Simultaneously stabilizing controller design for a class of MIMO systems

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Abstract

It is shown that a class of linear, time-invariant, multi-input multi-output plants can be simultaneously stabilized. This class of plants all have the same number of zeros at infinity, at zero, or both, but no other zeros in the unstable region. If they have zeros at zero or infinity, then their gain matrices at zero and infinity also satisfy a positive-definiteness condition. There is no restriction on the poles of the plants considered in this class. An explicit design procedure is proposed to achieve simultaneously stabilizing controllers. All simultaneously stabilizing controllers for this class of plants are also characterized in terms of a parameter matrix that satisfies a unimodularity condition. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Simultaneous stabilization of a set of linear time-invariant (LTI), single-input single-output (SISO) or multi-input multi-output (MIMO) plants is a challenging control problem. Controller design for a number of different plants is encountered in many applications, such as when considering a class of systems generated by a given nominal plant in different modes of operation, or when the actual plant is only known to belong to a finite set of plants, or when partial failures of sensors or actuators change the original plant description so drastically that the resulting systems cannot be described as small perturbations of the nominal plant.

In the case of two plants, the well-known parametrization of all stabilizing controllers leads to explicit necessary and sufficient conditions for existence of simultaneously stabilizing controllers (Vidyasagar, 1985). These simple conditions require that a pseudo-plant associated with the two given plants has the parity-interlacing-property (PIP), i.e., it has an even number of poles between consecutive pairs of real-axis zeros in the region of instability. However, there are no known necessary and sufficient conditions to check for existence of simultaneously stabilizing controllers for a completely general class of three or more plants.

Although it is necessary for the plants to satisfy the PIP pairwise, this is not sufficient in the case of three or more plants. In fact, conditions restricted to checking the real-axis pole/zero locations are not sufficient to guarantee that a single controller can stabilize all of the plants simultaneously (Blondel, Gevers, Mortini, & Rupp, 1994) and the simultaneous stabilization problem for three or more plants is in general rationally undecidable (Blondel, 1994). Since necessary and sufficient conditions applicable to a completely general class of three or more different plants are not available, it is important to identify classes for which simultaneous stabilization is possible to achieve. Identifying such classes has proven to be a very difficult problem even in the SISO case. Some important sufficient conditions for simultaneous stabilizability have been considered in the literature. For example, a special class of uncertain SISO plants was considered in Barmish and Wei (1986), where it was shown that a class of SISO minimum-phase, strictly proper plants that have the same high-frequency gain sign are simultaneously...
stabilizable by a stable and strictly proper controller. This result was extended to discrete-time systems in Wei and Barmish (1988), and modified for MIMO systems in Wei (1993). Various special cases of sufficient conditions presented in the literature for the simultaneous stabilization of SISO plants were generalized in Bredemann (1995) (see also the references therein), where the simultaneous stabilizability conditions for two plants were formulated in terms of the difference of the plants, and conditions for simultaneous stabilizability were derived for SISO plants with zeros either at zero or at infinity (but not both). A sufficient condition for strong simultaneous stabilizability of SISO systems that have the same relative degree and the same unstable zeros was presented in Abdallah, Dorato, and Bredemann (1997). It was shown in Gündüz and Kabuli (1999) that a class of MIMO plants that have no other unstable poles except at zero are simultaneously stabilizable and that the controller can be chosen stable and strictly proper. While conditions guaranteeing simultaneous stabilizability have been studied extensively, most of the previous results are applicable to SISO systems. Furthermore, most of these results derive existence conditions and do not tackle explicit controller design, which is a difficult problem even in the case of two plants.

In this paper, we consider the class \( \mathcal{P} = \{P_0, P_1, \ldots, P_n\} \) of \( n + 1 \) LTI, MIMO plants that have no other zeros in the region of instability except at infinity and/or zero. These plants all have \( w \) blocking-zeros at infinity and \( m \) blocking-zeros at zero, where \( w \) and \( m \) are non-negative integers. In addition, when \( w \neq 0 \), or \( m \neq 0 \), the high frequency gains or the low frequency gains of the plants in this class are related by positive-definite matrices (see Assumptions 2.1 for a complete description of the class). There are no restrictions on the poles of the plants considered here. In the special case of SISO plants \((\eta=1)\), Assumption 2.1(iii) is equivalent to \((P_0/P_1)(\infty) > 0\); Assumption 2.1(iv) is equivalent to \((P_0/P_1)(0) > 0\), for all \( j \in \{0, 1, \ldots, n\}\). Note that more restricted classes of plants with zeros either at zero or at infinity (but not both) were considered in previous literature and controller design for these classes has proven to be a challenging task even in the SISO case. Although the class considered here includes finitely many MIMO plants as “centers”, and proposes simultaneously stabilizing controller design that guarantees stabilization of these centers, “small” perturbations around these centers are also stabilized using the same controller as in standard robustness results. However, the finitely many plants in the class are not related to one another by any small-gain restrictions.

The main result of this paper (Proposition 2.3) gives a simple design procedure of constructing a simultaneously stabilizing controller \( C_0 \), called the nominal controller. This nominal controller \( C_0 \) is biproper; in fact, it has a stable inverse. All simultaneously stabilizing controllers are obtained from this nominal controller in terms of a stable controller-parameter that satisfies an additional unimodularity condition. Following the main result, we apply the design method of Proposition 2.3 to a class of MIMO plants in Example 2.5. We provide a proof of Proposition 2.3 in the appendix.

Due to the algebraic framework described in the following notation, the results apply to continuous-time as well as discrete-time systems. A continuous-time setting was assumed throughout for simplicity; in the discrete-time case, all evaluations and discussions involving poles and zeros at \( s = 0 \) should be interpreted at \( z = 1 \).

**Notation.** Let \( \mathcal{U} \) be the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). The sets of real numbers, rational functions (with real coefficients), proper and strictly proper rational functions, proper rational functions that have no poles in the region of instability \( \mathcal{U} \) are denoted by \( \mathbb{R}, \mathbb{R}_p, \mathbb{R}_p^s \). The set of matrices whose entries are in \( \mathcal{U} \) is denoted by \( \mathcal{M}(\mathcal{U}) \); \( M \) is called stable if \( M \in \mathcal{M}(\mathcal{U}) \); a square \( M \in \mathcal{M}(\mathcal{U}) \) is called unimodular if \( M^{-1} \in \mathcal{M}(\mathcal{U}) \). For \( M \in \mathcal{M}(\mathcal{U}) \), the norm \( ||M|| \) is defined as \( ||M|| = \sup_{x \in \mathcal{U}} \sigma(M(s)); \sigma \) denotes the maximum singular value and \( \partial \mathcal{U} \) denotes the boundary of \( \mathcal{U} \). The product notation used with matrices \( M_1, M_2 \in \mathcal{M}(\mathcal{U}) \) assumes an ascending order in the index, i.e., \( \prod_{i=1}^{n} M_i = M_1 M_2 \cdots M_n \).

