Reliable Decentralized Integral-Action Controller Design

A. N. Gündeş and M. G. Kabuli

***Abstract*—Reliable stabilizing controller design with integral-action is considered for linear time-invariant, multi-input–multi-output decentralized systems with stable plants. Design methods are proposed to achieve reliable closed-loop stability with integral-action in each output channel for asymptotic tracking of step-input references applied at each input. The design approaches guarantee stability and integral-action in the active channels when all controllers are operational and when any of the controllers is set equal to zero due to failure.**

***Index Terms*—Decentralized control, integral action, reliable stabilization.**

I. Introduction

Reliable stabilizing controller design with integral-action is considered for linear time-invariant (LTI), multi-input–multi-output (MIMO), multichannel decentralized systems with stable plants. The goal is to achieve closed-loop stability with integral-action in each output channel so that step-input references applied at each input are asymptotically tracked (with zero steady-state error). Reliable stabilization with integral-action maintains stability and integral-action when any of the controllers fail. The model of controller failure used here assumes that a controller that fails is replaced by zero; the failure is recognized and the corresponding controller is taken out of service (i.e., the states in the controller implementation are all set to zero, the initial conditions and the outputs of the channel that failed are set to zero for all inputs). Clearly, stability is still maintained when all controllers are set to zero since the open-loop plant is stable. If some of the controllers fail, integral-action is still present in the outputs of the channels with active controllers due to the integrators in those controllers.

The reliable stabilization problem was introduced in [9] and [10] and has been studied with full-feedback and decentralized controllers [13], [14], [6], [11]. Reliable decentralized designs were given in [8] and [12], which guarantee stability and satisfy performance criteria based on a given H_∞ norm bound despite complete sensor or actuator outages for any subset of a prescribed set of control channels; integral-action is not a criterion in these designs. In [7], [2], [1], integral-action was considered in the decentralized configuration with single-input single-output (SISO) channels that have gain uncertainty between zero and one, and conditions on the steady-state gain of the plant were presented. Although reliable decentralized stabilizability conditions with integral action were given for two-channel and multichannel decentralized configurations with stable plants (for example, in [4]), explicit design approaches were not explored in detail.

In this note, the goals are: 1) to present necessary and sufficient conditions that guarantee existence of reliable decentralized integral-action controllers, and 2) to propose explicit algebraic design procedures that actually achieve reliable stability with integral action. Although the only criteria incorporated into the design approaches developed here are stability under possible failure of controllers and integral action, other performance criteria may be included in the designs due to the freedom in choosing certain control parameters given in the explicit design steps. The results are explored in detail for two-, three-, and four-channel decentralized systems, where the channels are assumed to be MIMO. Simplifications are also presented for the fully decentralized case with SISO channels. The proposed design methods can be extended to more than four channels. The main results are the existence conditions for reliable decentralized integral-action controllers given in Lemma 1 and the controller design methodology developed in Proposition 1. Corollary 1 states necessary and sufficient conditions for existence of pure integral controllers and proposes an explicit design when all channels except for channel one are restricted to have a single output, which includes the special case of SISO channels. A simple example is given to illustrate the design method of Proposition 1 for a three-channel plant with SISO channels.

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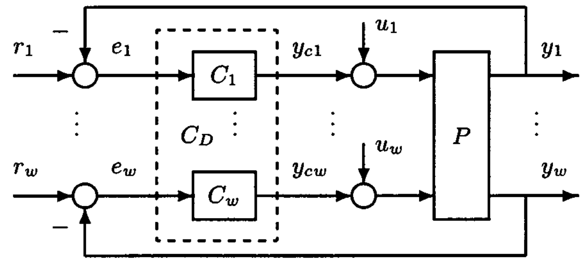

Fig. 1. The w-channel decentralized system $\mathcal{S}(P, C_D)$.

Due to the algebraic framework, the results apply to continuous-time and discrete-time systems. A continuous-time setting was assumed throughout; all evaluations and discussions involving poles and zeros at $s = 0$ should be interpreted at $z = 1$ in the discrete-time case.

1) Notation and Algebraic Framework: The region of instability $\mathcal{U}$ is the extended closed right-half plane (for continuous-time systems) or the complement of the open unit disk (for discrete-time systems). The sets of real numbers, proper rational functions with no $\mathcal{U}$ poles, proper and strictly-proper rational functions with real coefficients are denoted by $\mathbb{R}$, $\mathcal{R}$, R_p, and R_s. The set of matrices with entries in $\mathcal{R}$ is denoted by $\mathcal{M}(\mathcal{R})$; M is called stable iff $M \in \mathcal{M}(\mathcal{R})(M \in \mathcal{R}^{n_y \times n_u}$ indicates the matrix order); $M \in \mathcal{M}(\mathcal{R})$ is called unimodular iff $M^{-1} \in \mathcal{M}(\mathcal{R})$. A block-diagonal matrix whose entries are the matrices N_i, N_j is denoted by $\operatorname{diag}[N_i, N_j]$. A right-inverse of $A \in \mathcal{M}(\mathbb{R})$ is denoted by $A^I \in \mathcal{M}(\mathbb{R})$. For $M \in \mathcal{M}(\mathcal{R})$, the norm $\|\cdot\|$ is defined as $\|M\| = \sup_{s\in\partial\mathcal{U}} \overline{\sigma}(M(s))$, where $\overline{\sigma}$ denotes the maximum singular value, $\partial\mathcal{U}$ denotes the boundary of $\mathcal{U}$. Let $P \in \mathcal{M}(\mathcal{R})$, where $\operatorname{rank} P = \rho$; $s_o \in \mathcal{U}$ is called a (transmission) $\mathcal{U}$-zero of P iff $\operatorname{rank} P(s_o) < \rho$; s_o is called a blocking $\mathcal{U}$-zero of P iff $P(s_o) = 0$.

II. Analysis

Consider the LTI, MIMO, w-channel decentralized feedback system $\mathcal{S}(P, C_D)$ shown in Fig. 1. $P \in R_p^{n_y \times n_u}$ and $C_D \in R_p^{n_u \times n_y}$ represent the transfer functions of the plant and the decentralized controller, partitioned as

$$P = \begin{bmatrix} P_{11} & \cdots & P_{1w} \\ \vdots & & \vdots \\ P_{w1} & \cdots & P_{ww} \end{bmatrix} \in \mathcal{R}^{n_y \times n_u}$$
$$C_D = \operatorname{diag}[C_1, \ldots, C_w] \in R_p^{n_u \times n_y} \tag{1}$$

where $P_{ii} \in \mathcal{R}^{n_{yi}\times n_{ui}}$, $C_i \in R_p^{n_{ui}\times n_{yi}}$, $i = 1, \ldots, w$, $n_y = \sum_{i=1}^{w} n_{yi}$, $n_u = \sum_{i=1}^{w} n_{ui}$. It is assumed that $\mathcal{S}(P, C_D)$ is a well-posed system (i.e., all closed-loop transfer functions are proper). It is also assumed that P and C_D have no hidden modes corresponding to eigenvalues in $\mathcal{U}$. Although $P \in \mathcal{M}(\mathcal{R})$, the decentralized controller C_D is unstable (due to poles at zero for the integral-action requirement and other possible $\mathcal{U}$ poles). Let H_{er} denote the (input-error) transfer function from r to e, where $r := [r_1^T \ldots r_w^T]^T$, $u := [u_1^T \ldots u_w^T]^T$, $e := [e_1^T \ldots e_w^T]^T$, $y := [y_1^T \ldots y_w^T]^T$, $y_c := [y_{c1}^T \ldots y_{cw}^T]^T$. A controller that fails is set equal to zero; the failure is recognized and the corresponding controller is taken out of service. When $w = 2$, the only possible failures are due to one controller failure. When $w = 3$, the failures are due to one or two controller failure. When $w = 4$, the failures are due to one, two or three controller failure. For $i = 1, \ldots, w$, $\mathcal{S}(P, C_i)$ denotes the system with only the ith controller active and all others failed. For $j = 2, \ldots, w$, $i = 1, \ldots, i-1$, $\mathcal{S}(P, C_i, C_j)$ denotes the system with the ith and jth controllers active and all others failed. For $k = 3, \ldots, w$, $j = 2, \ldots, k-1$, $i = 1, \ldots, j-1$, $\mathcal{S}(P, C_i, C_j, C_k)$ denotes the system with ith, jth and kth controllers active and the remaining controller failed. In these systems, the outputs of the inactive channels are not observed; i.e., for $\ell = 1, \ldots, w$, $y_{c\ell}$, $\ell \neq i$ of $\mathcal{S}(P, C_i)$, $y_{c\ell}$, $\ell \neq i$, $\ell \neq j$ of $\mathcal{S}(P, C_i, C_j)$, $y_{c\ell}$, $\ell \neq i$, $\ell \neq j$, $\ell \neq k$ of $\mathcal{S}(P, C_i, C_j, C_k)$ are not observed.

