^{-1}(0)(I + sX)\sum_{i=1}^{w}(1/s^i)\prod_{\ell=1}^{i} f_\ell] s^m / \prod_{i=1}^{m}(s+\beta_i) = D_j C_0^{-1} + N$ is unimodular for $j \in \{0, \ldots, n\}$. Define C_0 as in (19); $C_0^{-1} \in \mathscr{M}(\mathscr{R})$ since $(s^m/\prod_{i=1}^{m}(s+\beta_i))(I + sX)\sum_{i=1}^{m}\frac{1}{s^i} \in \mathscr{R}$. Since M_j is unimodular, $\tilde{M}_j := N + C_0^{-1} D_j$ is also unimodular. Writing an identity similar to (32), the rest of the proof follows similar steps as in part (a) where Ψ and $\tilde{\Psi}_j$ are replaced by M_j and $\tilde{M}_j$, In this case, the controller C is proper if and only if $\det(C_0^{-1} - Q)(\infty) = \det(f_1 D_0^{-1}(0) X(\infty) - Q(\infty)) \neq 0$.

(d) If $m = 0$, $w = 0$, then $N = I$, $P_j = D_j^{-1}$. Choose any $A \in \mathscr{M}(\mathscr{R})$ such that $\|A\| < \min_j \|D_j\|^{-1}$ and $\det A(\infty) \neq 0$; then $\Psi_j := I + D_j A$ and $\tilde{\Psi}_j := I + A D_j$ are unimodular for $j \in \{0, \ldots, n\}$. Define $C_0 := A^{-1}$; then C_0 is proper. The proof then follows from an identity similar to (32). □

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