## 2. Main results

Consider the standard LTI, MIMO, unity-feedback system \( \mathcal{G}(P_j, C) \), where \( P_j : \text{e}^{\text{p}t} \rightarrow \text{y}, \ C : \text{e}^{-t} \rightarrow \text{y}, \ e = \text{r} - y, \ e_p = y_c + w; \ P_j \in \mathbb{R}_p^{m \times n} \) and \( C \in \mathbb{R}_p^{n \times n} \) represent the transfer-functions of the plant and the controller, respectively. It is assumed that \( P_j \) and \( C \) have no hidden modes corresponding to eigenvalues in the region of instability \( \mathcal{U} \).

### 2.1. Assumptions

The plant \( P_j \in \mathbb{R}_p^{m \times s} \) belongs to the class \( \mathcal{P} := \{P_0, P_1, \ldots, P_n\} \). For \( j \in \{0, 1, \ldots, n\} \), \( P_j \in \mathcal{P} \) satisfies the following assumptions: (i) (normal) rank \( P_j = n \); (ii) let \( w \) and \( m \) be non-negative integers; \( P_j \) has \( w \) blocking-zeros at \( \infty \) and \( m \) blocking-zeros at zero (i.e., \( s^{w-1}P_j(\infty) = 0, s^mP_j(\infty) \neq 0 \), \( s^{-1}P_j(0) = 0, s^{-n}P_j(0) \neq 0 \)) but it has no other transmission-zeros in \( \mathcal{U} \); (iii) when \( w \neq 0, (s^wP_j)(\infty) \neq 0 \) for some symmetric positive-definite matrix \( A_j \in \mathbb{R}_p^{m \times s} \); (iv) when \( m \neq 0, (s^nP_j)(0) \neq 0 \) for some symmetric positive-definite matrix \( A_j \in \mathbb{R}_p^{n \times n} \).

In the special case of SISO plants \((\eta=1)\), Assumption 2.1(iii) is equivalent to \((P_0/P_1)(\infty) > 0\); Assumption 2.1(iv) is equivalent to \((P_0/P_1)(0) > 0\), for all \( j \in \{0, 1, \ldots, n\} \).
By Assumption 2.1, each \( P_j \in R_p^{n \times n} \) in the class \( \mathcal{P} \) has a left-co-prime-factorization (LCF) \( P_j = D_j^{-1}N_j \) given by

\[
P_j = D_j^{-1}N_j = \frac{s^m}{\prod_{k=1}^{s^m} (s + \alpha_k)} \prod_{k=1}^{s^m} (s + \beta_k) I,
\]

where \( D_j \in \mathbb{R}^{n \times n} \), det \( D_j(\infty) \neq 0 \), \( -\alpha_k \in \mathbb{R} \backslash \mathbb{U} \) for \( i = 1, \ldots, m \), \( -\beta_k \in \mathbb{R} \backslash \mathbb{U} \) for \( i = 1, \ldots, m \) (i.e., for continuous-time systems, \( \alpha_k > 0 \), \( \beta_k > 0 \)). Note that (1) the numerator (matrix) \( N \) is the same for every plant \( P_j \in \mathcal{P} \) since the plants \( P_j \) all have the same blocking-zeros and no other transmission-zeros in \( \mathbb{U} \); (2) \( P_j = ND_j^{-1} \) is a right-co-prime-factorization (RCF) since \( N \) commutes with \( D_j^{-1} \); (3) if \( m \neq 0 \), then \( \text{det}(D_j(0)) \neq 0 \), i.e., since \( P_j \) has blocking-zeros at \( s = 0 \), it does not have any poles at \( s = 0 \). Furthermore, by (1), the assumption \((s^m P_j)(\infty)A_j = (s^m P_0)(\infty)\) is equivalent to \( D_j(\infty)D_j^{-1}(\infty) = A_j \), and the assumption \((s^{-m} P_j)(0)\theta_j = (s^{-m} P_0)(0)\) is equivalent to \( D_j(0)D_j^{-1}(0) = \theta_j \).

### 2.2. Definitions

The system \( \mathbb{S}(P_j, C) \) is said to be stable iff the transfer-function from \((r, u)\) to \((y, y_c)\) is stable. The controller \( C \) is said to be a stabilizing controller for the plant \( P_j \) (or \( C \) stabilizes \( P_j \)) iff \( C \in \mathcal{M}(R_j) \) and the system \( \mathbb{S}(P_j, C) \) is stable; \( C \) is said to simultaneously stabilize all \( P_j \in \mathcal{P} \) if the system \( \mathbb{S}(P_j, C) \) is stable for all \( P_j \in \mathcal{P} \).

Let \( P = \hat{D}_1^{-1} \hat{N} \) be any LCF of \( P \in \mathcal{M}(R_p) \) (\( \hat{D}, \hat{N} \in \mathcal{M}(R) \), det \( \hat{D}(\infty) \neq 0 \)). It is well-known that the controller \( C \in \mathcal{M}(R_p) \) stabilizes the plant \( P \in \mathcal{M}(R_p) \) if and only if \( (\hat{D} \hat{D}_c + \hat{N} N_c) \) is unimodular for any RCF \( C = N_c \hat{D}_c^{-1} \) (with \( D_c, N_c \in \mathcal{M}(R) \), det \( D_c(\infty) \neq 0 \)) (Vidyasagar, 1985). In the problem studied here, for the particular RCF \( P_j = D_j^{-1}N_j \) given in (1), the controller \( C = N_c D_c^{-1} \) simultaneously stabilizes all \( P_j \in \mathcal{P} \) if and only if

\[
D_j D_c + N_c N_j \quad \text{is unimodular}
\]