We use the following standard definitions of stability and integral-action (see [13] and [7]).

Definitions: (Stability, Reliable Decentralized Integral-Action Controller): 1) The system $\mathcal{S}(P, C_D)$ is *stable* iff the transfer function from (r, u) to (y, y_c) is stable. The stable $\mathcal{S}(P, C_D)$ has *integral-action* iff $H_{er}(0) = 0$. For $i = 1, \ldots, w$, the system $\mathcal{S}(P, C_i)$ is stable iff the transfer function from (r_i, u) to (y, y_{ci}) is stable. The stable $\mathcal{S}(P, C_i)$ has integral action iff the transfer function from r_i to e_i has blocking zeros at zero. For $j = 2, \ldots, w$, $i = 1, \ldots, j-1$, the system $\mathcal{S}(P, C_i, C_j)$ is stable iff the transfer function from (r_i, r_j, u) to (y, y_{ci}, y_{cj}) is stable. The stable $\mathcal{S}(P, C_i, C_j)$ has integral-action iff the transfer function from (r_i, r_j) to (e_i, e_j) has blocking zeros at zero. For $k = 3, \ldots, w$, $j = 2, \ldots, k-1$, $i = 1, \ldots, j-1$, the system $\mathcal{S}(P, C_i, C_j, C_k)$ is stable iff the transfer-function from (r_i, r_j, r_k, u) to $(y, y_{ci}, y_{cj}, y_{ck})$ is stable. The stable $\mathcal{S}(P, C_i, C_j, C_k)$ has integral-action iff the transfer-function from (r_i, r_j, r_k) to (e_i, e_j, e_k) has blocking-zeros at zero. *2)* The controller $C_D = \mathrm{diag}\,[C_1, \ldots, C_w]$ is a *stabilizing controller* for the plant P (or C_D stabilizes P) iff $C_D \in \mathcal{M}(R_p)$ and the system $\mathcal{S}(P, C_D)$ is stable. *3)* The controller $C_D = \mathrm{diag}\,[C_1, \ldots, C_w]$ is a *reliable decentralized integral-action controller* iff the system $\mathcal{S}(P, C_D)$ is stable with integral-action when all controllers are active and when any subset of the controllers are set equal to zero; i.e., when $w = 2$, all three systems $\mathcal{S}(P, C_D)$, $\mathcal{S}(P, C_i)$, $i = 1, 2$, are stable with integral action, when $w = 3$, all seven systems $\mathcal{S}(P, C_D)$, $\mathcal{S}(P, C_i)$, $i = 1, 2, 3$ and $\mathcal{S}(P, C_i, C_j)$, $j = 2, 3$, $i = 1, \ldots, j-1$, are stable with integral action, when $w = 4$, all fifteen systems $\mathcal{S}(P, C_D)$, $\mathcal{S}(P, C_i)$, $i = 1, \ldots, 4$, $\mathcal{S}(P, C_i, C_j)$, $j = 2, \ldots, 4$, $i = 1, \ldots, j-1$, and $\mathcal{S}(P, C_i, C_j, C_k)$, $k = 3, 4$, $j = 2, \ldots, k-1$, $i = 1, \ldots, j-1$, are stable with integral-action.□

It is well known that $C_D \in R_p^{n_u\times n_y}$ stabilizes $P \in \mathcal{R}^{n_y\times n_u}$ if and only if $(\hat{D}_c + PN_c)$ is unimodular for any right-coprime-factorization (RCF) $C_D = N_c\hat{D}_c^{-1}$ (N_c, $\hat{D}_c \in \mathcal{M}(\mathcal{R})$, $\det \hat{D}_c(\infty) \neq 0$) [13], [3]. For any stabilizing controller C_D, the (input-error) transfer-function $H_{er}(0) = (I_{n_y} + PC)^{-1}(0) = \hat{D}_c(\hat{D}_c + PN_c)^{-1}(0) = 0$ if and only if $\hat{D}_c(0) = 0$. Let $N_i\hat{D}_i^{-1}$ be any RCF of C_i, $i = 1, \cdots, w$. Let $N_c := \mathrm{diag}\,[N_1, \cdots, N_w]$, $\hat{D}_c := \mathrm{diag}\,[\hat{D}_1, \ldots, \hat{D}_w]$; then $N_c\hat{D}_c^{-1}$ is an RCF of C_D. The denominator $\hat{D}_c(0) = \mathrm{diag}\,[\hat{D}_1, \ldots, \hat{D}_w](0) = 0$ if and only if $\hat{D}_c = (s/(s+\alpha))D_c$ for some $D_c \in \mathcal{M}(\mathcal{R})$, i.e., $\hat{D}_i = (s/(s+\alpha))D_i$ for some $D_i \in \mathcal{M}(\mathcal{R})$, where $-\alpha \in \mathbb{R}\backslash\mathcal{U}$ (i.e., $\alpha > 0$ for the continuous-time case and $|\alpha| < 1$ for the discrete-time case). Therefore, the system $\mathcal{S}(P, C_D)$ is stable with integral-action if and only if

$$\Theta := PN_c + \frac{s}{s+\alpha} D_c = \begin{bmatrix} P_{11}N_1 + \frac{s}{s+\alpha} D_1 & \cdots & P_{1w}N_w \\ \vdots & \ddots & \vdots \\ P_{w1}N_1 & \cdots & P_{ww}N_w + \frac{s}{s+\alpha} D_w \end{bmatrix} \tag{2}$$

is unimodular. Similarly, the system $\mathcal{S}(P, C_i)$ is stable with integral action if and only if

$$\Theta_i := P_{ii}N_i + \frac{s}{s+\alpha} D_i \tag{3}$$

is unimodular. The system $\mathcal{S}(P, C_i, C_j)$ is stable with integral-action if and only if

$$\Theta_{ij} := \begin{bmatrix} P_{ii} & P_{ij} \\ P_{ji} & P_{jj} \end{bmatrix} \mathrm{diag}\,[N_i, N_j] + \frac{s}{s+\alpha} \mathrm{diag}\,[D_i, D_j] \tag{4}$$

is unimodular. The system $\mathcal{S}(P, C_i, C_j, C_k)$ is stable with integral-action if and only if Θ_{ijk}, defined similarly by taking the i, j, kth rows and columns of Θ, is unimodular. If $\mathcal{S}(P, C_D)$ is stable with integral action, then $H_{er}(0) = I_{n_y} - PC(I_{n_y} + PC)^{-1}(0) = 0$ implies $\mathrm{rank}\, P(0) = n_y \leq n_u$. Since $N_i\hat{D}_i^{-1}$ is an RCF of C_i, if $\hat{D}_i(0) = 0$, then $\mathrm{rank}\, N_i(0) = n_{yi} \leq n_{ui}$. Since these are necessary for stability and integral action in $\mathcal{S}(P, C_D)$, we assume that $P \in \mathcal{R}^{n_y\times n_u}$ is full row-rank, has no (transmission) zeros at zero ($\mathrm{rank}\, P(0) = n_y \leq n_u$), and the number of outputs does not exceed the number of inputs in each channel ($n_{yi} \leq n_{ui}$, $i = 1, \ldots, w$). If $\mathcal{S}(P, C_i)$ is stable with integral action, $\Theta_i \in \mathcal{R}^{n_{yi}\times n_{yi}}$ unimodular implies $\mathrm{rank}\,(P_{ii}N_i)(0) = n_{yi}$. Therefore, for $\mathcal{S}(P, C_i)$ to be stable with integral action, $\mathrm{rank}\, P_{ii}(0) = n_{yi} \leq n_{ui}$.

