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Multi-input Multi-output PI Controller Design *

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Abstract– The design of MIMO PI controller is formulated as an LQR problem. The weighting matrices of the quadratic performance index are chosen so that tuning can be done for each input-output channel and for tradeoff between transient response and robustness with respect to modeling error. The number of tuning parameters is the same as that of a decentralized PI controller. A design example is given to demonstrate the feasibility of the proposed approach.

1 Introduction

The PI (proportional plus integral) controller is probably the most commonly used controller in the industry. that Arguably the PI controller is the simplest practical controller that provides integral action which is required in many process control applications for asymptotic tracking of set-point commands and rejection of constant load disturbances [3],[11],[8]. There is much research on the design (or tuning) of PI controllers for SISO systems but very little is done on MIMO design. Proportional plus integral state feedback design, in the LQR framework, is discussed in [1] and [2], however the state estimator included for output feedback implementation gives away the simplicity of PI control. So far almost all the MIMO PI controllers proposed have a decentralized structure, although some design include static precompensation to achieve diagonal dominance at steady-state [13], [4], [5]. In general decentralized structure limits performance although, being simpler, it may have some advantage in real-time implementations, e. g., fewer tuning parameters and easier to make the design fault-tolerant [10], [7].

In this paper, the design of MIMO PI controller is formulated as an LQR problem. The weighting matrices of the quadratic performance index are chosen so that tuning can be done for each input-output channel and for tradeoff between transient response and robustness with respect to modeling error.There are two tuning parameters for each input-output channel. For low order plants, the number of inputs equals the number of states, the PI controller implements exactly the optimal state feedback. For high order plants, the design involves approximations: either model reduction of the plant or approximation of the feedback gain matrix or both. The error in the approximation is taken into account in robustness consideration and tuning can be done accordingly. A design example is given to demonstrate the feasibility of the proposed approach.

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The paper is organized as follows. Section 2 describes the design problem and shows that, under the assumptions, a stabilizing PI controller exits . Section 3 discusses the proposed design approaches A design example is given in Section 4 and concluding remarks is given in Section 5.

2 Problem Formulation

Consider the linear time-invariant multi-input multi-output plant

$$\dot{x} = Ax + Bu \tag{2.1}$$

$$y = Cx \tag{2.2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$. The plant has m inputs and m outputs and the input-output transfer matrix is $P(s) = C(sI - A)^{-1}B$. We make the following assumptions throughout
(A1) The plant is controllable, observable and (asymptotically) stable, and
(A2) The plant has no transmission zero at $s = 0$, that is, $P(0) = -CA^{-1}B$ is nonsingular.
It follows from (A2) that A is nonsingular and both B and C are full rank. We note also that (A.2) is equivalent to that $\begin{bmatrix} -A & B \\ C & 0 \end{bmatrix}$ is nonsingular.
The problem studied in this paper is the following. Given the MIMO plant (2.1) and (2.2) and a fixed PI controller structure, how do we design the PI gain matrices so that the closed-loop system is stable and achieves some performance requirements? The block diagram of the feedback system is shown in Figure 1, where K_p and K_i are respectively the proportional gain matrix and the integral gain matrix. We note that in decentralized PI controllers these gain matrices are constrained to be diagonal; in our proposed design they are in general full matrices.

Refer to Figure 1, the equation of the PI controller are

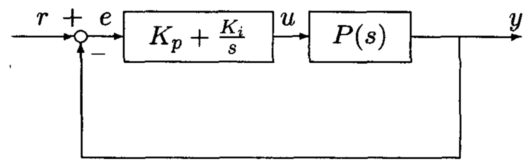

Figure 1: Closed-loop system with PI controller

$$e = r - y \tag{2.3}$$

$$\dot{v} = e \tag{2.4}$$

$$u = K_p e + K_i v \tag{2.5}$$

where r is the command input, e is the tracking error, $K_p \in \mathbb{R}^{m\times m}$ and $K_i \in \mathbb{R}^{m\times m}$. For design purpose we will assume that the command input is a vector of step functions, i.e., $r(t) = \bar{r}1(t)$ and $\bar{r} \in \mathbb{R}^m$.