for all \( j \in \{0, 1, \ldots, n\} \). Let one of the plants \( P_0 = D_0^{-1}N_0 \in \mathcal{P} \) be called the nominal plant. For the left-co-prime pair \( (D_0, N) \), there exist \( \hat{P}, \hat{U} \in \mathcal{M}(R) \) such that \( \hat{D}_0 \hat{P} + \hat{N} \hat{U} = I \). All stabilizing controllers for \( P \) are \( C = (\hat{U} + D_0 Q)(\hat{P} - N Q)^{-1} \), where \( Q \in \mathcal{M}(R) \) is such that \( \text{det}(\hat{P} - N Q)(\infty) \neq 0 \) (this condition guarantees that \( C \) is proper and it holds for all \( Q \in \mathcal{M}(R) \) when \( w \neq 0 \), i.e., \( P_j \in \mathcal{M}(R_j) \)). In the case of two plants, i.e., \( \mathcal{P} = \{P_0, P_1\} \), by (2), there exists a controller that simultaneously stabilizes \( P_0 \) and \( P_1 \) if and only if there exists a \( Q \in \mathcal{M}(R) \) such that \( D_1 \hat{P} + N \hat{U} = I + (D_1 - D_0) \hat{P} + (D_0 - D_1) N \) is unimodular. From the parity-interlacing-property (PIP), such \( Q \in \mathcal{M}(R) \) exists if and only if \( \text{det}(I + (D_1 - D_0) \hat{P}(s_0)) \) has the same sign at all blocking-zeros \( s_0 \in \mathbb{U} \) of \((D_0 - D_1)N\). In the case of three or more plants, i.e., \( \mathcal{P} = \{P_0, P_1, \ldots, P_n\} \), \( n > 1 \), PIP between any pairs of plants is a necessary condition for existence of controllers that simultaneously stabilize all \( P_j \in \mathcal{P} \); however, it is not sufficient. We now verify that \( P_0 \) and \( P_1 \) for any \( P_j \in \mathcal{P} \) actually satisfy the PIP under Assumptions 2.1: For \( s_0 \in \mathbb{U} \) such that \((D_0 - D_1) \hat{P}(s_0) = 0 \), we have \( \text{det}(I + (D_1 - D_0) \hat{P}(s_0)) = 1 \). For \( s_0 \in \mathbb{U} \) such that \( N(s_0) = 0 \) (i.e., \( s_0 = \infty \) or \( s_0 = 0 \)), we have \( (D_0 \hat{P} + N \hat{U})(s_0) = I \). Let \( D_0 \hat{P} + N \hat{U}(s_0) = \hat{D}_1 \hat{P}(s_0) \), det \( \hat{D}_1 \hat{P}(s_0) = \text{det}(D_1 \hat{P}(s_0)) \). By Assumption 2.1, since \( \Delta_j \) and \( \Theta_j \) are positive-definite, \( D_j(\infty)D_j^{-1}(\infty) = \Delta_j > 0 \), \( D_j(0)D_j^{-1}(0) = \Theta_j > 0 \), and hence, PIP holds.

Checking that the PIP holds pairwise for the plants \( P_j \in \mathcal{P} \) only confirms that the necessary condition is satisfied. It does not guarantee existence of simultaneously stabilizing controllers. Furthermore, this check is still a long way from explicit construction of simultaneously stabilizing controllers. In Proposition 2.3, an explicit design procedure is given that achieves a simultaneously stabilizing controller for the class \( \mathcal{P} \). In addition to defining one such controller explicitly, all controllers can also be characterized based on the nominal plant \( P_0 \). The selection of the nominal plant \( P_0 \in \mathcal{P} \) is completely arbitrary. Four possible cases are considered in Proposition 2.3: (a) \( P_j \) has blocking-zeros at infinity and zero \((w > 0, m > 0)\), (b) \( P_j \) has blocking-zeros only at infinity \((w > 0, m = 0)\), (c) \( P_j \) has blocking-zeros only at zero \((w = 0, m > 0)\), (d) \( P_j \) has no blocking-zeros at infinity and at zero \((w = 0, m = 0)\). Since these plants have no other transmission-zeros in the region of instability \( \mathbb{U} \), the last case corresponds to the class of minimum-phase plants. Existence of simultaneously stabilizing controllers for case (d) is rather obvious and is included here only for completeness.

### 2.3. Proposition

Let \( P_j \in R_p^{n \times n}, P_j \in \mathcal{P} \) satisfy Assumptions 2.1, \( j \in \{0, 1, \ldots, n\} \), i.e., \( P_j = D_j^{-1}N_j \) as in (1).

(a) If \( w > 0, m > 0 \), let \( k_1 \in \mathbb{R} \) satisfy (3); if \( w > 1 \), let \( k_2 \in \mathbb{R} \) satisfy (4); if \( w > 2 \), let \( k_e \in \mathbb{R}, v = 3, \ldots, w \), satisfy (5):

\[
k_1 > \max_{j \in \{0, \ldots, n\}} \|s(A_j - D_j D_0^{-1}(\infty))\|,
\]

\[
k_2 > \max_{j \in \{0, \ldots, n\}} \|s^{v-1} I + D_j D_0^{-1}(\infty) s^{v-1} I_k^{-1}\|,
\]

\[
k_e > \max_{j \in \{0, \ldots, n\}} \|s I + D_j D_0^{-1}(\infty) s^{v-1} I_k^{-1}\| \times \left(I + D_j D_0^{-1}(\infty) s^{v-2} I_k^{-1}\right)^{-1}.
\]
Define $K \in \mathbb{R}^{n \times n}$ as

$$K := D_0^{-1}(\infty) \sum_{i=1}^{w} s^{j_i} \prod_{\ell=1}^{i} \frac{1}{k_{\ell}}.$$  

(6)

Let $f_1 \in \mathbb{R}$, $f_1 \geq 0$ satisfy (7); if $m > 1$, let $f_2 \in \mathbb{R}$, $f_2 > 0$ satisfy (8); if $m > 2$, let $f_3 \in \mathbb{R}$, $f_3 > 0$, $v = 3, \ldots, m$, satisfy (9):

$$f_1 < \min_{j_0[0,\ldots,n]} \left\| s^{-1} \left[ \Theta_I \prod_{i=1}^{w} \frac{f_i}{k_i} - (I + D_I K)^{-1} D_I D_0^{-1}(0) \right. \right.$$

$$\times \left. \prod_{i=1}^{w} \frac{(s + z_i)}{k_i} \right\|^{-1},$$

(7)

$$f_2 < \min_{j_0[0,\ldots,n]} \left\| s^{-1} \left[ I + D_I (K + D_0^{-1}(0)s^{-1} f_1 \right.$$

$$\times \left. \prod_{i=1}^{w} \frac{(s + z_i)}{k_i} \right]^{-1} (I + D_I K) \right\|^{-1},$$

(8)

$$f_3 < \min_{j_0[0,\ldots,n]} \left\| s^{-1} \left[ I + D_I \left( K + D_0^{-1}(0) \sum_{i=1}^{w} \frac{1}{s} \prod_{\ell=1}^{i} f_{\ell} \right. \right.$$

$$\times \left. \prod_{i=1}^{w} \frac{(s + z_i)}{k_i} \right]^{-1} \left[ I + D_I \left( K + D_0^{-1}(0) \sum_{i=1}^{w} \frac{1}{s} \prod_{\ell=1}^{i} f_{\ell} \right. \right.$$  

$$\times \left. \prod_{i=1}^{w} \frac{(s + z_i)}{k_i} \right]^{-1} \right\|^{-1}.$$

(9)

A controller $C_0 \in \mathcal{R}_p^{m \times n}$ that simultaneously stabilizes all PEs is

$$C_0 = \left( \prod_{i=1}^{w} \frac{s^m}{(s + z_i)} \prod_{i=1}^{m} \frac{s^m}{(s + \beta_i)} \right) K$$

$$+ \prod_{i=1}^{w} \frac{s^m}{(s + \beta_i)} D_0^{-1}(0) \sum_{i=1}^{m} \frac{1}{s} \prod_{\ell=1}^{i} f_{\ell} \prod_{i=1}^{w} \frac{1}{k_i} - 1. $$

(10)