III. Design

Reliable decentralized integral-action controller design requires that the systems $\mathcal{S}(P, C_D)$, $\mathcal{S}(P, C_i)$, $\mathcal{S}(P, C_i, C_j)$, $\mathcal{S}(P, C_i, C_j, C_k)$ are all stable with integral-action. In Lemma 1, we state the conditions for existence of w-channel reliable decentralized integral-action controllers for $w = 2, 3, 4$. In Proposition 1, we propose a reliable decentralized integral-action controller design approach. We define the following to be used in the subsequent results.

For $i = 1, \ldots, w$, let $C_i = N_i(s/(s+\alpha)\, D_i)^{-1}$ be an RCF of $C_i \in R_p^{n_{ui}\times n_{yi}}$ (N_i, $D_i \in \mathcal{M}(\mathcal{R})$, $\det D_i(\infty) \neq 0$), $-\alpha \in \mathbb{R}\backslash\mathcal{U}$. Let $P_{ii}^I(0) \in \mathbb{R}^{n_{ui}\times n_{yi}}$ denote a right-inverse of $P_{ii}(0) \in \mathbb{R}^{n_{yi}\times n_{ui}}$. For $j = 2, \ldots, w$, $i = 1, \ldots, j-1$, define $X_{ij} \in \mathcal{R}^{n_{yj}\times n_{uj}}$, $X_{ij}(0) \in \mathbb{R}^{n_{yj}\times n_{uj}}$ as (5); with k_j to be specified as in Proposition 1, define $W_{ij} \in \mathcal{R}^{n_{yj}\times n_{yj}}$ as (6)

$$X_{ij} := P_{jj} - P_{ji}N_iP_{ij}, \qquad X_{ij}(0) := (P_{jj} - P_{ji}P_{ii}^IP_{ij})(0); \tag{5}$$

$$W_{ij} := I + (X_{ij} - P_{jj})(I + s^{-1}k_jP_{jj}^I(0)X_{ij})^{-1}Q_j. \tag{6}$$

When $w \geq 3$, for $k = 1, \ldots, w-2$, $\ell, m = k+1, \ldots, w$, and $\ell \neq m$, define $Y_{\ell m}^k \in \mathcal{R}^{n_{y\ell}\times n_{um}}$, $Y_{\ell m}^k(0) \in \mathbb{R}^{n_{y\ell}\times n_{um}}$ as (7); for $v = 3, \ldots, w$, $q = 1, \ldots, v-2$, $r = q+1, \ldots, v-1$, define

$Z^q_{rv} \in \mathcal{R}^{n_{yv} \times n_{uv}}$, $Z^q_{rv}(0) \in \mathbb{R}^{n_{yv} \times n_{uv}}$ as (8) and (9), and $W^q_{rv} \in \mathcal{R}^{n_{yv} \times n_{yv}}$ as (10)

$$\begin{aligned} Y^k_{\ell m} &:= P_{\ell m} - P_{\ell k} N_k P_{km} \\ Y^k_{\ell m}(0) &:= (P_{\ell m} - P_{\ell k} P^I_{kk} P_{km})(0) \qquad (7) \\ Z^q_{rv} &:= X_{qv} - Y^q_{vr} N_r (I - P_{rq} N_q P_{qr} N_r)^{-1} Y^q_{rv} \\ &= X_{qv} - Y^q_{vr} N_r (I + (X_{qr} - P_{rr}) N_r)^{-1} Y^q_{rv} \qquad (8) \\ Z^q_{rv}(0) &:= (X_{qv} - Y^q_{vr} P^I_{rr} (X_{qr} P^I_{rr})^{-1} Y^q_{rv})(0) \qquad (9) \\ W^q_{rv} &:= I + (Z^q_{rv} - P_{vv})(I + s^{-1} k_v P^I_{vv}(0) Z^q_{rv})^{-1} Q_v. \qquad (10) \end{aligned}$$

When $w = 4$, define $G \in \mathcal{R}^{n_{y4} \times n_{u4}}$, $G(0) \in \mathbb{R}^{n_{y4} \times n_{u4}}$ as (11) and (12), $W_g \in \mathcal{R}^{n_{y4} \times n_{y4}}$ as (13):

$$\begin{aligned} G &:= Z^1_{24} - (Y^1_{43} - Y^1_{42} N_2 (I - P_{21} N_1 P_{12} N_2)^{-1} Y^1_{23}) N_3 \\ &\quad \cdot (I + (Z^1_{23} - P_{33}) N_3)^{-1} \\ &\quad \cdot (Y^1_{34} - Y^1_{32} N_2 (I - P_{21} N_1 P_{12} N_2)^{-1} Y^1_{24}) \qquad (11) \\ G(0) &:= Z^1_{24}(0) - (Y^1_{43} - Y^1_{42} P^I_{22} (X_{12} P^I_{22})^{-1} Y^1_{23}) P^I_{33} \\ &\quad \cdot (Z^1_{23} P^I_{33})^{-1} (Y^1_{34} - Y^1_{32} P^I_{22} (X_{12} P^I_{22})^{-1} Y^1_{24})(0) \qquad (12) \\ W_g &:= I + (G - P_{44})(I + s^{-1} k_4 P^I_{44}(0) G)^{-1} Q_4. \qquad (13) \end{aligned}$$

Lemma 1 (Existence Conditions for Reliable Decentralized Integral-Action Controllers): Let $P \in \mathcal{R}^{n_y \times n_u}$ be as in (1). Let $P^I_{ii}(0)$ be a right-inverse of $P_{ii}(0) \in \mathbb{R}^{n_{yi} \times n_{ui}}$, $i = 1, \ldots, w$.

a) *Necessary Conditions*: If there exist reliable decentralized integral-action controllers C_D, then the following conditions hold: *i)* $\operatorname{rank} P(0) = n_y$, $\operatorname{rank} P_{ii}(0) = n_{yi}$, $i = 1, \ldots, w$, and *ii)* $\det(X_{ij}(0) P^I_{jj}(0)) \neq 0$, for some right-inverse $P^I_{ii}(0)$ of $P_{ii}(0)$, $P^I_{jj}(0)$ of $P_{jj}(0)$, $j = 2, \ldots, w$, $i = 1, \ldots, j-1$, and *iii)* when $w \geq 3$, $\det(Z^q_{rv}(0) P^I_{vv}(0)) \neq 0$, for some right-inverse $P^I_{vv}(0)$ of $P_{vv}(0)$, $P^I_{rr}(0)$ of $P_{rr}(0)$, $v = 3, \ldots, w$, $q = 1, \ldots, v-2$, $r = q+1, \ldots, v-1$, and *iv)* when $w = 4$, $\det(G(0) P^I_{44}(0)) \neq 0$.