If the closed-loop system is stable, then K_i is nonsingular and the system reaches constant steady-state as $t \to \infty$. In steady-state, $y(\infty) = \bar{r}$, $e(\infty) = 0$, $u(\infty) = -(CA^{-1}B)^{-1}\bar{r}$, $v(\infty) = -K_i^{-1}(CA^{-1}B)^{-1}\bar{r}$ and $x(\infty) = A^{-1}B(CA^{-1}B)^{-1}\bar{r}$. Define the deviation variables $\tilde{x} = x - x(\infty)$, $\tilde{v} = v - v(\infty)$, $\tilde{u} = u - u(\infty)$ and $\tilde{y} = y - \bar{r}$ and rewrite the state equations of the closed-loop system as

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{v}} \end{bmatrix} = A_o \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix} + B_o \tilde{u} \tag{2.6}$$

$$\tilde{y} = [C \;\; 0] \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix} \tag{2.7}$$

$$\tilde{u} = -K_p C\tilde{x} + K_i \tilde{v} \tag{2.8}$$

where

$$A_o = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \qquad B_o = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{2.9}$$

We note that $e = -\tilde{y}$. We can think of the design problem as one of constrained state feedback design or if we take $\tilde{v}$ as part of the output (in addition to $\tilde{y}$), as one of output feedback design. The design goal here is to obtain good dynamic response of the tracking error e while maintaining a certain degree of robustness. Before discussing the design approaches, we show first that a stabilizing design exists, that is, there is always a nontrivial ($K_i \neq 0$) PI controller which stabilizes the closed-loop system.

Proposition 1 *Under the assumptions (A.1) and (A.2), there is a PI controller with K_i nonsingular so that the closed-loop system shown in Figure 1 is stable.*

Remarks: (a) Assumption (A2) is also necessary since otherwise there would be unstable pole-zero cancellations at $s = 0$. (b) The result is also true for rectangular plants with more inputs than outputs.

Proof: Let $K \in \mathbb{R}^{m\times m}$ be positive definite. Then, for any K_i nonsingular,

$$K_p+\frac{K_i}{s} = [(K_p s+K_i)(sI+K)^{-1}][s(sI+K)^{-1}]^{-1}$$

is a right coprime factorization of the controller. Since $P(s)$ is stable, the closed-loop system is stable if and only if $M(s) := P(s)(K_p s + K_i)(sI + K)^{-1} + s(sI + K)^{-1}$ is unimodular. Choose $K_i = P(0)^{-1}K$, which is nonsingular, and $K_p = \tilde{K}K$ for some $\tilde{K} \in \mathbb{R}^{m\times m}$. Then

$$M(s) = I+(\frac{P(s)(\tilde{K}s + P(0)^{-1}) - I}{s})sK(sI+K)^{-1}$$

Since $\|s(sI + K)^{-1}\|_\infty \leq 1$, $M(s)$ is unimodular if $\sigma_{\max}(K) < \|\frac{P(s)(\tilde{K}s+P(0)^{-1})-I}{s}\|_\infty^{-1}$. Hence there is a stabilizing PI controller with K_i nonsingular. □

3 Design Approach

We discuss the determination of the gain matrices K_p and K_i by LQR design . Consider the system defined in (2.6). Let $G = \mathrm{diag}[\alpha_1^2, \cdots, \alpha_m^2] > 0$ and let the quadratic performance be defined as

$$J = \int_0^\infty ([\tilde{x}(t)^T \ \tilde{v}(t)^T]Q \begin{bmatrix} \tilde{x}(t) \\ \tilde{v}(t) \end{bmatrix} + \tilde{u}(t)^T P(0)^T RP(0)\tilde{u}(t))dt$$

where

$$Q = \begin{bmatrix} C^TGC & 0 \\ 0 & I \end{bmatrix} \text{ and } R = \mathrm{diag}[\beta_1^2, \ldots, \beta_m^2] > 0$$

Since $e = -\tilde{y} = -C\tilde{x}$, the first term of the performance index is simply the weighted sum of square tracking error and sum of square integrated error. The choice of the second term requires some explanation. Define

$$\hat{u}(t) = P(0)\tilde{u}(t)$$

then the second term inside the integral becomes $\hat{u}(t)^T R\hat{u}(t)$. If we think of $\tilde{u}$ as the input to the plant $P(s)$ with output $\tilde{y}$, then $\hat{u}$ is the input to the 'normalized plant' $P(0)^{-1}P(s)$ to produce the same output $\tilde{y}$. Since $P(0)^{-1}P(s)$ is diagonal at $s = 0$, it is nearly decoupled at low frequencies. Hence, roughly speaking, weighting a component of $\hat{u}$ has the effect of weighting the control input required for the performance of the corresponding component of output $\tilde{y}$. The performance index can be written as

$$J = \int_0^\infty \Sigma_{i=1}^m (\alpha_i^2|e_i(t)|^2 + |\tilde{v}_i(t)|^2 + \beta_i^2|\hat{u}_i(t)|^2)dt$$

where the subscripts i indicate the ith component of the respective vector. The parameters α_i and β_i are to be selected for the trade-off between the tracking error response and the control effort required in each channel. If the response of every channel is of the same importance then G and R can be chosen as a multiple of identity matrix (to start with.) Roughly, increasing R and decreasing G improves robustness at the expense of deteriorating dynamic response. The LQR solution gives a state feedback control law

$$\tilde{u}(t) = -(K_1\tilde{x}(t) + K_2\tilde{v}(t)) \qquad (3.10)$$

where $K_1 \in \mathbb{R}^{m\times n}$ and $K_2 \in \mathbb{R}^{m\times m}$. The following result shows that the LQR control law (3.10) always gives a stable closed-loop system.

Proposition 2 *$\{A_o,\ B_o\}$ is controllable and $\{Q^{1/2},\ A_o\}$ is observable.*

Proof: We will show that $\text{rank}[\lambda I - A_o\ \ B_o] = m + n$ for all $\lambda \in \mathbb{C}$ From (2.9),

$$[\lambda I - A_o\ \ B_o] = \begin{bmatrix} \lambda I - A & 0 & B \\ C & \lambda I & 0 \end{bmatrix}$$

It follows from (A.2) that the rank condition holds for $\lambda = 0$. Since $\{A, B\}$ is controllable, the rank condition holds for all $\lambda \neq 0$. The proof of observability is similar. □

Comparing (2.8) and (3.10), if we assign $K_i := -K_2$ and if the proportional gain matrix is such that

$$K_p C = K_1 \tag{3.11}$$

then the PI control law and the state feedback law are identical. The equation (3.11) has a unique solution if $m = n$, the number of outputs equals the number of states. In this case, $K_p = K_1 C^{-1}$. We note that there are many processes which can be adequately represented by low order models staisfying the above condition[10], including models for rapid thermal processing systems [12], [9].

If the plant has more states than outputs, that is, $m < n$, then the equation (3.11) have no solution in general. So the LQR control law can not be implemented as a PI controller. One way to approach this problem is the following. Perform a balanced model reduction on the plant to obtain a reduced plant model with the number of states equals the number of outputs, and then determine the gain matrices K_p and K_i by the LQR design with reduced model. The PI controller designed will guarantee stability and performance of the reduced system. How good the design is for the original system depends on the model reduction error. If the plant can be approximated well by the low order model, then this seems a good approach. Suppose $\Delta(s)$ is the additive model reduction error transfer matrix, then a condition for robust stability is

$$\sigma_{\max}(H_{ur}(j\omega)) < \frac{1}{\sigma_{\max}(\Delta(j\omega))} \quad \text{for all } \omega \tag{3.12}$$

If condition (3.12) is satisfied with some margin, then the design can be expected to perform well for the original model. In general increase R will decrease $\|H_{ur}\|_\infty$. Of course we have to make sure that the reduced model (which is stable) has a nonsingular dc-gain. A sufficient condition for the balanced reduced model to have a nonsingular dc-gain is given as follows. Suppose $\sigma_1 \geq \cdots \geq \sigma_m > \sigma_{m+1} \geq \cdots \geq \sigma_n$ are the Hankel singular values of the plant and $\bar{\sigma}_1 \geq \cdots \geq \bar{\sigma}_m$ are singular values of $P(0)$. If $\bar{\sigma}_m > \Sigma_{j=m+1}^{n} \sigma_j$, then the reduced model obtained using balanced realization by keep m states has a nonsingular dc-gain.

Another way to determine the gain matrices is to set $K_i = -K_2$ and K_p as the least square solution of (3.11), that is,

$$K_p = K_1 C^T (CC^T)^{-1}$$

where $K = [K_1\ \ K_2]$ is the gain matrix obtained by solving the original LQR problem. Error is now introduced to the supposedly optimal state feedback controller. Performance of this approximated design depends on the error $K_1(I - C^T(CC^T)^{-1}C)$, small error ensures good performance. A combination of the two approaches above is to do model reduction keeping enough states ($> m$) to ensure small reduction error and to determine K_p by least square approximation.

4 A Design Example

We illustrate the proposed design approach by the following example.

Example[10, p. 441] Consider the 2-input 2-output stable transfer matrix describing a high-purity distillation column near certain operating point,

$$P(s) = \begin{bmatrix} \frac{87.8}{1+194s} & -\frac{87.8}{1+194s} + \frac{1.4}{1+15s} \\ \frac{108.2}{1+194s} & -\frac{108.2}{1+194s} - \frac{1.4}{1+15s} \end{bmatrix}$$

The design specifications are:(a) Each channel should have a step response that settles to within 10% of the desired final value within 40 minutes. (b) The design should allow for a worst-case time delay of one minute on the control action and for $\pm 20\%$ uncertainty in the actuator gains.

We note that the plant is ill-conditioned with the condition number 140 at $s = 0$. Since the uncertainty and unmodeled dynamics occur at the input, plant input relative uncertainty model is used. We will take the worst plant model as

$$P_a(s) = P(s)(I + \Delta_a)\exp(-s) = P(s)(I + L_i(s))$$

where $\Delta_a = \mathrm{diag}[0.2 \;\; 0.2]$ and $L_i(s) = \exp(-s)(\Delta_a + I) - I$. The design is to remain stable for the worst plant and to satisfy the time response specification. Let $C(s) = K_p + \frac{K_i}{s}$. A sufficient condition for roust stability is

$$\sigma_{\max}(CP(I+CP)^{-1}(j\omega)) < \frac{1}{\sigma_{\max}(L_i(j\omega))} \quad \text{for all } \omega \tag{4.13}$$

This condition is checked as we tune the design parameters. A minimal realization is

$$A = \begin{bmatrix} -0.0052 & 0 \\ 0 & -0.0667 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.4526 & 0.0933 \\ 0.5577 & -0.0933 \end{bmatrix}$$

Since the channels are treated as of equal importance, G and R are chosen scalar multiples of identity matrix. Initially they are chosen equal. The final design is

$$R = \begin{bmatrix} 37.2 & 0 \\ 0 & 39.4 \end{bmatrix} \quad G = \begin{bmatrix} 1463 & 0 \\ 0 & 1640 \end{bmatrix}$$

$$K_p = \begin{bmatrix} 2.105 & -2.089 \\ 2.052 & -2.133 \end{bmatrix} \quad K_i = \begin{bmatrix} 0.060 & -0.057 \\ 0.059 & -0.057 \end{bmatrix}$$

Figure 2 shows the robustness condition (4.13) is satisfied. The one minute time delay practically limits the bandwidth of the closed-loop system to about 1 rad/min. Step responses of the closed-loop system is shown in Figure 3 and Figure 4. The design satisfies the time response requirement. Note that the response is better for the case where the step commands are of opposite signs.

5 Conclusions

We have described a MIMO PI controller design method based on LQR formulation. The choice of tuning parameters allow tuning of individual input-output channel and tradeoff between dynamic response and robustness. The number of tuning parameters is exactly the same as that of a decentralized PI controller. Although only set-point command are considered in the design, the same formulation is also applicable to design for load disturbance rejection.

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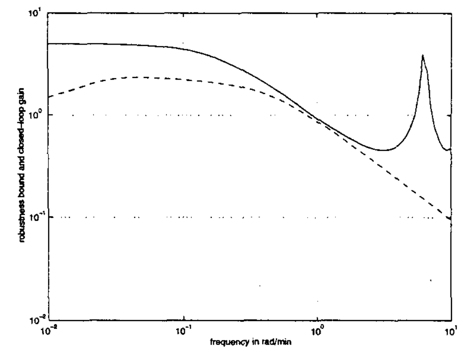

Figure 2: Design for robustness

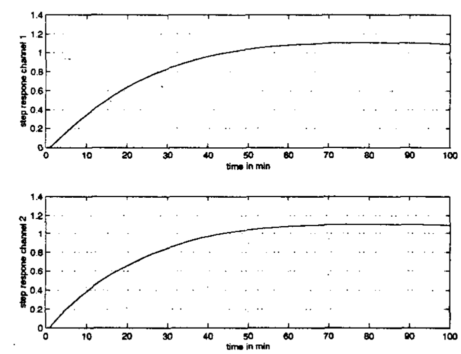

Figure 3: Step responses with positive step in each channel

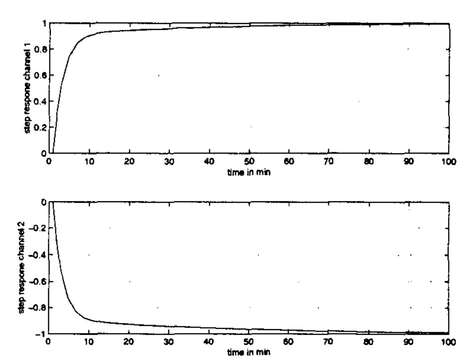

Figure 4: Step responses with positive and negative step commands