Furthermore, all simultaneously stabilizing controllers $C$ are

$$C = \left( C_0^{-1} - \prod_{i=1}^{w} \frac{s^m}{(s + z_i)} \prod_{i=1}^{m} \frac{s^m}{(s + \beta_i)} Q \right)^{-1} (I + QD_0),$$

(11)

for $j \in \{1, \ldots, n\}$, $Q \in \mathcal{R}_p^{m \times n}$ is such that $G_j$ is unimodular:

$$G_j := I + (D_0 - D_I) \left( I + C_0^{-1} D_I \prod_{i=1}^{w} \frac{(s + z_i)}{k_i} \right)^{-1} Q.$$  

(12)

(b) If $w > 0$, $m > 0$, let $k_\ell$, $\ell = 1, \ldots, w$, satisfy (3), (4), (5). A controller $C_0 \in \mathcal{R}_p^{m \times n}$ that simultaneously stabilizes all PEs is

$$C_0 = \left( \prod_{i=1}^{w} \frac{s^m}{(s + z_i)} \right)^{-1} \left( \prod_{i=1}^{m} \frac{s^m}{(s + \beta_i)} \right)^{-1} K$$

$$= D_0(\infty) \left( \sum_{i=1}^{w} s^i \prod_{\ell=1}^{i} \frac{1}{k_\ell} \right)^{-1} \prod_{i=1}^{w} \left( s + z_i \right).$$

(13)

Furthermore, all simultaneously stabilizing controllers $C$ are

$$C = \left( C_0^{-1} - \prod_{i=1}^{w} \frac{s^m}{(s + z_i)} \prod_{i=1}^{m} \frac{s^m}{(s + \beta_i)} Q \right)^{-1} (I + QD_0),$$

(14)

for $j \in \{1, \ldots, n\}$, $Q \in \mathcal{R}_p^{m \times n}$ is such that $G_j$ is unimodular:

$$G_j := I + (D_0 - D_I) \left( I + C_0^{-1} D_I \prod_{i=1}^{w} \frac{(s + z_i)}{k_i} \right)^{-1} Q.$$  

(15)

(c) If $w = 0$, $m > 0$, choose $X \in \mathcal{R}_p^{m \times n}$ such that $\det X(\infty) \neq 0$. Let $f_1 \in \mathbb{R}$, $f_1 \geq 0$ satisfy (16); if $m > 1$, let $f_2 \in \mathbb{R}$, $f_2 > 0$ satisfy (17); if $m > 2$, let $f_3 \in \mathbb{R}$, $f_3 > 0$, $v = 3, \ldots, m$, satisfy (18):

$$f_1 < \min_{j_0[0,\ldots,n]} \left\| s^{-1} \left[ \Theta_I - D_I D_0^{-1}(0) \right] - D_I D_0^{-1}(0)X \right\|^{-1},$$

(16)

$$f_2 < \min_{j_0[0,\ldots,n]} \left\| s^{-1} \left[ I + D_I D_0^{-1}(0)(I + sX)s^{-1} f_1 \right. \right.$$  

$$\times \left. \prod_{i=1}^{w} \frac{(s + z_i)}{k_i} \right]^{-1} \right\|^{-1},$$

(17)

$$f_3 < \min_{j_0[0,\ldots,n]} \left\| s^{-1} \left[ I + D_I D_0^{-1}(0)(I + sX) \sum_{i=1}^{m} \frac{1}{s} \prod_{\ell=1}^{i} f_{\ell} \right. \right.$$  

$$\times \left. \prod_{i=1}^{w} \frac{(s + z_i)}{k_i} \right]^{-1} \right\|^{-1}.$$  

(18)

A controller $C_0 \in \mathcal{R}_p^{m \times n}$ that simultaneously stabilizes all PEs is

$$C_0 = \left( \prod_{i=1}^{w} \frac{s^m}{(s + \beta_i)} D_0^{-1}(0) \sum_{i=1}^{m} \frac{1}{s} \prod_{\ell=1}^{i} f_{\ell} \right)^{-1}.$$

(19)

Furthermore, all simultaneously stabilizing controllers $C$ are

$$C = \left( C_0^{-1} - \prod_{i=1}^{w} \frac{s^m}{(s + z_i)} \prod_{i=1}^{m} \frac{s^m}{(s + \beta_i)} Q \right)^{-1} (I + QD_0),$$

(20)

$Q \in \mathcal{R}_p^{m \times n}$ is such that $\det(f_1 D_0^{-1}(0)X - Q(\infty)) \neq 0$ and for $j \in \{1, \ldots, n\}$, $G_j$ is unimodular:

$$G_j := I + (D_0 - D_I) \left( I + C_0^{-1} D_I \prod_{i=1}^{w} \frac{(s + \beta_i)}{s^m} \right)^{-1} Q.$$  

(21)

(d) If $m = 0$ and $w = 0$, let $A \in \mathcal{R}_p^{m \times n}$ be such that $\|A\| < \min_{j_0[0,\ldots,n]} \|D_I\|^{-1}$ and $\det A(\infty) \neq 0$. A controller $C_0 \in \mathcal{R}_p^{m \times n}$ that simultaneously stabilizes all PEs is $C_0 = A^{-1}$. Furthermore, all simultaneously stabilizing controllers are $C = (A - Q)^{-1}(I + QD_0)$; $Q \in \mathcal{R}_p^{m \times n}$ satisfies $\det(A - Q(\infty)) \neq 0$ and $(I + D_0 - D)(I + AD)Q^{-1}$ is unimodular for $j \in \{1, \ldots, n\}$. 


2.4. Comments

(i) In Proposition 2.3, the choices of $-a_i \in \mathbb{R} \setminus \mathcal{H}$ for $i = 1, \ldots, w$, and $-\beta_i \in \mathbb{R} \setminus \mathcal{H}$ for $i = 1, \ldots, m$, are completely arbitrary (in the continuous-time case, $x_i > 0$ and $\beta_i > 0$). However, these choices will determine some of the zeros of the designed controllers: When $w > 0$, $m > 0$, $C_0$ has zeros at $-x_i$ and $-\beta_i$; when $w > 0$, $m = 0$, $C_0$ has zeros at $-x_i$; when $w = 0$, $m > 0$, $C_0$ has zeros at $-\beta_i$.