b) *Necessary and Sufficient Conditions*: *i)* There exist reliable decentralized integral-action controllers C_D if *1)* the necessary conditions in *a)* hold and *2)* $\det(X_{ij}(0) P^I_{jj}(0)) > 0$ for some right-inverse $P^I_{ii}(0)$ of $P_{ii}(0)$, $P^I_{jj}(0)$ of $P_{jj}(0)$, $j = 2, \ldots, w$, $i = 1, \ldots, j-1$, and *3)* when $w \geq 3$, $\det(Z^q_{rv}(0) P^I_{vv}(0)) > 0$ for some right-inverse $P^I_{vv}(0)$ of $P_{vv}(0)$, $P^I_{rr}(0)$ of $P_{rr}(0)$, $v = 3, \ldots, w$, $q = 1, \ldots, v-2$, $r = q+1, \ldots, v-1$, and *4)* when $w = 4$, $\det(G(0) P^I_{44}(0)) > 0$. *ii)* When the plant's off-diagonal sub-blocks P_{ij}, or P_{ji}, $j = 2, \ldots, w$, $i = 1, \ldots, j-1$, are strictly-proper, or when any $w-1$ of the w controllers $C_1, \ldots, C_w$ are strictly-proper, or when these transfer-functions have blocking $\mathcal{U}$-zeros, conditions *i2)–i4)* become necessary: In these cases, there exist reliable decentralized integral-action controllers C_D if and only if conditions *i1)–i4)* hold.

c) *Sufficient Conditions*: There exist reliable decentralized integral-action controllers C_D if *1)* the necessary conditions in *a)* hold, and *2)* $X_{ij}(0) P^I_{jj}(0)$ is symmetric, positive-definite for some right-inverse $P^I_{ii}(0)$, $P^I_{jj}(0)$ of $P_{ii}(0)$, $P_{jj}(0)$, $j = 2, \ldots, w$, $i = 1, \ldots, j-1$, and *3)* when $w \geq 3$, $Z^q_{rv}(0) P^I_{vv}(0)$ is symmetric, positive-definite for some right-inverse $P^I_{vv}(0)$, $P^I_{rr}(0)$ of $P_{vv}(0)$, $P_{rr}(0)$, $v = 3, \ldots, w$, $q = 1, \ldots, v-2$, $r = q+1, \ldots, v-1$, and *4)* when $w = 4$, $G(0) P^I_{44}(0)$ is symmetric, positive-definite. □

In some cases, the conditions of Lemma 1-c) and b) are equivalent [for example, when channels 2 through w each have only a single output, i.e., $n_{yj} = 1$ for $j = 2, \ldots, w$ because $X_{ij}(0) P^I_{jj}(0) \in \mathbb{R}$, or when $P_{ij}(0) = 0$ or $P_{ji}(0) = 0$, $j = 2, \ldots, w$, $i = 1, \ldots, j-1$].

Proposition 1 (Reliable Decentralized Integral-Action Controller Design): Let $P \in \mathcal{R}^{n_y \times n_u}$ be as in (1). Let $\operatorname{rank} P(0) = n_y \leq n_u$; for $i = 1, \ldots, w$, let $\operatorname{rank} P_{ii}(0) = n_{yi} \leq n_{ui}$. For $j = 2, \ldots, w$, $i = 1, \ldots, j-1$, let $X_{ij}(0) P^I_{jj}(0) \in \mathbb{R}^{n_{yj} \times n_{yj}}$ be symmetric, positive-definite for some right-inverse $P^I_{ii}(0)$ of $P_{ii}(0)$, $P^I_{jj}(0)$ of $P_{jj}(0)$. When $w \geq 3$, for $v = 3, \ldots, w$, $q = 1, \ldots, v-2$, $r = q+1, \ldots, v-1$, let $Z^q_{rv}(0) P^I_{vv}(0) \in \mathbb{R}^{n_{yv} \times n_{yv}}$ be symmetric, positive-definite for some right-inverse $P^I_{vv}(0)$ of $P_{vv}(0)$, $P^I_{rr}(0)$ of $P_{rr}(0)$. When $w = 4$, let $G(0) P^I_{44}(0) \in \mathbb{R}^{n_{y4} \times n_{y4}}$ be symmetric, positive-definite. For $i = 1, \ldots, w$, define

$$\begin{aligned} N_i := &(I + s^{-1} k_i P^I_{ii}(0) P_{ii})^{-1} \\ &\cdot (s^{-1} k_i P^I_{ii}(0) + Q_i) \in \mathcal{R}^{n_{ui} \times n_{yi}}. \qquad (14) \end{aligned}$$

Under these assumptions, $C_D = \operatorname{diag}[C_1, \ldots, C_w]$ is a reliable decentralized integral-action controller, where $C_i \in R_p^{n_{ui} \times n_{yi}}$, $i = 1, \ldots, w$, is given by

$$C_i = (I - Q_i P_{ii})^{-1} \left(\frac{k_i P^I_{ii}(0)}{s} + Q_i \right) \qquad (15)$$

where $k_i \in \mathbb{R}$ and $Q_i \in \mathcal{R}^{n_{ui} \times n_{yi}}$ are chosen as follows. Let $k_1 \in \mathbb{R}$ be such that

$$0 < k_1 < \| s^{-1} (P_{11}(s) P^I_{11}(0) - I) \|^{-1} \qquad (16)$$

and let $Q_1 \in \mathcal{R}^{n_{u1} \times n_{y1}}$ satisfy $\det(I - Q_1 P_{11})(\infty) \neq 0$. Fix Q_1. Let $k_2 \in \mathbb{R}$ be such that

$$\begin{aligned} 0 < k_2 < \min\{ &\| s^{-1} (P_{22}(s) P^I_{22}(0) - I) \|^{-1} \\ &\| s^{-1} (X_{12}(s) - X_{12}(0)) P^I_{22}(0) \|^{-1} \} \qquad (17) \end{aligned}$$

and let $Q_2 \in \mathcal{R}^{n_{u2} \times n_{y2}}$ satisfy $\det(I - Q_2 P_{22})(\infty) \neq 0$ and W_{12} is unimodular. When $w \geq 3$, fix Q_2. Let $k_3 \in \mathbb{R}$ be such that, for $i = 1, 2$,

$$\begin{aligned} 0 < k_3 < \min\{ &\| s^{-1} (P_{33}(s) P^I_{33}(0) - I) \|^{-1} \\ &\| s^{-1} (X_{i3}(s) - X_{i3}(0)) P^I_{33}(0) \|^{-1} \\ &\| s^{-1} (Z^1_{23}(s) - Z^1_{23}(0)) P^I_{33}(0) \|^{-1} \} \qquad (18) \end{aligned}$$

and let $Q_3 \in \mathcal{R}^{n_{u3} \times n_{y3}}$ satisfy $\det(I - Q_3 P_{33})(\infty) \neq 0$, W_{i3} and W^1_{23} are unimodular. When $w = 4$, fix Q_3. Let $k_4 \in \mathbb{R}$ be such that, for $i = 1, 2, 3$, $q = 1, 2$, $r = q+1, \ldots, 3$

$$\begin{aligned} 0 < k_4 < \min\{ &\| s^{-1} (P_{44}(s) P^I_{44}(0) - I) \|^{-1} \\ &\| s^{-1} (X_{i4}(s) - X_{i4}(0)) P^I_{44}(0) \|^{-1} \\ &\| s^{-1} (Z^q_{r4}(s) - Z^q_{r4}(0)) P^I_{44}(0) \|^{-1} \\ &\| s^{-1} (G(s) - G(0)) P^I_{44}(0) \|^{-1} \} \qquad (19) \end{aligned}$$

and let $Q_4 \in \mathcal{R}^{n_{u4} \times n_{y4}}$ satisfy $\det(I - Q_4 P_{44})(\infty) \neq 0$, W_{i4}, W^q_{r4}, W_g are unimodular. □

If $Q_i = 0$, then $C_i = (K_i/s)$ is a "pure integral" controller, $K_i = k_i P^I_{ii}(0)$. By Lemma 1-c), the assumptions of Proposition 1 are sufficient for existence of reliable decentralized integral-action controllers. Corollary 1 states that these conditions are *necessary and* sufficient for existence of pure integral controllers when channels 2–w each have only a single output.