(ii) In all of the four cases considered in Proposition 2.3, the explicitly defined nominal controller $C_0$ has a stable inverse although $C_0$ itself is not necessarily stable. The nominal controller is biproper since $C_0$ is proper and $C_0^{-1}$ is stable. In case (d), the controller $C_0$ is stable if and only if $A \in \mathcal{M}(\mathcal{H})$ satisfying $||A|| < \text{min} ||D_j||^{-1}$ is chosen unimodular. In case (b), where the plants have $w > 0$ blocking zeros at infinity but no zeros at zero, $C_0$ has a pole at $s = 0$ (and possibly other $\mathcal{H}$-poles depending on the constants $k_1, \ldots, k_w$). In particular, when $w = 1, C_0 = k_1 D_0(\infty) + (x_1/k_1) D_0(\infty)$ is in the form of a proportional-plus-integral controller. This design with a pole at $s = 0$ in case (b) provides integral-action in the closed-loop system. In case (c), where the plants have $m > 0$ blocking zeros at zero but no zeros at infinity, $C_0$ may have $\mathcal{H}$-poles depending on the choice of $X \in \mathcal{M}(\mathcal{H})$ and the constants $f_1, \ldots, f_m$. When $w = 1$, if $X$ is chosen constant nonsingular, where the eigenvalues of $X^{-1}$ have positive real parts, then the corresponding $C_0 = f_1^{-1}(s + \beta_1)(sI + X^{-1})^{-1}X^{-1} D_0(0)$ is stable.

(iii) The choice of $Q = 0$ satisfies the unimodularity conditions on $(12), (15), (21)$ and $I + (D_0 - D_j)(I + AD_j)^{-1}Q$ in Proposition 2.3(a)-(d). With $Q = 0$, the controller $C$ becomes the nominal controller $C_0$. Other simple choices satisfying these unimodularity conditions include choices for $Q \in \mathcal{M}(\mathcal{H})$ based on the small-gain approach as follows: A sufficient condition for $G_j$ in (12), $G_j$ in (15), $G_j$ in (15) unimodular is if $Q \in \mathcal{M}(\mathcal{H})$ satisfies

$$||Q|| < ||(D_0 - D_j)||\left(I + C_0^{-1}D_j \frac{\prod_{i=1}^{w}(s + x_i)}{s^w} \frac{\prod_{i=1}^{m}(s + \beta_i)}{s^m}\right)^{-1}$$

or

$$||Q|| < ||(D_0 - D_j)\left(I + C_0^{-1}D_j \frac{\prod_{i=1}^{w}(s + x_i)}{s^w}\right)^{-1}$$

or

$$||Q|| < ||(D_0 - D_j)\left(I + C_0^{-1}D_j \frac{\prod_{i=1}^{m}(s + \beta_i)}{s^m}\right)^{-1}$$

respectively. A sufficient condition for $I + (D_0 - D_j)(I + AD_j)^{-1}Q$ unimodular is if $Q \in \mathcal{M}(\mathcal{H})$ satisfies

$$||Q|| < ||(D_0 - D_j)(I + AD_j)^{-1}||^{-1}$$

(iv) In Proposition 2.3(c), $X \in \mathcal{M}(\mathcal{H})$ is chosen biproper in order to guarantee that the designed nominal controller $C_0$ in (19) is proper. The additional condition $\text{det}(f_1 D_0^{-1}(0)X - q(\infty) \neq 0$ on $Q \in \mathcal{M}(\mathcal{H})$ ensures that the controller $C$ in (20) is proper. A sufficient condition to satisfy this constraint is to choose $Q \in \mathcal{M}(\mathcal{H})$ strictly proper.

2.5. Example

Consider the class $\mathcal{P} = \{P_0, P_1, P_0H, P_1F, FP_1\}$ of five $2 \times 2$ plants:

$$P_0 = \begin{bmatrix} s & -s^2 \\ s^2 + 64 & s^2 - 1(s - 3) \\ (s - 2)(s + 10) & (s + 1)(s - 2) \\ \end{bmatrix}$$

$$H = \begin{bmatrix} (s + 4)(s + 16) & s \\ s^2 + 16 & 40(s - 1)(s - 3) \\ 0 & 40(s - 5)(s - 6) \\ \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 13s & -s(s + 4) \\ (s - 2)(s - 8) & s^2 - 4(s - 8) \\ 0 & s(5s + 16) \\ (s + 2)^2 + 1(s + 1) & (s + 1)(s - 2)(s + 5) \\ \end{bmatrix}$$

$$F = \text{diag} \begin{bmatrix} 3(s - 2), 3s + 14, 3s + 7 \end{bmatrix}.$$
satisfying (7). By (10), the nominal controller is

\[ C_0 = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \left( \frac{s^2}{k_1(s + \alpha_1)(s + \beta_1)}D_0^{-1}(\infty) + \frac{f_1}{k_1(s + \beta_1)}D_0^{-1}(0) \right)^{-1}, \]

where

\[ C_{11} = \frac{70(s + 5)(s + 10)(s^2 - 9.25s - 46.25)}{2s^4 - 9.8859s^3 - 52.1035s^2 - 26.7823s - 66.8457}, \]
\[ C_{12} = \frac{70(s + 5)(s + 10)s^2}{2s^4 - 9.8859s^3 - 52.1035s^2 - 26.7823s - 66.8457}, \]
\[ C_{21} = \frac{-70(s + 5)(s + 10)(s^2 - 9.25s - 46.25)}{2s^4 - 9.8859s^3 - 52.1035s^2 - 26.7823s - 66.8457}, \]
\[ C_{22} = \frac{70(s + 5)(s + 10)s^2 + 0.2891s + 1.4453}{2s^4 - 9.8859s^3 - 52.1035s^2 - 26.7823s - 66.8457}. \]

Since \( C_0^{-1} \in \mathcal{M}(\mathcal{R}) \), an RCF of \( C_0 \) is \( C_0 = I(C_0^{-1})^{-1} \). Using (2), it can be shown that \( C_0 \) simultaneously stabilizes all \( P_j \in \mathcal{P} \) since \( D_j C_0^{-1} + N = :P_j \) is unimodular for \( j \in \{0, 1, \ldots, 4\} \). By (11), all simultaneously stabilizing controllers are

\[ C = \frac{(s + 5)(s + 10)}{s} \times \left( \frac{s}{70}D_0^{-1}(\infty) + \frac{0.37(s + 5)}{70s}D_0^{-1}(0) - Q \right)^{-1}(I + QD_0), \]

where \( Q \in \mathcal{R}^{2 \times 2} \) satisfies \( G_j \).

\[ G_j = I + (D_0 - D_j) \times \left[ I + \left( \frac{s}{70}D_0^{-1}(\infty) + \frac{0.37(s + 5)}{70s}D_0^{-1}(0) \right)D_j \right]^{-1}Q \]

is unimodular, \( j \in \{1, \ldots, 4\} \).

3. Conclusions

We considered simultaneous stabilization of a class of LTI, square MIMO plants that all have the same number of blocking-zeros at infinity and/or at zero; these plants have no other zeros in the region of instability, and satisfy positive-definiteness assumptions on the high frequency and the low frequency gain matrices as described in Assumptions 2.1. We proved that this class of plants are simultaneously stabilizable and proposed an explicit design method to find a nominal controller. We characterized all simultaneously stabilizing controllers based on this nominal controller by choosing a controller-parameter satisfying additional unimodularity conditions. The design method was illustrated by an example of a class of 2 × 2 MIMO plants that all have one blocking-zero at infinity and one blocking-zero at zero. As these results show, although necessary and sufficient conditions are not available to check simultaneous stabilizability of a completely general class of more than two plants, it may be possible to identify classes of three or more plants for which simultaneously stabilizing controllers exist and to design controllers explicitly for such classes.

Appendix

Proof of Proposition 2.3. By assumption, \( P_j = D_j^{-1}N \) is an RCF of \( P_j \), where

\[ N := \prod_{l=1}^{d_+} P_{l+1}^{-1}(s + \alpha_l)I. \]

Furthermore, \( D_j(\infty)D_0^{-1}(\infty) = A_j \), and \( D_j(0)D_0^{-1}(0) = \Theta_j \).

(a) Choose any \( k_1 \in \mathbb{R} \) satisfying (3). Define

\[ W_{1j} := \frac{k_1}{(s + \alpha_1)}I + D_jD_0^{-1}(\infty) \times \frac{s}{(s + \alpha_1)} \]

\[ = (k_1I + D_jD_0^{-1}(\infty)s(sA_j + k_1I))^{-1}(sA_j + k_1I) \]

\[ = [I - s(A_j - D_jD_0^{-1}(\infty))(sA_j + k_1I)]^{-1}(sA_j + k_1I) \]

\[ = [I - D_jD_0^{-1}(\infty)D_j]^{-1}(sA_j + k_1I) \]

\[ (s + \alpha_1). \]

(22)

for \( j \in \{0, \ldots, n\} \). \( W_{1j} \in \mathcal{M}(\mathcal{R}) \) is unimodular since \( k_1 \) satisfies (3), \( \alpha_1 > 0 \), \( A_j \) is symmetric, positive definite and \( D_j(\infty)D_0^{-1}(\infty) = A_j \) implies \( sA_j - D_jD_0^{-1}(\infty)s(s + \alpha_1) \) is (not)(note that by (22), \( [I - W_{1j}^{-1}D_jD_0^{-1}(\infty)s(s + \alpha_1)] = W_{1j}^{-1}[W_{1j} - D_jD_0^{-1}(\infty)s(s + \alpha_1)] = W_{1j}^{-1}k_1 \)

\[ /s(s + \alpha_1) = [I + D_jD_0^{-1}(\infty)s(s + \alpha_1)]^{-1}. \]

Define

\[ W_{2j} := \frac{k_2}{(s + \alpha_2)}I + W_{1j}^{-1}D_jD_0^{-1}(\infty) \times \frac{s}{(s + \alpha_1)} \]

\[ = \left[ I - s[I - W_{1j}^{-1}D_jD_0^{-1}(\infty)s(s + \alpha_1)]^{-1} \right]^{-1} \times \frac{s}{(s + \alpha_1)} \]

\[ = \left[ I - D_jD_0^{-1}(\infty)s(s + \alpha_1)]^{-1} \right]^{-1} \times \frac{s}{(s + \alpha_1)} \]

\[ = \left[ I - D_jD_0^{-1}(\infty)s(s + \alpha_1)]^{-1} \right]^{-1} \times \frac{s}{(s + \alpha_1)}. \]

(23)

for \( j \in \{0, \ldots, n\} \). \( W_{2j} \in \mathcal{M}(\mathcal{R}) \) is unimodular since \( k_2 \) satisfies (4), \( \alpha_2 > 0 \), and \( [W_{2j}^{-1}]D_0^{-1}(\infty)s(s + \alpha_2)]^{-1} \)

\( (\infty) = I \) implies \( s[I - W_{1j}^{-1}D_jD_0^{-1}(\infty)s(s + \alpha_1)]^{-1} \) is (not)(note that by (22), \( [I - W_{1j}^{-1}D_jD_0^{-1}(\infty)s(s + \alpha_1)] = W_{1j}^{-1}[W_{1j} - D_jD_0^{-1}(\infty)s(s + \alpha_1)] = W_{1j}^{-1}k_1 \)

\[ /s(s + \alpha_1) = [I + D_jD_0^{-1}(\infty)s(s + \alpha_1)]^{-1}. \]

for \( v = 3, \ldots, n \) with \( k_v \) satisfying (5), i.e., when \( v = v \) for \( j \in \{0, \ldots, n\} \), \( k_v \) satisfies \( k_v > \max_j[s[I - D_jD_0^{-1}(\infty)s(s + \alpha_1)]^{-1}D_jD_0^{-1}(\infty)s(s + \alpha_1)]^{-1} \)

\[ = \max_j[s[I - (\prod_{l=1}^{d_+} P_{l+1}^{-1}(s + 1/k_v))]^{-1}] = \max_j[s[I - (\prod_{l=1}^{d_+} P_{l+1}^{-1}(s + 1/k_v))]^{-1} \]

\[ = \max_j[s[I - (\prod_{l=1}^{d_+} P_{l+1}^{-1}(s + 1/k_v))]^{-1}] = \max_j[s[I - (\prod_{l=1}^{d_+} P_{l+1}^{-1}(s + 1/k_v))]^{-1}]. \]
\[ s(s + \alpha_m^{-1})^{-1} \] Define
\[ W_{wj} := \frac{k_w}{(s + \alpha_m)} I + \left( \prod_{i=1}^{w-1} W_{ij} \right)^{-1} D_j D_0^{-1} (s) \left( \prod_{i=1}^{w} I \right) \]
\[ = \left[ I - s \left( I - \left( \prod_{i=1}^{w-1} W_{ij} \right)^{-1} D_j D_0^{-1} (s) \left( \prod_{i=1}^{w} I \right) \right) \left( s + k_w \right) \right]^{-1} \frac{(s + k_w)}{(s + z_i)}^{-1} \]
\[ \prod_{i=1}^{w} W_{ij} \left( s + z_i \right)^{-1} \left( s + k_w \right) \]
\[ (24) \]
for \( j \in \{0, \ldots, n\}, W_{wj} \in \mathcal{M}(\mathcal{H}) \) is unimodular since \( k_w \) satisfies (5), \( \alpha_m > 0 \), and \( \left( \prod_{i=1}^{w} W_{ij} \right)^{-1} D_j D_0^{-1} (s) \left( \prod_{i=1}^{w} I \right) \in \mathcal{M}(\mathcal{H}) \). Since \( W_{1j}, W_{2j} \) are unimodular, \( (W_{1j}, W_{2j}) = [k_1 k_2 I + D_j D_0^{-1} (s + \alpha_m)]^{-1} \) is unimodular. Similarly, \( W_j \) is unimodular for \( j \in \{0, \ldots, n\} \).