Corollary 1 (Pure Integral Controller Design): Let $P \in \mathcal{R}^{n_y \times n_u}$ be as in (1). Let $n_{yi} = 1$, $i = 2, \ldots, w$. There exists a reliable decentralized integral-action controller $C_D = \operatorname{diag}[(K_1/s), \ldots, (K_w/s)]$, $K_i \in \mathbb{R}^{n_{ui} \times n_{yi}}$, $i = 1, \ldots, w$, if and only if: *1)* $\operatorname{rank} P(0) = n_y$, $\operatorname{rank} P_{ii}(0) = n_{yi}$, $i = 1, \ldots, w$, and *2)* for $j = 2, \ldots, w$, $i = 1, \ldots, j-1$, $X_{ij}(0)P_{jj}^I(0) > 0$, for some right-inverse $P_{ii}^I(0)$ of $P_{ii}(0)$, $P_{jj}^I(0)$ of $P_{jj}(0)$, and *3)* when $w \geq 3$, for $v = 3, \ldots, w$, $q = 1, \ldots, v-2$, $r = q+1, \ldots, v-1$, $Z_{rv}^q(0)P_{vv}^I(0) > 0$, for some right-inverse $P_{vv}^I(0)$ of $P_{vv}(0)$, $P_{rr}^I(0)$ of $P_{rr}(0)$, and *4)* when $w = 4$, $G(0)P_{44}^I(0) > 0$. Furthermore, $s^{-1}K_i$ can be chosen as $s^{-1}k_iP_{ii}^I(0)$, with $k_i \in \mathbb{R}$ as in (16)–(19) for $i = 1, 2, 3, 4$. □

In Proposition 1, $Q_i = 0$ satisfies the unimodularity conditions; then $C_D = \operatorname{diag}[(K_1/s), \ldots, (K_w/s)]$, $K_i = k_iP_{ii}^I(0) \in \mathbb{R}^{n_{ui}\times n_{yi}}$. If $Q_2, Q_3, Q_4 \in \mathcal{M}(\mathcal{R})$ satisfy $\|Q_2\| < \|(X_{12} - P_{22})\ (I + s^{-1}k_2P_{22}^I(0)X_{12})^{-1}\|^{-1}$, $\|Q_3\| < \min\{\|(X_{i3} - P_{33})\ (I + s^{-1}k_3P_{33}^I(0)X_{i3})^{-1}\|^{-1}, \|(Z_{23}^1 - P_{33})\ (I + s^{-1}k_3P_{33}^I(0)Z_{23}^1)^{-1}\|^{-1}\}$, $\|Q_4\| < \min\{\|(X_{i4} - P_{44})\ (I + s^{-1}k_4P_{44}^I(0)X_{i4})^{-1}\|^{-1}, \|(Z_{r4}^q - P_{44})\ (I + s^{-1}k_4P_{44}^I(0)Z_{r4}^q)^{-1}\|^{-1}, \|(G - P_{44})\ (I + s^{-1}k_4P_{44}^I(0)G)^{-1}\|^{-1}\}$, then $W_{12}, W_{i3}, W_{23}^1, W_{i4}, W_{r4}^q, W_g$ are unimodular. The condition $\det(I - Q_iP_{ii})(\infty) \neq 0$ is satisfied for all $Q_i \in \mathcal{M}(\mathcal{R})$ when $P_{ii} \in \mathcal{M}(R_s)$; $C_i \in \mathcal{M}(R_s)$ if and only if $Q_i \in \mathcal{M}(\mathcal{R}) \cap \mathcal{M}(R_s)$.

We now apply the reliable decentralized integral-action controller design procedure in Proposition 1 to a three-channel plant with SISO channels that was considered in [2].

Example 1 (3-Channel Reliable Decentralized Integral-Action Controller Design): Let

$$P = \begin{bmatrix} 1 & 0 & 2 \\ (s+1)^{-1} & 1 & -4(s+1)^{-1} \\ 0 & 4 & 1 \end{bmatrix}, w = 3,\ n_{yi} = 1,$$
$$n_{ui} = 1,\ \operatorname{rank} P(0) = 3, \operatorname{rank} P_{ii}(0) = 1,\ i = 1, 2, 3.$$

From (5), (7), (9), $X_{12}(0) = 1$, $X_{13}(0) = 1$, $X_{23}(0) = 1$, $Y_{32}^1(0) = 4$, $Y_{23}^1(0) = -2$, $Z_{23}^1(0) = 9$. The assumptions hold since $X_{ij}(0)P_{jj}^{-1}(0) = 1$, $j = 2, 3$, $i = 1, j-1$, and $Z_{23}^1(0)P_{33}^{-1}(0) = 9$. By (14), $N_i := (s+k_i)^{-1}(k_i + sQ_i)$; finding $X_{12}, X_{13}, X_{23}, Y_{32}^1, Y_{23}^1, Z_{23}^1$ from (5), (7), (8), any $k_1 > 0$, $k_2 > 0$ satisfy (16), (17); $Q_1 \in \mathcal{R}$ must satisfy $Q_1(\infty) \neq 1$, and $W_{12} = 1$ implies $Q_2 \in \mathcal{R}$ must satisfy $Q_2(\infty) \neq 1$. We choose $k_1 = 10$, $k_2 = 0.5$, $Q_1 = 0$, $Q_2 = 0$. From (18), k_3 must satisfy $0 < k_3 < 0.1136$. Choosing $k_3 = 0.1$, $Q_3 = 0$, the corresponding reliable decentralized controller $C_D = \operatorname{diag}[10/s,\ 1/2s,\ 1/10s]$ is of the pure integral form; the poles of $\mathcal{S}(P, C_D)$ are $\{-9.9953, -0.5333 \pm j0.7432, -0.5381\}$. For an alternate design, let $k_1 = 10$, $Q_1 = ((7.7s^3 - 46s^2 + 13s - 14)/(s^3 + 7s^2 + 16s + 20))$; then (17) and (18) hold for any $k_2 > 0$, $0 < k_3 < 0.1068$. Choosing $k_2 = 0.5$, $k_3 = 0.1$, $Q_2 = 0$, $Q_3 = 0$, we obtain $C_1 = (7.7s^4 - 36s^3 + 83s^2 + 146s + 200)/s(-6.7s^3 + 53s^2 + 3s + 34)$, $C_2 = 1/2s$, $C_3 = 1/10s$. The poles of $\mathcal{S}(P, C_D)$, are $\{-10.1223, -3.36881, -1.0585, -1.0159 \pm j0.4816, -1.0093 \pm j0.9767\}$. If other design specifications are given, then the design can be modified to satisfy these requirements by choosing the controller parameters Q_1, Q_2, Q_3, subject to the unimodularity constraints. □

IV. Conclusion

We presented conditions for existence of reliable decentralized integral-action controllers, and proposed explicit design approaches that achieve reliable w-channel decentralized stability with integral-action for $w = 2, 3, 4$. Although the results explored two-, three-, and four-channel decentralized systems in detail, the proposed design methods can be extended to more than four channels by imposing additional conditions (similar positive-definiteness assumptions) on the dc-gain matrices of higher-order minors of the plant.

Appendix
Proofs

Proof of Lemma 1: a) i) If $\mathcal{S}(P, C_D)$ is stable with integral-action, then Θ in (2) unimodular for any RCF $C_D = N_c((s/(s+\alpha))D_c)^{-1}$ implies $\det(P(0)N_c(0)) \neq 0$, $\operatorname{rank} P(0) = n_y$. If $\mathcal{S}(P, C_i)$ is stable with integral-action, $i = 1, \ldots, w$, (3) implies $\Theta_i = I_{n_{yi}}$ for some RCF $C_i = N_i((s/(s+\alpha))D_i)^{-1}$; therefore, $\det(P_{ii}(0)N_i(0)) \neq 0$ implies $\operatorname{rank} P_{ii}(0) = n_{yi}$, $N_i(0) = P_{ii}^I(0)$ for some right-inverse $P_{ii}^I(0)$ of $P_{ii}(0)$. *ii)* If $\mathcal{S}(P, C_i)$, $i = 1, \ldots, w$, $\mathcal{S}(P, C_i, C_j)$, $j = 2, \ldots, w$, $i = 1, \ldots, j-1$, are stable with integral-action, then Θ_{ij} in (4) unimodular for some RCF $C_i = N_i((s/(s+\alpha))D_i)^{-1}$ satisfying $\Theta_i = I$ implies $\det(I - P_{ji}N_iP_{ij}N_j)\ (0) = \det(X_{ij}(0)P_{jj}^I(0)) \neq 0$, where $N_i(0) = P_{ii}^I(0)$, $N_j(0) = P_{jj}^I(0)$. *iii)* When $w \geq 3$, if $\mathcal{S}(P, C_i)$, $\mathcal{S}(P, C_i, C_j)$, $\mathcal{S}(P, C_q, C_r, C_v)$, $v = 3, \ldots, w$, $q = 1, \ldots, v-2$, $r = q+1, \ldots, v-1$, are stable with integral-action, then Θ_{qrv} unimodular implies $\det(Z_{rv}^q(0)P_{vv}^I(0)) \neq 0$, where $N_q(0) = P_{qq}^I(0)$, $N_r(0) = P_{rr}^I(0)$, $N_v(0) = P_{vv}^I(0)$. *iv)* When $w = 4$, if $\mathcal{S}(P, C_D)$ is also stable with integral-action, then $\det(P(0)N_c(0)) \neq 0$ implies $\det(G(0)P_{44}^I(0)) \neq 0$, where $N_i(0) = P_{ii}^I(0)$, $i = 1, 2, 3, 4$. *b) i)* Let the necessary conditions in *a)* hold. Let $K_i \in \mathbb{R}^{n_{ui}\times n_{yi}}$ be such that

$$(s+\alpha)^{-1}sI + P_{ii}(s+\alpha)^{-1}\,K_i =: M_i \tag{20}$$

is unimodular. Defining $\hat{N}_i := (s+\alpha)^{-1}K_iM_i^{-1}$, all solutions of $P_{ii}N_i + (s/s+\alpha)\,D_i = I$ are

$$\begin{aligned} N_i &:= \hat{N}_i + (I - \hat{N}_iP_{ii})Q_i: \\ &= \left(I + \frac{K_i}{s}\,P_{ii}\right)^{-1}\left(\frac{K_i}{s} + Q_i\right) \end{aligned} \tag{21}$$

$$\begin{aligned} D_i &= M_i^{-1}(I - P_{ii}Q_i) \\ &= \frac{(s+\alpha)}{s}\left(I + P_{ii}\,\frac{K_i}{s}\right)^{-1}(I - P_{ii}Q_i) \end{aligned} \tag{22}$$

for some $Q_i \in \mathcal{R}^{n_{ui}\times n_{yi}}$. Therefore, $\mathcal{S}(P, C_i)$ is stable with integral-action if and only if $C_i = N_i((s/(s+\alpha))D_i)^{-1} \in Rp^{n_{ui}\times n_{yi}}$, where N_i, D_i are given by (21)–(22) and Q_i is such that $\det D_i(\infty) = \det(I - P_{ii}Q_i)(\infty) \neq 0$. The system $\mathcal{S}(P, C_i, C_j)$ is also stable with integral-action if and only if Θ_{ij} in (4) is unimodular; equivalently

$$\begin{aligned} I - P_{ji}N_iP_{ij}N_j &= I + (X_{ij} - P_{jj})N_j \\ &= I - P_{ji}N_iP_{ij}\hat{N}_j - P_{ji}N_iP_{ij}(I - \hat{N}_jP_{jj})Q_j \end{aligned} \tag{23}$$

is unimodular for some $Q_j \in \mathcal{R}^{n_{uj}\times n_{yj}}$. Since $(I - P_{ji}N_iP_{ij}\hat{N}_j, -P_{ji}N_iP_{ij}(I - \hat{N}_jP_{jj}))$ is left-coprime, there exists Q_j such that (23) is unimodular if and only if $\det(I - P_{ji}N_iP_{ij}\hat{N}_j)$ has the same sign at all blocking $\mathcal{U}$-zeros of $P_{ji}N_iP_{ij}(I - \hat{N}_jP_{jj})$. This pair is left coprime if and only if $\operatorname{rank}[I - P_{ji}N_iP_{ij}\hat{N}_j \quad -P_{ji}N_iP_{ij}(I - \hat{N}_jP_{jj})] = n_{yj}$ for all $s \in \mathcal{U}$. Equivalently, by (20), $\operatorname{rank}[(s+\alpha)^{-1}sM_j^{-1}X_{ij}] = n_{yj}$, which holds for $s \neq 0$; it also holds for $s = 0$ since $\det(X_{ij}(0)P_{jj}^I)(0) \neq 0$ implies $\operatorname{rank} X_{ij}(0) = n_{yj}$. The only blocking $\mathcal{U}$-zeros of $P_{ji}N_iP_{ij}(I - \hat{N}_jP_{jj})$ are at $s_o \in \mathcal{U}$ such that $(P_{ji}N_iP_{ij})(s_o) = 0$ and possibly at $s = 0$ [note $\det(I - \hat{N}_jP_{jj}) = \det((s+\alpha)^{-1}sM_j^{-1})$]. If $s_o \in \mathcal{U}$ is a blocking-zero of $P_{ji}N_iP_{ij}$, then $\det(I - P_{ji}N_iP_{ij}\hat{N}_j(s_o)) = 1$. Therefore, $\det(I - P_{ji}N_iP_{ij}\hat{N}_j)(0) > 0$ is sufficient; if $P_{ji}N_iP_{ij}$ has no blocking $\mathcal{U}$-zeros, then the sign at $s = 0$ does not matter. By

(20), $\hat{N}_j(0) = P_{jj}^I(0)$; hence, $\det(X_{ij}(0)P_{jj}^I(0)) > 0$ is sufficient for existence of Q_j such that (23) is unimodular. When $w \ge 3$, $\mathcal{S}(P, C_q, C_r, C_v)$ is also stable with integral-action if and only if Θ_{qrv} is unimodular. Equivalently

$$\begin{aligned} I &+ (Z_{rv}^q - P_{vv})N_v \\ &= I + (Z_{rv}^q - P_{vv})\hat{N}_v + (Z_{rv}^q - P_{vv})(I - \hat{N}_v P_{vv})Q_v \end{aligned} \tag{24}$$

is unimodular for some $Q_v \in \mathcal{R}^{n_{uv} \times n_{yv}}$. By similar steps, $\det(Z_{rv}^q(0)P_{vv}^I(0)) > 0$ is sufficient for existence of Q_v such that (24) is unimodular. When $w = 4$, $\mathcal{S}(P, C_1, C_2, C_3, C_4)$ is also stable with integral-action if and only if Θ in (2) is also unimodular. Equivalently

$$\begin{aligned} I &+ (G - P_{44})N_4 \\ &= I + (G - P_{44})\hat{N}_4 + (G - P_{44})(I - \hat{N}_4 P_{44})Q_4 \end{aligned} \tag{25}$$

is unimodular for some $Q_4 \in \mathcal{R}^{n_{u4} \times n_{y4}}$. For existence of Q_4 such that (25) is unimodular, $\det(G(0)P_{44}^I(0)) > 0$ is sufficient. Since $\det(X_{ij}(0)P_{jj}^I(0)) > 0$, $\det(Z_{rv}^q(0)P_{vv}^I(0)) > 0$, $\det(G(0)P_{44}^I(0)) > 0$ are sufficient for existence of Q_j, Q_v, Q_4 such that (23)–(25) are unimodular, the four conditions of Lemma 1-b)-*i)* are sufficient for existence of reliable decentralized integral-action controllers. *ii)* Since $C_i = N_i((s/(s+\alpha))D_i)^{-1}$ is an RCF, $C_i(s_o) = 0$ for $s_o \in \mathcal{U}$ if and only if $N_i(s_o) = 0$. If P_{ij}, or P_{ji}, $j = 2, \ldots, w$, $i = 1, \ldots, j-1$, or if any $w - 1$ of $C_1, \ldots, C_w$ have blocking $\mathcal{U}$-zeros (including infinity), then at such $s_o \in \mathcal{U}$, $\det(I - P_{ji}N_iP_{ij}N_j(s_o)) = \det(I + (X_{ij} - P_{jj})N_j(s_o)) = 1$, $\det(I + (Z_{rv}^q - P_{vv})N_v(s_o)) = 1, \det(I + (G - P_{44})N_v(s_o)) = 1$. In the presence of these blocking $\mathcal{U}$-zeros, if (23)–(25) are unimodular, then these determinants are positive at all $s \in \mathcal{U}$ including $s = 0$. Therefore, $\det(X_{ij}(0)P_{jj}^I(0)) > 0$, $\det(Z_{rv}^q(0)P_{vv}^I(0)) > 0$, $\det(G(0)P_{44}^I(0)) > 0$ are necessary. *c)* The sufficiency follows from b)-*i)* since the determinants are positive in this case. □

Proof of Proposition 1: Let $P_{ii}^I(0)$ be any right-inverse of $P_{ii}(0)$, $i = 1, \ldots, w$, such that the following are symmetric, positive-definite: $X_{ij}(0)P_{jj}^I(0)$, $j = 2, \ldots, w$, $i = 1, \ldots, j-1$; when $w \ge 3$, $Z_{rv}^q(0)P_{vv}^I(0)$, $v = 3, \ldots, w$, $q = 1, \ldots, v-2$, $r = q+1, \ldots, v-1$; when $w = 4$, $G(0)P_{44}^I(0)$. Let $K_i = k_iP_{ii}^I(0) \in \mathbb{R}^{n_{ui} \times n_{yi}}$, where k_1, k_2, k_3, k_4 satisfy (16)–(19). For $i = 1, \ldots, 4$, $0 < k_i < \|s^{-1}(P_{ii}(s)P_{ii}^I(0) - I)\|^{-1}$ implies $M_i = (s+\alpha)^{-1}sI + P_{ii}(s+\alpha)^{-1}k_iP_{ii}^I(0) \in \mathcal{R}^{n_{yi} \times n_{yi}}$ is unimodular. By (17)–(19), for $j = 2, 3, 4$, $0 < k_j < \|s^{-1}(X_{ij} - X_{ij}(0))P_{jj}^I(0)\|^{-1}$ implies $M_{ij} := (s+\alpha)^{-1}sI + X_{ij}(s+\alpha)^{-1}k_jP_{jj}^I(0) \in \mathcal{R}^{n_{yj} \times n_{yj}}$ is unimodular. When $w \ge 3$, by (18), (19), for $v = 3, 4$, $0 < k_v < \|s^{-1}(Z_{rv}^q - Z_{rv}^q(0))P_{vv}^I(0)\|^{-1}$ implies $M_{rv}^q := (s+\alpha)^{-1}sI + Z_{rv}^q\,(s+\alpha)^{-1}k_vP_{vv}^I(0) \in \mathcal{R}^{n_{yv} \times n_{yv}}$ is unimodular. When $w = 4$, $G(0)P_{44}^I(0)$ is symmetric, positive-definite; by (19), $0 < k_4 < \|s^{-1}(G - G(0))P_{44}^I(0)\|^{-1}$ implies $M_g := (s+\alpha)^{-1}sI + G\,(s+\alpha)^{-1}k_4P_{44}^I(0) \in \mathcal{R}^{n_{y4} \times n_{y4}}$ is unimodular. As in the proof of Lemma 1-b), since M_i is unimodular with $K_i = k_iP_{ii}^I(0)$, all C_i such that $\mathcal{S}(P, C_i)$ is stable with integral-action are $C_i = N_i((s/(s+\alpha))D_i)^{-1} = (I - Q_iP_{ii})^{-1}(s^{-1}K_i + Q_i)$. Defining $\hat{N}_i = (s+\alpha)^{-1}k_iP_{ii}^I(0)M_i^{-1}$, $N_i = \hat{N}_i + (I - \hat{N}_iP_{ii})Q_i$ is given by (21), equivalently (14), and $Q_i \in \mathcal{R}^{n_{ui} \times n_{yi}}$ is such that $\det(I - P_{ii}Q_i)(\infty) \ne 0$. Fix N_1, C_1 by choosing Q_1 (choosing $Q_1 = 0$ satisfies this constraint). In addition to $\mathcal{S}(P, C_i)$, $\mathcal{S}(P, C_1, C_2)$ is stable with integral-action if and only if (23) is unimodular for some $Q_2 \in \mathcal{M}(\mathcal{R})$, i.e., $I + (X_{12} - P_{22})N_2 = M_{12}M_2^{-1} + (X_{12} - P_{22})(I - \hat{N}_2P_{22})Q_2 = M_{12}M_2^{-1}(I - P_{22}Q_2 + M_2M_{12}^{-1}X_{12}Q_2)$, equivalently, W_{12} in (6) are unimodular. Fix N_2, C_2 by choosing Q_2 such that $\det(I - P_{22}Q_2)(\infty) \ne 0$ and W_{12} is unimodular ($Q_2 = 0$ satisfies these two constraints). When $w \ge 3$, $\mathcal{S}(P, C_i, C_3)$, $i = 1, 2$, is also stable with integral-action if and only if (23) is unimodular for some $Q_3 \in \mathcal{M}(\mathcal{R})$, i.e., $I + (X_{i3} - P_{33})N_3 = M_{i3}M_3^{-1}(I - P_{33}Q_3 + M_3M_{i3}^{-1}X_{i3}Q_3)$, equivalently, W_{i3} in (6) are unimodular, $i = 1, 2$. The system $\mathcal{S}(P, C_1, C_2, C_3)$ is also stable with integral-action if and only if (24) is unimodular for some $Q_3 \in \mathcal{M}(\mathcal{R})$, i.e., $I + (Z_{23}^1 - P_{33})N_3 = M_{23}^1M_3^{-1}\,(I - P_{33}Q_3 + M_3(M_{23}^1)^{-1}Z_{23}^1Q_3)$, equivalently, W_{23}^1 in (10) are unimodular. Fix N_3, C_3 by choosing Q_3 such that $\det(I - P_{33}Q_3)(\infty) \ne 0$, W_{i3}, $i = 1, 2$, W_{23}^1 are unimodular ($Q_3 = 0$ satisfies these four constraints). When $w = 4$, $\mathcal{S}(P, C_i, C_4)$, $i = 1, 2, 3$, is also stable with integral-action if and only if (23) is unimodular for some $Q_4 \in \mathcal{M}(\mathcal{R})$, i.e., $I + (X_{i4} - P_{44})N_4 = M_{i4}M_4^{-1}(I - P_{44}Q_4 + M_4M_{i4}^{-1}X_{i4}Q_4)$, equivalently, W_{i4} in (6) are unimodular, $i = 1, 2, 3$. The system $\mathcal{S}(P, C_q, C_r, C_4)$, $q = 1, 2$, $r = q+1, 3$, is also stable with integral-action if and only if (24) is unimodular for some $Q_4 \in \mathcal{M}(\mathcal{R})$, i.e., $I + (Z_{r4}^q - P_{44})N_4 = M_{r4}^qM_4^{-1}(I - P_{44}Q_4 + M_4(M_{r4}^q)^{-1}Z_{r4}^qQ_4)$, equivalently, W_{r4}^q in (10) are unimodular. Finally, $\mathcal{S}(P, C_1, C_2, C_3, C_4)$ is also stable with integral-action if and only if (25) is unimodular for some $Q_4 \in \mathcal{M}(\mathcal{R})$, i.e., $I + (G - P_{44})N_4 = M_gM_4^{-1}\,(I - P_{44}Q_4 + M_4M_g^{-1}GQ_4)$, equivalently, W_g in (13) is unimodular. In C_4, $Q_4 \in \mathcal{M}(\mathcal{R})$ is chosen such that $\det(I - P_{44}Q_4)(\infty) \ne 0$, W_{i4}, $i = 1, 2, 3$, W_{r4}^q, $q = 1, 2$, $r = q+1, 3$, W_g are unimodular ($Q_4 = 0$ satisfies these eight constraints). With C_i as in (15), C_D is a reliable decentralized integral-action controller. □

Proof of Corollary 1: By Lemma 1-b)-*ii)*, when $C_i = (K_i/s) \in \mathcal{M}(R_s)$, there exist reliable decentralized integral-action controllers if and only if the conditions of Lemma 1-b)-*i)* hold, which are equivalent to those of Lemma 1-c) since $n_{yi} = 1$, $i = 2, \ldots, w$. Since the assumptions of Proposition 1 hold, C_i can be chosen as $C_i = s^{-1}k_iP_{ii}^I(0)$ with $Q_i = 0$. □

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Jump Linear Quadratic Regulator with Controlled Jump Rates

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***Abstract*—This note deals with the class of continuous-time linear systems with Markovian jumps. We assume that jump rates are controlled. Our purpose is to study the jump linear quadratic (JLQ) regulator of the class of systems. The structure of the optimal controller is established. For a one-dimensional (1-D) system, an algorithm for solving the corresponding set of coupled Riccati equations of this optimal control problem is provided. Two numerical examples are given to show the usefulness of our results.**

***Index Terms*—Controlled jump Markov process, dynamic programming, jump linear quadratic regulator, jump linear system.**

I. Introduction

Since the introduction of the framework of the class of jump linear system (JLS) by Krasovskii and Lidskii [6], we have seen an increasing interest for this class of systems. It was used to model different physical systems, like manufacturing systems, power systems, economics systems, etc. For more information regarding the use of this class of systems, we refer the reader to Mariton [7], Sethi and Zhang [8], and the references therein.

Roughly speaking, a JLS is a hybrid one with state vector that has two components $x(t)$ and $r(t)$. The first one is in general referred to as the state and the second one is referred to as the mode. In its operation, the JLS will jump from one mode to another in a random way, which makes this class of systems a stochastic one. The switching between the modes is governed by a Markov process with discrete and finite state space. When the system mode is fixed, the system evolves like a deterministic linear system. This kind of system can be used to describe abrupt phenomena, such as component and interconnection failures. Because of its extensive application, JLS has attracted a lot of researchers, and therefore, vast literature in this field has appeared (see Mariton [7], Costa and Boukas [9], and references therein).

The optimal control problem of JLS has been studied by many authors (see Mariton [7], Costa and Boukas [9], and the references therein). All of the authors focused on the feedback optimal regulator of JLS and thus solved it successfully when the jump rates of the Markov process are not controlled. Boukas and Haurie have proposed some models in manufacturing systems in which the jumps rates are control variables. Boukas and Haurie [1] model the continuous flow control problem in a manufacturing system by a multimode system in which the state of the system is modeled by a Markov process. The production rate and the jump rates (function of the age of the machines) are determined by minimizing a given criterion. Instead of using the age of the machines as in [1], Boukas [3] divides the age of the machine to many regions, between which the state of the machine switches in a logic sense and the production rate and the jump rate are optimized at the same time.

The goal of this paper is to study the jump linear quadratic (JLQ) regulator of the class of JLS with controlled jump rates. We deal with a one-dimensional (1-D) problem and establish the optimal control law for this optimization problem.

The paper is organized as follows. In Section II, the optimization problem is formulated. In Section III, the main results of this paper are given and two numerical examples are provided to show the usefulness of the proposed results.

II. Problem Statement

Consider the class of linear systems with Markovian jumps, and let $x_i \in \mathbb{R}^n$, $t \in [0, \infty)$ be the dynamic state of our system with multiple modes taking values in a set denoted by $S = \{1, 2, \ldots, s\}$. Let the dynamics of our system be described by the following differential equation:

$$\hat{x}_t = A(r(t))x_t + B(r(t))u_t, \quad x_0 \text{ given} \tag{1}$$

where u_t is the control variable at time t, $\{r(t), t \geq 0\}$, $r(t) \in \mathcal{S}$, is a Markov process giving the mode of the system at time t, $A(r(t)) \in \mathbb{R}^{n \times n}$, $B(r(t)) \in \mathbb{R}^{n \times m}$, $\forall\, r(t) \in \mathcal{S}$.

Assume that the Markov process $r(t)$ has a generator $\mathcal{Q}(w(t)) = (q_{ij}(w_j^i(t)))$, $i, j \in \mathcal{S}$, where $q_{ij}(\cdot)$ are nonnegative and increasing functions and $w_j^i(t)$, $j \in \mathcal{S}$, where $j \neq i$ are control variables for each i. The transition probabilities of the modes are described as follows:

$$\begin{aligned} &P(r(t+\Delta) = j | r(t) = i) \\ &\quad = \begin{cases} q_{ij}(w_j^i(t))\Delta + o(\Delta), & \text{if } j \neq i \\ 1 + q_i(w^i(t))\Delta + o(\Delta), & \text{otherwise} \end{cases} \end{aligned}$$

where $w^i(t) = (w_1^i(t), \ldots, w_{i-1}^i(t), w_{i+1}^i(t), \ldots, w_s^i(t))$, $w_j^e(t) \in W$ (a given set) are control variables.

The optimization problem is to seek a control law, $(u(t), w(t))$, that minimizes the following cost function:

$$\begin{aligned} &J(x_0, r_0, u(\cdot), w(\cdot)) \\ &\quad = \mathrm{E}\left[\int_0^\infty \left[x_t^\top M(r(t))x_t + u_t^\top N(r(t))u_t\right]\right. \\ &\qquad\qquad \left. \cdot dt \,|x(0) = x_0, r(0) = r_0 \vphantom{\int_0^\infty}\right] \end{aligned} \tag{2}$$

where $M(r(t))$ are positive semidefinite and $N(r(t))$ are positive definite for any $r(t) \in \mathcal{S}$.

In the rest of this paper, we assume all of the required classical assumptions for the existence of the solution of the optimization problem we are dealing with (see Wonham [4]). We will also, for notation simplicity, use A_i to represent $A(i)$ when the mode $r(t)$ is equal to i.

III. Main Results

The above optimal control problem falls into the framework of stochastic optimal control. Two ways to solve this problem exist, dynamic

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