\[ \left( I + D_j K \right) \prod_{i=1}^{w} \frac{k_i}{(s + z_i)} \]
\[ (25) \]
For simplicity, define \( \alpha := \prod_{i=1}^{w} \frac{z_i}{k_i} \). If \( m > 0 \), choose any \( f_1 \in \mathbb{R} \) satisfying (7). Define
\[ M_{1j} := \frac{s}{(s + \beta_1)} I + W_j^{-1} D_j D_0^{-1} (0) \frac{f_1}{(s + \beta_1)} \]
\[ = (sI + W_j^{-1} D_j D_0^{-1} (0) f_1) (sI + \alpha f_1 D_j D_0^{-1} (0))^{-1} (sI + \alpha f_1 D_j D_0^{-1} (0)) \]
\[ = \left[ I - s^{-1} \left( \beta_1 \right) (I - \left( \prod_{i=1}^{w} W_{ij} \right)^{-1} D_j D_0^{-1} (s) \left( \prod_{i=1}^{w} I \right) \right) \left( s + \beta_1 \right) \right]^{-1} \frac{(s + \beta_1)}{(s + \beta_1)}^{-1} \]
\[ s \left( I - \left( \prod_{i=1}^{w} W_{ij} \right)^{-1} D_j D_0^{-1} (s) \left( \prod_{i=1}^{w} I \right) \right) \left( s + \beta_1 \right) \]
\[ (26) \]
for \( j \in \{0, \ldots, n\}, M_{1j} \in \mathcal{M}(\mathcal{H}) \) is unimodular since \( f_1 \) satisfies (7), \( \beta_1 > 0 \), \( \beta_1 \) is symmetric, positive definite, and \( D_j (D_0^{-1} (0) = \Theta_j \) implies \( s^{-1} (\beta_1 (I - \left( \prod_{i=1}^{w} W_{ij} \right)^{-1} D_j D_0^{-1} (0)) \in \mathcal{M}(\mathcal{H}) \). If \( m > 1 \), choose any \( f_2 \in \mathbb{R} \) satisfying (8), i.e., for \( j \in \{0, \ldots, n\} \), let \( f_2 \approx \min \left\{ \left\| \left[ I - M_{1j}^{-1} W_j^{-1} D_j D_0^{-1} (0) f_1 (s + \beta_1)^{-1} \right] \right\| \right\} \). Define
\[ M_{2j} := \frac{s}{(s + \beta_2)} I + M_{1j}^{-1} W_j^{-1} D_j D_0^{-1} (0) \frac{f_1}{(s + \beta_2)} \]
\[ = \left[ I - s^{-1} \left( I - M_{1j}^{-1} W_j^{-1} D_j D_0^{-1} (0) f_1 (s + \beta_1)^{-1} \right) \right]^{-1} \frac{(s + \beta_2)}{(s + \beta_2)} \]
\[ f_2, s \left( s + f_2 \right) \frac{(s + f_2)}{(s + \beta_2)} \]
\[ (27) \]
for \( j \in \{0, \ldots, n\}, M_{2j} \in \mathcal{M}(\mathcal{H}) \) is unimodular since \( f_2 \) satisfies (8), \( \beta_2 > 0 \), and \( \left( \prod_{i=1}^{w} M_{2ij} \right)^{-1} f_2 (s + \beta_1)^{-1} \in \mathcal{M}(\mathcal{H}) \). If \( m > 1 \), continue similarly for \( v = 3, \ldots, m \), with \( f_v \) satisfying (9). Define
\[ M_{mj} := \frac{s}{(s + \beta_v)} I + \left( \prod_{i=1}^{v-1} M_{ij} \right)^{-1} \times W_j^{-1} D_j D_0^{-1} (0) \frac{f_v}{(s + \beta_v)} \]
\[ = \left[ I - s^{-1} \left( I - \left( \prod_{i=1}^{v-1} M_{ij} \right)^{-1} \right) \right]^{-1} \times W_j^{-1} D_j D_0^{-1} (0) \frac{f_v}{(s + \beta_v)} \]
\[ (28) \]
for \( j \in \{0, \ldots, n\}, M_{mj} \in \mathcal{M}(\mathcal{H}) \) is unimodular since \( f_m \) satisfies (9), \( \beta_m > 0 \), and \( \left( \prod_{i=1}^{m} \right)^{-1} W_j^{-1} D_j D_0^{-1} (0) \left( \prod_{i=1}^{m} f_i (s + \beta_i)^{-1} \right) \in \mathcal{M}(\mathcal{H}) \). Since \( M_{1j}, M_{2j} \) are unimodular, \( (M_{1j}, M_{2j}) = (sI + W_j^{-1} D_j D_0^{-1} (0) (f_1 s + f_2) (s + \beta_v)^{-1} (s + \beta_v)^{-1} \in \mathcal{M}(\mathcal{H}) \). Similarly, \( M_{j} \) is unimodular for \( j \in \{0, \ldots, n\} \).

\[ \left( I + W_j^{-1} D_j D_0^{-1} (0) \left( \prod_{i=1}^{m} \frac{1}{s} \left( \prod_{i=1}^{m} \frac{f_i}{s} \right) \right) \prod_{i=1}^{m} \frac{f_m s}{(s + \beta_m)} \right) \]
\[ (29) \]
Define \( C_0 \) as in (10); \( C_0 \) satisfies (9), \( \beta_0 > 0 \), and \( \left( \prod_{i=1}^{m} f_i s (s + \beta_i)^{-1} \right) \in \mathcal{M}(\mathcal{H}) \). Since \( M_{1j}, M_{2j} \) are unimodular, \( (M_{1j}, M_{2j}) = (sI + W_j^{-1} D_j D_0^{-1} (0) \left( f_1 s + f_2 \right) (s + \beta_v)^{-1} (s + \beta_v)^{-1} \in \mathcal{M}(\mathcal{H}) \). Similarly, \( M_{j} \) is unimodular for \( j \in \{0, \ldots, n\} \):

\[ W_j M_j = \left( I + D_j K \right) \prod_{i=1}^{w} \frac{k_i}{(s + z_i)} \]
\[ \prod_{i=1}^{m} \frac{k_i}{(s + \beta_i)} \]
\[ (30) \]
Define \( \Psi_j := D_j C_0^{-1} + N, \quad \tilde{\Psi}_j := C_0^{-1} D_j + N \).

\[ (31) \]
Note that \( C_0^{-1} \Psi_j^{-1} = \tilde{\Psi}_j^{-1} C_0^{-1}, \quad \tilde{\Psi}_j^{-1} D_j = D_0 \Psi_j^{-1} \).

\[ (32) \]
By (31), we have the following important identity:

\[ \left[ C_0^{-1} \right] \quad I \quad D_0 \Psi_j^{-1} \quad \tilde{\Psi}_j^{-1} \quad C_0^{-1} \tilde{\Psi}_j^{-1} \quad I. \]

\[ (32) \]
In (32), \( P_0 = (N \tilde{\Psi}_j^{-1} (D_0 \Psi_j^{-1})^{-1} = (I - C_0^{-1} \Psi_0^{-1} D_0) \Psi_0^{-1} D_0^{-1} \) is another RCF of \( P_0 = ND_0^{-1} \). By (32), the controller \( C \) stabilizes \( P_0 \) if and only if \( C \) is given by (11), where \( Q \in \mathcal{M}(\mathcal{H}) \), \( C \) becomes \( C_0 \) given in (10) when \( Q = 0 \). An RCF for \( C \) can also be obtained from (32) as...
\[ C := N_t D_c^{-1} = (\Psi_0^{-1}(I + D_0 Q)(\Psi_0^{-1}(C_0^{-1} - NQ))^{-1}. \]

The controller \( C \) is proper if and only if \( \det(C_0^{-1} - NQ)(\infty) = \det C_0^{-1}(\infty) \neq 0 \), which follows by construction since \( \det C_0(\infty) = \det D_0(\infty) \prod_{k=1}^{n} \lambda_k \neq 0 \). The controller \( C \) stabilizes all \( P_j = D_j^{-1}N_j \in \mathcal{P} \) in addition to \( P_0 = D_0^{-1}N \) if and only if \( Q \in \mathcal{M}(\mathcal{R}) \) is such that \( D_j D_c + N_j N_c = I + (D_j - D_0)D_j^{-1} = I + (D_j - D_0)[(C_0^{-1} \Psi_0^{-1} - N \Psi_0^{-1})Q] \) is unimodular. Since \( N \) is diagonal, \( \Psi_0^{-1} = \Psi_0^{-1}N \); since \( C_0^{-1} \Psi_0^{-1} = \Psi_0^{-1}C_0^{-1} \), \( D_0 \Psi_0^{-1} = \Psi_0^{-1}D_0 \), using the definitions of \( \Psi_0, \Psi_j, \Psi_0, \Psi_j \) given by (31), it follows that this last unimodularity is equivalent to \( ((D_j - D_0)[(C_0^{-1} \Psi_0^{-1} - N \Psi_0^{-1})Q] = \Psi_j \Psi_0^{-1}[I + D_0 Q - \Psi_0 \Psi_j^{-1}D_j Q] = \Psi_j \Psi_0^{-1}[I + D_0 (I - C_0^{-1} D_j^{-1} Q - ND_j \Psi_j^{-1}Q)] = \Psi_j \Psi_0^{-1}[I + (D_j - D_0)N \Psi_j^{-1}Q] = \Psi_j \Psi_0^{-1} G_j \) is unimodular, equivalently, \( G_j \) is (32) is unimodular.

(b) If \( w > 0 \) but \( m = 0 \), then \( N = \left( \prod_{i=1}^{n} (s + z_i) \right)^{-1} I \). The proof is as in (a) above, with \( M_j = M_c^{-1} \).

(c) If \( w = 0 \) and \( m > 0 \), then \( N = s^{m} \left( \prod_{i=1}^{n} (s + z_i) \right)^{-1} I \).

Choose any biproper \( X \in \mathcal{M}(\mathcal{R}) \) and choose any \( f_i \in \mathbb{R} \) satisfying (7). Define

\[
M_{1j} := \frac{s}{(s + \beta_1)} I + D_{j} D_0^{-1}(0)(I + sX) - \frac{f_1}{(s + \beta_1)} = \left[ I - (s^{-1}[\Theta_j - D_{j} D_0^{-1}(0)]) - D_{j} D_0^{-1}(0)X \right] f_1 s(s + f_1 \Theta_j)^{-1} \left( \frac{s + f_1 \Theta_j}{s + \beta_1} \right),
\]

for \( j = 0, \ldots, n \), \( M_{1j} \in \mathcal{M}(\mathcal{R}) \) is unimodular since \( f_1 \) satisfies (16), \( \beta_j > 0 \), \( \Theta_j \) is symmetric, positive definite, and \( s^{-1}[\Theta_j - D_{j} D_0^{-1}(0)] \in \mathcal{M}(\mathcal{R}) \). If \( m > 0 \), choose any \( f_2 \in \mathbb{R} \) satisfying (17), i.e., for \( j = 0, \ldots, n \), let \( f_2 < \min \|s^{-1}[I - M_{1j}^{-1} D_{j} D_0^{-1}(0)(I + sX)f_1(s + \beta_1)^{-1}]\| = \min \|M_{1j}(s + \beta_1)^{-1}\| \). Define

\[
M_{2j} := \frac{s}{(s + \beta_2)} I + M_{1j}^{-1} D_{j} D_0^{-1}(0)(I + sX) - \frac{f_1}{(s + \beta_1)(s + \beta_2)} = \left[ I - (s^{-1}(I - M_{1j}^{-1} D_{j} D_0^{-1}(0)) - (s + f_2) f_1(s + \beta_1)^{-1}) \right] \frac{f_2 s}{(s + f_2)} \left( \frac{s + f_2}{s + \beta_2} \right)^{-1},
\]

for \( j = 0, \ldots, n \), \( M_{2j} \in \mathcal{M}(\mathcal{R}) \) is unimodular since \( f_2 \) satisfies (17), \( \beta_j > 0 \), and \( [M_{1j}^{-1} D_{j} D_0^{-1}(0)(I + sX)f_1(s + \beta_1)^{-1}] \) is unimodular since \( s^{-1}[I - M_{1j}^{-1} D_{j} D_0^{-1}(0)(I + sX)f_1(s + \beta_1)^{-1}] \in \mathcal{M}(\mathcal{R}) \). If \( m > 2 \), continue similarly for \( v = 3, \ldots, m \), with \( f_v \) satisfying (9). Define

\[
M_{mj} := \frac{s}{(s + \beta_m)} I + \left( \prod_{i=1}^{m-1} M_i \right)^{-1} D_{j} D_0^{-1}(0)(I + sX) \prod_{i=1}^{m} \frac{f_i}{(s + \beta_i)}
\]

for \( j = 0, \ldots, n \), \( M_{mj} \in \mathcal{M}(\mathcal{R}) \) is unimodular since \( f_m \) satisfies (18), \( \beta_m > 0 \), and \( \left( \prod_{i=1}^{m-1} M_i \right)^{-1} D_{j} D_0^{-1}(0)(I + sX) \prod_{i=1}^{m} \frac{f_i}{(s + \beta_i)} \left( \frac{s + f_m}{s + \beta_m} \right)^{-1} \]

In (32), the rest of the proof follows similar steps as in (a) where \( \Psi \) and \( \Psi_j \) are replaced by \( M_j \) and \( M_1 \), respectively. In this case, the controller \( C \) is proper if and only if \( \det(C_0^{-1} - Q)(\infty) = \det(f_1 D_0^{-1}(0)(X(\infty) - Q(\infty)) \neq 0 \).

(d) If \( m = 0, w = 0, \) then \( N = I, P_j = D_j^{-1} \), choose any \( A \in \mathcal{M}(\mathcal{R}) \) such that \( \|A\| = \min \|D_j\|^{-1} \) and \( \det A(\infty) \neq 0 \); then \( \Psi_j := I + D_{j} A \) and \( \Psi_j := I + A D_{j} \) are unimodular for \( j = 0, \ldots, n \). Define \( C_0 := A^{-1}; \) then \( C_0 \) is proper. The proof then follows from an identity similar to (32).

Reference:


