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Simultaneously stabilizing controllers for a class of linear plants ☆

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Abstract

It is shown that a class of linear, time-invariant, multi-input–multi-output plants that all have poles at zero but do not have other unstable poles can be simultaneously stabilized. A procedure is proposed to design a stable and strictly proper simultaneously stabilizing controller. All simultaneously stabilizing controllers for this class are also characterized in terms of a parameter matrix that has to satisfy a unimodularity condition. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Simultaneous stabilization; Controller design

1. Introduction

Controller design for simultaneous stabilization of a set of linear time-invariant (LTI), multi-input–multi-output (MIMO) plants is a challenging problem. The well-known parametrization of all stabilizing controllers in the standard unity-feedback system leads to explicit necessary and sufficient conditions for existence of controllers that simultaneously stabilize two given plants [5]. These remarkably simple conditions require that a pseudo-plant associated with the two given plants has the parity-interlacing property (PIP), i.e., it has an even number of poles between consecutive pairs of real-axis zeros in the region of instability. However, there are no known necessary and sufficient conditions for existence of simultaneously stabilizing controllers for a class of three or more arbitrary plants. Although it is obviously necessary for all pairs of plants in the given class to satisfy the PIP, conditions restricted to checking the real-axis pole-zero locations are not sufficient to guarantee that a single controller can stabilize all of the plants simultaneously [1,3]. However, in the absence of necessary and sufficient conditions applicable to a completely general class of (three or more) plants, it may be possible to conclusively answer the question of simultaneous stabilizability for some special classes [2,4,6,7].

As a special case, we consider the class $\mathscr{P} = \{P_o, P_1, \ldots, P_n\}$ of $n+1$ LTI MIMO plants that have no other poles in the region of instability except at $s=0$; furthermore, for $j = 0, \ldots, n$, $(s^m P_j)$ have full-rank DC-gain matrices that are symmetric positive-definite multiples of $(s^m P_o)(0)$ (see Assumptions 2.1 for a formal description of the class $\mathscr{P}$). For the case of single-input–single-output (SISO) systems, this class can

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be described more explicitly: for $j=1,\dots,n$, the plants P_j all have exactly m poles at $s=0$ but no other unstable poles; furthermore, since all $(s^m P_j)$ have the same sign at zero, the lowest order coefficients of the numerator and denominator polynomials of $(s^m P_j)$ are of the same sign although the signs of all other coefficients can be arbitrary. These plants are simultaneously stabilizable; in fact, there exist stable and strictly proper simultaneously stabilizing controllers.

A particularly interesting special class of SISO plants was considered in [6] (and extended to discrete-time systems in [7]), where the SISO plants are all minimum-phase, strictly proper have the same high-frequency gain sign. It was shown that these plants are simultaneously stabilizable and that the simultaneously stabilizing controller can be stable and strictly proper. The simultaneous stabilization procedure proposed in the present paper applies to a rather different class of MIMO plants that are not necessarily minimum-phase or strictly proper; their relative degrees (in the special SISO case) need not all be the same; however, their unstable poles are at zero only. Although the class considered here includes finitely many MIMO plants as "centers" and proposes simultaneously stabilizing controller design that guarantees stabilization of these centers, "small" perturbations around these centers are also stabilized using the same controller as in standard robustness results.

The main result of this paper (Proposition 2.2) is a simple design procedure based on calculating m positive real constants $k_1,\dots,k_m$ that define a stable simultaneously stabilizing controller, whose poles can be arbitrarily pre-assigned. All simultaneously stabilizing controllers can be obtained from this central controller in terms of a stable controller parameter that satisfies a restrictive unimodularity condition. Following the main result, two simple examples are included to illustrate the design proposed in Proposition 2.2; the first example is for a class of three SISO plants each with two poles at $s=0$, and the second example is for a class of ten 2×2 MIMO plants. The proof of Proposition 2.2 is provided in the appendix.

Due to the algebraic framework described in the following notation, the results apply to continuous-time as well as discrete-time systems; for the case of discrete-time systems, all evaluations and poles at $s=0$ would be interpreted at $z=1$.

Notation. Let $\mathscr{U}$ be the extended closed right half-plane (for continuous-time systems) or the complement of the open unit disk (for discrete-time systems). The sets of real numbers, rational functions (with real coefficients), proper and strictly proper rational functions, proper rational functions that have no poles in the region of instability $\mathscr{U}$ are denoted by $\mathbb{R}, R, R_{\mathrm{p}}, R_{\mathrm{s}}, \mathscr{R}$, respectively. The set of matrices whose entries are in $\mathscr{R}$ is denoted by $\mathscr{M}(\mathscr{R})$; M is called stable iff $M\in\mathscr{M}(\mathscr{R})$ (a notation of the form $M\in\mathscr{R}^{i\times j}$ is used where it is important to indicate the order of a matrix explicitly); a stable M is called $\mathscr{R}$-unimodular iff $M^{-1}\in\mathscr{M}(\mathscr{R})$. For $M\in\mathscr{M}(\mathscr{R})$, the norm $\|\cdot\|$ is defined as $\|M\|=\sup_{s\in\partial\mathscr{U}}\bar{\sigma}(M(s))$, where $\bar{\sigma}$ denotes the maximum singular value and $\partial\mathscr{U}$ denotes the boundary of $\mathscr{U}$.

2. Main results

Consider the standard LTI, MIMO, unity-feedback system $\mathscr{S}(P_j,C)$ (see Fig. 1); $\mathscr{S}(P_j,C)$ is a well-posed system, where $P_j\in R_{\mathrm{p}}^{n_y\times n_u}$ and $C\in R_{\mathrm{p}}^{n_u\times n_y}$ represent the transfer functions of the plant and the controller. It is assumed that P_j and C have no hidden modes corresponding to eigenvalues in the region of instability $\mathscr{U}$.

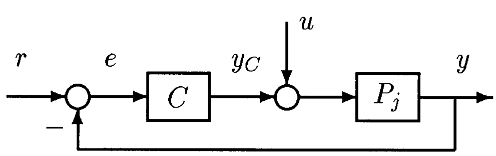

Fig. 1. The system $\mathscr{S}(P_j,C)$.

2.1. Assumption (*Assumptions on P_j*). The plant $P_j \in R_p^{n_y \times n_u}$ belongs to the class $\mathscr{P}:=\{P_o, P_1, \ldots, P_n\}$; for $j \in \{0, 1, \ldots, n\}$, each $P_j \in \mathscr{P}$ satisfies the following assumptions:

(i) $s^m P_j$ has no poles in $\mathscr{U}$;

(ii) $\operatorname{rank}(s^m P_j)(0) = \operatorname{rank}(s^m P_o)(0) = \min\{n_y, n_u\}$;

(iii) if $n_y \leqslant n_u$, then $(s^m P_j)(0) = \Theta_j\, (s^m P_o)(0)$, for some symmetric, positive-definite $\Theta_j \in \mathbb{R}^{n_y \times n_y}$; if $n_y > n_u$, then $(s^m P_j)(0) = (s^m P_o)(0) \Psi_j$, for some symmetric, positive-definite $\Psi_j \in \mathbb{R}^{n_u \times n_u}$.

Assumption 2.1(i) implies that the only $\mathscr{U}$-poles of P_j are at $s = 0$; Assumption 2.1(ii) implies that each P_j in the class $\mathscr{P}$ has at least one entry that has exactly m poles at $s = 0$; furthermore, P_j has no (transmission) zeros at $s = 0$.

In the special case of SISO plants, it is possible to relate Assumptions 2.1 to the assumptions in [6] using a transformation as follows: [1] Let $P_j(s) \in R_{\mathrm{p}}$ belong to the class $\mathscr{P}:=\{P_o, P_1, \ldots, P_n\}$. For $j = \{1, \ldots, n\}$, define $\hat{P}_j \in R_{\mathrm{s}}$ as $\hat{P}_j := (P_j(1/s))^{-1}$; then the plants $\hat{P}_j$ in the class $\hat{\mathscr{P}}:=\{\hat{P}_o, \hat{P}_1, \ldots, \hat{P}_n\}$ are all strictly proper and minimum phase (i.e., have no finite zeros in the region of instability) and have the same high-frequency gain sign. Therefore, the finite class $\hat{\mathscr{P}}$ of SISO plants satisfies the assumptions in [6] whenever the class $\mathscr{P}$ of SISO plants satisfies Assumptions 2.1.

We use the following standard stability definitions: The system $\mathscr{S}(P_j, C)$ is said to be stable iff the transfer-function H from (r, u) to (y, y_C) is stable, i.e., $H \in \mathscr{M}(\mathscr{R})$. The controller C is said to be a stabilizing controller for the plant P_j (or C stabilizes P_j) iff $C \in \mathscr{M}(R_{\mathrm{p}})$ and the system $\mathscr{S}(P_j, C)$ is stable. The stabilizing controller C is said to simultaneously stabilize all $P_j \in \mathscr{P}$ iff the system $\mathscr{S}(P_j, C)$ is stable for all $j \in \{0, \ldots, n\}$.

In Proposition 2.2, we propose a design procedure for controllers that simultaneously stabilize all $P_j \in \mathscr{P}$. In addition to finding one such controller explicitly, all controllers can also be characterized based on one of the plants, P_o, which we call the nominal plant. The choice of the nominal plant in the class $\mathscr{P}$ is completely arbitrary.

2.2. Proposition (Controllers stabilizing $P_j \in \mathscr{P}$). *Let $P_j \in R_{\mathrm{p}}^{n_y \times n_u}$ belong to the class $\mathscr{P}:=\{P_o, P_1, \ldots, P_n\}$ satisfying Assumptions* 2.1. *For $i = 1, \ldots, m$, let $-\alpha_i \in \mathbb{R} \setminus \mathscr{U}$. For $j \in \{0, 1, \ldots, n\}$, define*

$$N_j := \frac{s^m}{\prod_{i=1}^m (s + \alpha_i)} P_j \in \mathscr{R}^{n_y \times n_u}. \tag{1}$$

(a) *If* $\operatorname{rank}(s^m P_o)(0) = n_y \leqslant n_u$, *let $N_o(0)^I$ be any right-inverse of $N_o(0)$. Let $k_1 \in \mathbb{R}$ be such that*

$$0 < k_1 < \min_{j \in \{0, \ldots, n\}} \left\| s^{-1}(\Theta_j I - N_j N_o(0)^I) \right\|^{-1}. \tag{2}$$

For $v = 2, \ldots, m$, let $k_v \in \mathbb{R}$ be such that

$$0 < k_v < \min_{j \in \{0, \ldots, n\}} \left\| s^{-1} \left(I + N_j N_o(0)^I \sum_{i=1}^{v-1} \frac{1}{s^i} \prod_{\ell=1}^{i} k_\ell \right)^{-1} \left(I + N_j N_o(0)^I \sum_{i=1}^{v-2} \frac{1}{s^i} \prod_{\ell=1}^{i} k_\ell \right) \right\|^{-1}. \tag{3}$$

All $P_j \in \mathscr{P}$ can be simultaneously stabilized by the stable controller $C_o \in \mathscr{R}^{n_u \times n_y}$ given by

$$C_o = \frac{N_o(0)^I}{\prod_{i=1}^m (s + \alpha_i)} \sum_{i=1}^{m} s^{m-i} \prod_{\ell=1}^{i} k_\ell. \tag{4}$$

Furthermore, all controllers C that simultaneously stabilize all $P_j \in \mathscr{P}$ are given by

$$C = (I - Q N_o)^{-1} \left(\frac{s^m}{\prod_{i=1}^m (s + \alpha_i)} Q + C_o \right), \tag{5}$$

[1] This transformation relating Assumptions 2.1 to the assumptions in [6] was pointed out by an anonymous reviewer.

where $Q \in \mathscr{R}^{n_u \times n_y}$ *is such that*

$$D_j := I + Q(N_j - N_o)\left(I + N_o(0)^I N_j \sum_{i=1}^{m} \frac{1}{s^i} \prod_{\ell=1}^{i} k_\ell\right)^{-1} \tag{6}$$

is $\mathscr{R}$*-unimodular*, $j \in \{1, \ldots, n\}$, *and* $(I - QN_o)$ *is biproper* (*which holds for all* $Q \in \mathscr{M}(\mathscr{R})$ *when* $P_o \in \mathscr{M}(R_s)$).

(b) *If* $\operatorname{rank}(s^m P_o)(0) = n_u < n_y$, *let* $N_o(0)^{\mathrm{L}}$ *be any left-inverse of* $N_o(0)$. *Let* $\hat{k}_1 \in \mathbb{R}$ *be such that*

$$0 < \hat{k}_1 < \min_{j \in \{0,\ldots,n\}} \| s^{-1}(\Psi_j I - N_o(0)^{\mathrm{L}} N_j) \|^{-1}. \tag{7}$$

For $v = 2, \ldots, m$, *let* $\hat{k}_v \in \mathbb{R}$ *be such that*

$$0 < \hat{k}_v < \min_{j \in \{0,\ldots,n\}} \left\| s^{-1}\left(I + N_o(0)^{\mathrm{L}} N_j \sum_{i=1}^{v-1} \frac{1}{s^i} \prod_{\ell=1}^{i} \hat{k}_\ell\right)^{-1}\left(I + N_o(0)^{\mathrm{L}} N_j \sum_{i=1}^{v-2} \frac{1}{s^i} \prod_{\ell=1}^{i} \hat{k}_\ell\right)\right\|^{-1}. \tag{8}$$

All $P_j \in \mathscr{P}$ *can be simultaneously stabilized by the stable controller* $\hat{C}_o \in \mathscr{R}^{n_u \times n_y}$ *given by*

$$\hat{C}_o = \frac{N_o(0)^{\mathrm{L}}}{\prod_{i=1}^{m}(s + \alpha_i)} \sum_{i=1}^{m} s^{m-i} \prod_{\ell=1}^{i} \hat{k}_\ell. \tag{9}$$

Furthermore, all controllers C *that simultaneously stabilize all* $P_j \in \mathscr{P}$ *are given by*

$$C = \left(\frac{s^m}{\prod_{i=1}^{m}(s + \alpha_i)}\hat{Q} + \hat{C}_o\right)(I - N_o\hat{Q})^{-1}, \tag{10}$$

where $\hat{Q} \in \mathscr{R}^{n_u \times n_y}$ *is such that*

$$\hat{D}_j := I + \hat{Q}\left(I + N_j N_o(0)^{\mathrm{L}} \sum_{i=1}^{m} \frac{1}{s^i} \prod_{\ell=1}^{i} \hat{k}_\ell\right)^{-1}(N_j - N_o) \tag{11}$$

is $\mathscr{R}$*-unimodular*, $j \in \{1, \ldots, n\}$, *and* $(I - N_o\hat{Q})$ *is biproper* (*which holds for all* $\hat{Q} \in \mathscr{M}(\mathscr{R})$ *when* $P_o \in \mathscr{M}(R_s)$).

2.3. Comments. (a) The simultaneously stabilizing controller in (5) for the plant class $\mathscr{P}$ is based on the simple calculation of m scalar constants $k_1, \ldots, k_m$ as defined in (2) and (3). For the case of $n_y \leqslant n_u$, the block diagram of the system $\mathscr{S}(P_j, C)$, where C is given by (5), is shown in Fig. 2; a similar block diagram can be obtained for the case of $n_u < n_y$ from (10).

(b) The simultaneously stabilizing controller C_o proposed in (4) is stable and strictly proper. The m real poles of the controller C_o are pre-assigned to any desired negative real-axis locations with the choice of the real constants (positive for continuous-time case) $\alpha_1, \ldots, \alpha_m$.

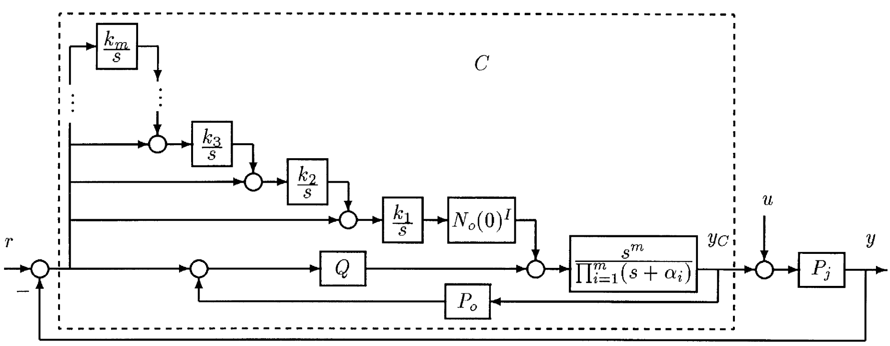

Fig. 2. The stable system $\mathscr{S}(P_j, C)$, where $P_j \in \mathscr{P}, n_y \leqslant n_u$, and Q satisfies (6).

(c) The stable controller-parameter Q in the simultaneously stabilizing controller characterization (5) must satisfy the unimodularity condition (6). This condition is obviously satisfied for $Q=0$, corresponding to the central controller C_o given in (4). Some additional possible choices for the stable controller-parameter Q satisfying (6), other than $Q=0$, can be obtained as follows: For convenience, define $K \in R^{n_u \times n_y}$ as

$$K := N_o(0)^I \sum_{i=1}^{m} \frac{1}{s^i} \prod_{\ell=1}^{i} k_\ell. \tag{12}$$

Since the constants $k_1,\ldots,k_m$ satisfy (2) and (3), the matrix $M_j \in \mathscr{R}^{n_y \times n_y}$ defined by

$$M_j := \frac{s^m}{\prod_{i=1}^{m}(s+\alpha_i)}(I+N_jK) \tag{13}$$

is $\mathscr{R}$-unimodular, for all $j \in \{0,\ldots,n\}$. Therefore, $(I+KN_j)^{-1} = I - C_o M_j^{-1} N_j$ is stable. A sufficient condition to satisfy (6) is to choose $Q \in \mathscr{M}(\mathscr{R})$ 'sufficiently small', i.e.,

$$\| Q \| < \min_{j \in \{1,\ldots,n\}} \left\| (N_j - N_o)(I+KN_j)^{-1} \right\|^{-1}. \tag{14}$$

In addition, choosing $Q \in \mathscr{M}(\mathscr{R})$ strictly proper is a sufficient condition for $(I-QN_o)$ to be biproper; note that the simultaneously stabilizing controller is strictly proper if and only if $Q \in \mathscr{M}(\mathscr{R})$ is strictly proper. Similar comments apply to the choice of $\hat{Q}$ satisfying (11) for the case of $n_u < n_y$.

(d) Using the characterization (5) of all simultaneously stabilizing controllers for the plant class $\mathscr{P}$, we obtain the following achievable closed-loop transfer functions for the stabilized system $\mathscr{S}(P_j, C)$: For $j \in \{0,\ldots,n\}$, the (input–output) transfer-function $H_{yj} = P_jC(I+P_jC)^{-1}$ from r to y and the (input-error) transfer-function $H_{ej} = I - H_{yj}$ from r to e are achievable using the controllers in (5) if and only if $H_{yj} = N_j(I + KN_j + Q(N_j - N_o))^{-1}(Q+K) = N_j(I+KN_j)^{-1}D_j^{-1}(Q+K)$, $H_{ej} = I - H_{yj} = (I + N_j(I-QN_o)^{-1}(Q+K))^{-1}$, where $Q \in \mathscr{R}^{n_u \times n_y}$ is such that the unimodularity condition (6) on $D_j \in \mathscr{R}^{n_u \times n_u}$ holds and $(I-QN_o)$ is biproper. The expressions for these transfer functions are simplified for the nominal plant P_o as $H_{yo} = (I+N_oK)^{-1}N_o(Q+K)$, $H_{eo} = (I+N_oK)^{-1}(I-N_oQ)$. Note that the stable system $\mathscr{S}(P_j, C)$ has integral action, i.e., $H_{ej}(0)=0$, due to the poles of P_j at zero. This guarantees asymptotic tracking of step inputs applied at each channel of r; in fact, polynomial inputs of order up to $m-1$ are tracked asymptotically with zero steady-state error due to the m plant poles at zero. Additional design goals may be achievable by appropriately selecting the controller-parameter Q in (5).

In Examples 1 and 2 below, we design simultaneously stabilizing controllers based on Proposition 2.2. Note that any member of the class $\mathscr{P}$ can be chosen as the nominal plant P_o.

Example 1. Consider the class $\mathscr{P}$ of SISO plants defined as

$$\mathscr{P} := \left\{ P_o = \frac{-120}{s^2},\ P_1 = \frac{(5s-72)(s+6)(s+10)}{s^2(s+4)(s+12)},\ P_2 = \frac{(2s^2-49s-140)(s+6)(s+10)}{s^2(7s^2+36s+75)} \right\}.$$

The plants $P_o, P_1, P_2 \in \mathscr{P}$ satisfy Assumptions 2.1(i)–(iii), where $m=2$, $(s^2P_o)(0) = -120$, $(s^2P_1)(0) = -90$, $(s^2P_2)(0) = -112$. In this case, Θ_j are positive real constants since $n_y = n_u = 1$; i.e., $\Theta_1 = \frac{3}{4} > 0$, $\Theta_2 = \frac{14}{15} > 0$. Choosing $\alpha_1 = 6$, $\alpha_2 = 10$, $N_o, N_1, N_2 \in \mathscr{R}$ defined in (1) are given by

$$N_o = \frac{-120}{(s+6)(s+10)}, \qquad N_1 = \frac{5s-72}{(s+4)(s+12)}, \qquad N_2 = \frac{2s^2-49s-140}{(7s^2+36s+75)}.$$

With $N_o(0) = -2$, we choose $k_1, k_2 \in \mathbb{R}$ satisfying (2) and (3) as

$$k_1 = 3 < \min_{j \in \{0,1,2\}} \left\| s^{-1}\left(\Theta_j + \frac{1}{2}N_j\right) \right\|^{-1}, \qquad k_2 = 1 < \min_{j \in \{0,1,2\}} \left\| s^{-1}\left(1 - \frac{k_1}{2s}N_j\right)^{-1} \right\|^{-1}.$$

By (4), the stable, strictly proper simultaneously stabilizing controller C_o is

$$C_o = \frac{-3(s+1)}{2(s+6)(s+10)}$$

and by (5), all simultaneously stabilizing controllers are given by

$$C = \left(1 + Q\frac{120}{(s+6)(s+10)}\right)^{-1}\left(\frac{s^2}{(s+6)(s+10)}Q + C_o\right),$$

where $Q \in \mathscr{R}$ is such that D_j in (6) is $\mathscr{R}$- unimodular; note that $(1 - QN_o)$ is biproper for all $Q \in \mathscr{R}$ since P_o is strictly proper in this example. □

Example 2. Consider the class $\mathscr{P} = \{P_o, P_1, P_2, P_3 = P_o\varDelta, P_4 = P_1\varDelta, P_5 = P_2\varDelta, P_6 = P_o + P_1, P_7 = P_o - P_2, P_8 = P_2 - P_1, P_9\}$ of ten MIMO plants, where

$$P_o = \begin{bmatrix} \frac{-5}{s(s+3)} & \frac{s-1}{s+8} \\ \frac{s-1}{s(s+4)} & \frac{2(s-1)}{s(s+1)} \end{bmatrix}, \quad P_1 = \begin{bmatrix} \frac{(s-2)(s+10)}{16s(s+1)(s+12)} & \frac{9}{16} \\ \frac{(3s^2+7s-3)(s+10)}{48s(s+5)(s+8)} & \frac{(s^3+5s-4)(s+5)}{8s(s+1)^2(s+20)} \end{bmatrix},$$

$$P_2 = \begin{bmatrix} \frac{5s-7}{s(s+1)(s+6)} & \frac{s^2}{(s+2)^2} \\ \frac{7(s-1)}{4s(s+10)} & \frac{9s^5-7}{s(s+5)(s+1)^4} \end{bmatrix}, \quad \varDelta = \begin{bmatrix} \frac{1}{(s+2)(s+5)} & 0 \\ \frac{5s}{s+9} & \frac{9-s}{10(s+9)} \end{bmatrix}, \quad P_9 = \begin{bmatrix} \frac{-(s+10)}{s} & 0 \\ \frac{-3(s+10)}{20s} & \frac{-6(s+10)}{5s} \end{bmatrix}.$$

The plants $P_j \in \mathscr{P}$ satisfy Assumptions 2.1(i)–(iii), where $m=1$. In this case, the symmetric, positive-definite matrices Θ_j are all diagonal; i.e., $(sP_j)(0) = \Theta_j(sP_o)(0)$ with $\Theta_j = \theta_j I_2$ for $\theta_1 = \frac{1}{16}$, $\theta_2 = 0.7$, $\theta_3 = 0.1$, $\theta_4 = \frac{1}{160}$, $\theta_5 = 0.07$, $\theta_6 = \frac{17}{16}$, $\theta_7 = 0.3$, $\theta_8 = \frac{51}{80}$, $\theta_9 = 6$. Choosing $\alpha_1 = 10$, $N_o(0) \in \mathbb{R}^{2\times 2}$ and its inverse $N_o(0)^{-1}$ are given by

$$N_o(0) = \begin{bmatrix} -1/6 & 0 \\ -1/40 & -1/5 \end{bmatrix}, \quad N_o(0)^{-1} = \begin{bmatrix} -6 & 0 \\ 3/4 & -5 \end{bmatrix}.$$

We choose $k_1 > 0$ satisfying (2) as

$$k_1 = 0.32 < \min_{j \in \{0,\ldots,9\}} \left\| s^{-1}(\theta_j I - N_j N_o(0)^{-1}) \right\|^{-1}.$$

By (4), the stable, strictly proper simultaneously stabilizing controller C_o is

$$C_o = \frac{k_1 N_o(0)^{-1}}{s + \alpha_1} = \frac{1}{s+10}\begin{bmatrix} -1.92 & 0 \\ 0.24 & -1.60 \end{bmatrix}$$

and all simultaneously stabilizing controllers are given by (5).

Appendix A

Proof of Proposition 2.2. (a) Let $n_y \leqslant n_u$. For $j \in \{0,\ldots,n\}$, $N_j(0) = \Theta_j N_o(0)$ implies $N_j(0)N_o(0)^I = \Theta_j I$ and hence, $s^{-1}(\Theta_j I - N_j N_o(0)^I) \in \mathscr{M}(\mathscr{R})$. Choose any positive constant $k_1 \in \mathbb{R}$ satisfying (2). Define X_{1j} and M_{1j} as

$$X_{1j} := (sI + k_1\Theta_j)^{-1}sI + (sI + k_1\Theta_j)^{-1}k_1 N_j N_o(0)^I = I - (sI + k_1\Theta_j)^{-1}k_1 s\left(\frac{\Theta_j I - N_j N_o(0)^I}{s}\right),$$

$$M_{1j} := \frac{(sI + k_1\Theta_j)}{(s+\alpha_1)}X_{1j} = \frac{s}{s+\alpha_1}I + N_j N_o(0)^I\frac{k_1}{s+\alpha_1}. \tag{A.1}$$

Since k_1 satisfies (2) and Θ_j is symmetric, positive definite, for $j \in \{0,\ldots,n\}$, $X_{1j} \in \mathscr{R}^{n_y\times n_y}$ is $\mathscr{R}$-unimodular, equivalently, $M_{1j} \in \mathscr{R}^{n_y\times n_y}$ is $\mathscr{R}$-unimodular. If $m > 1$, let $k_2 \in \mathbb{R}$ be a positive constant satisfying (3) for $v=2$.

For $j\in\{0,\ldots,n\}$, $((s+\alpha_1)^{-1}M_{1j}^{-1}N_jN_o(0)^Ik_1)(0)=I$ implies $s^{-1}(I-(s+\alpha_1)^{-1}M_{1j}^{-1}N_jN_o(0)^Ik_1)\in\mathscr{M}(\mathscr{R})$. Define X_{2j} and M_{2j} as

$$
\begin{aligned}
X_{2j} &:= \frac{s}{s+k_2}I+\frac{M_{1j}^{-1}N_jN_o(0)^Ik_1k_2}{(s+\alpha_1)(s+k_2)} = I-\frac{k_2 s}{(s+k_2)}\left[\frac{1}{s}\left(I-\frac{M_{1j}^{-1}N_jN_o(0)^Ik_1}{s+\alpha_1}\right)\right]\\
&= I-\frac{k_2 s M_{1j}^{-1}}{(s+k_2)(s+\alpha_1)} = I-\frac{k_2 s}{(s+k_2)s}\left(I+N_jN_o(0)^I\frac{k_1}{s}\right)^{-1},\\
M_{2j} &:= \frac{(s+k_2)}{(s+\alpha_2)}X_{2j} = \frac{s}{s+\alpha_2}I+\frac{M_{1j}^{-1}N_jN_o(0)^Ik_1k_2}{(s+\alpha_1)(s+\alpha_2)}.
\end{aligned}
\tag{A.2}
$$

Since by (3), k_2 satisfies $0<k_2<\min_{j\in\{0,\ldots,n\}}\left\|s^{-1}(I+N_jN_o(0)^Is^{-1}k_1)^{-1}\right\|^{-1}$, for $j\in\{0,\ldots,n\}$, $X_{2j}\in\mathscr{R}^{n_y\times n_y}$ is $\mathscr{R}$-unimodular, equivalently, M_{2j} is $\mathscr{R}$-unimodular.

Continue similarly with k_v satisfying (3), for $v=3,\ldots,m$. For $j\in\{0,\ldots,n\}$, $(\prod_{i=1}^{m-1}(s+\alpha_i)^{-1}(\prod_{i=1}^{m-1}M_{ij})^{-1}N_jN_o(0)^I\prod_{i=1}^{m-1}k_i)(0)=I$. Since

$$
\begin{aligned}
&\frac{1}{s}\left[I-\left(\prod_{i=1}^{m-1}M_{ij}\right)^{-1}N_jN_o(0)^I\prod_{i=1}^{m-1}\frac{k_i}{(s+\alpha_i)}\right] = \frac{M_{(m-1)j}^{-1}}{s+\alpha_{(m-1)}}\\
&= s^{-1}\left(I+N_jN_o(0)^I\sum_{i=1}^{m-1}\frac{1}{s^i}\prod_{\ell=1}^{i}k_\ell\right)^{-1}\left(I+N_jN_o(0)^I\sum_{i=1}^{m-2}\frac{1}{s^i}\prod_{\ell=1}^{i}k_\ell\right)
\end{aligned}
$$

and since k_m satisfies (3) for $v=m$, it can be shown as in the case of $v=2$ above that $M_{mj}\in\mathscr{R}^{n_y\times n_y}$ is $\mathscr{R}$-unimodular, where

$$
M_{mj}:=\frac{s}{s+\alpha_m}I+\left(\prod_{i=1}^{m-1}M_{ij}\right)^{-1}N_jN_o(0)^I\prod_{i=1}^{m}\frac{k_i}{(s+\alpha_i)}. \tag{A.3}
$$

By (A.1) and (A.2), since M_{1j} and M_{2j} are $\mathscr{R}$-unimodular, $(M_{1j}M_{2j})$ is $\mathscr{R}$-unimodular, where

$$
M_{1j}M_{2j}=\frac{s^2}{\prod_{i=1}^{2}(s+\alpha_i)}I+\frac{s^2}{\prod_{i=1}^{2}(s+\alpha_i)}N_jN_o(0)^I\sum_{i=1}^{2}\frac{1}{s^i}\prod_{\ell=1}^{i}k_\ell.
$$

Similarly, $M_j:=\prod_{i=1}^{m}M_{ij}$ is $\mathscr{R}$-unimodular for all $j\in\{0,\ldots,n\}$, where

$$
M_j=\frac{s^m}{\prod_{i=1}^{m}(s+\alpha_i)}I+\frac{s^m}{\prod_{i=1}^{m}(s+\alpha_i)}N_jN_o(0)^I\sum_{i=1}^{m}\frac{1}{s^i}\prod_{\ell=1}^{i}k_\ell =: \frac{s^m}{\prod_{i=1}^{m}(s+\alpha_i)}(I+N_jK). \tag{A.4}
$$

With K and C_o related as

$$
\frac{s^m}{\prod_{i=1}^{m}(s+\alpha_i)}K=\frac{s^m}{\prod_{i=1}^{m}(s+\alpha_i)}N_o(0)^I\sum_{i=1}^{m}\frac{1}{s^i}\prod_{\ell=1}^{i}k_\ell=\frac{N_o(0)^I}{\prod_{i=1}^{m}(s+\alpha_i)}\sum_{i=1}^{m}s^{m-i}\prod_{\ell=1}^{i}k_\ell=C_o,
$$

we write $M_o:=\prod_{i=1}^{m}M_{io}=\frac{s^m}{\prod_{i=1}^{m}(s+\alpha_i)}I+N_oC_o$ from (A.4) and obtain

$$
\begin{bmatrix} I & C_o \\ -N_o & \dfrac{s^m}{\prod_{i=1}^{m}(s+\alpha_i)}I \end{bmatrix}
\begin{bmatrix} I-C_oM_o^{-1}N_o & -C_oM_o^{-1} \\ M_o^{-1}N_o & M_o^{-1} \end{bmatrix} = I_{n_u+n_y}. \tag{A.5}
$$

By (A.5), it follows using standard arguments (see for example [5]) that the controller C stabilizes

$$
P_o=\left(\frac{s^m}{\prod_{i=1}^{m}(s+\alpha_i)}I\right)^{-1}N_o
$$

if and only if

$$C=(I-QN_o)^{-1}\frac{s^m}{\prod_{i=1}^m(s+\alpha_i)}(Q+K)=\left(C_oM_o^{-1}+(I-C_oM_o^{-1}N_o)Q\right)(M_o^{-1}(I-N_oQ))^{-1},$$

as claimed in (5). This controller stabilizes all

$$P_j=\left(\frac{s^m}{\prod_{i=1}^m(s+\alpha_i)}I\right)^{-1}N_j\in\mathscr{P}$$

in addition to P_o if and only if $Q\in\mathscr{M}(\mathscr{R})$ is such that

$$\begin{aligned}&\frac{s^m}{\prod_{i=1}^m(s+\alpha_i)}M_o^{-1}(I-N_oQ)+N_j(C_oM_o^{-1}+(I-C_oM_o^{-1}N_o)Q)\\&\quad=M_jM_o^{-1}(I-N_oQ+M_oM_j^{-1}N_jQ)\\&\quad=M_jM_o^{-1}[I+(N_j-N_o)(I+KN_j)^{-1}Q]\end{aligned}\tag{A.6}$$

is $\mathscr{R}$-unimodular, equivalently, $[I+(N_j-N_o)(I+KN_j)^{-1}Q]$ $\mathscr{R}$-unimodular (since $M_jM_o^{-1}$ is $\mathscr{R}$-unimodular) as claimed in (6). Note that (A.6) is unimodular for $Q=0$ and hence, C_o given in (4) simultaneously stabilizes all $P_j\in\mathscr{P}$. One last technicality is to choose the controller-parameter $Q\in\mathscr{M}(\mathscr{R})$ so that the stabilizing controllers are proper; the controller C in (5) is proper if and only if $(I-N_oQ)$ is biproper, equivalently $(I-QN_o)$ is biproper.

(b) The proof for the case of $n_u<n_y$ is entirely similar, so we briefly outline the important steps. For $j\in\{0,\dots,n\}, N_j(0)=N_o(0)\Psi_j$ implies $N_o(0)^{\mathrm{L}}N_j(0)=\Psi_j$. If $\hat{k}_1\in\mathbb{R}$ satisfies (7), then $\hat{M}_{1j}\in\mathscr{R}^{n_u\times n_u}$ is $\mathscr{R}$-unimodular for $j\in\{0,\dots,n\}$, where

$$\hat{M}_{1j}:=\frac{s}{s+\alpha_1}I+\frac{\hat{k}_1}{s+\alpha_1}N_o(0)^{\mathrm{L}}N_j.\tag{A.7}$$

Since $\hat{k}_m$ satisfies (8), $\hat{M}_{mj}\in\mathscr{R}^{n_u\times n_u}$ is $\mathscr{R}$-unimodular for $j\in\{0,\dots,n\}$, where

$$\hat{M}_{mj}:=\frac{s}{s+\alpha_m}I+\left(\prod_{i=1}^{m-1}\hat{M}_{ij}\right)^{-1}\prod_{i=1}^{m}\frac{\hat{k}_i}{(s+\alpha_i)}N_o(0)^{\mathrm{L}}N_j.\tag{A.8}$$

Continuing as in part (a) above, $\hat{M}_j:=\prod_{i=1}^m\hat{M}_{ij}$ is $\mathscr{R}$-unimodular, where

$$\hat{M}_j=\frac{s^m}{\prod_{i=1}^m(s+\alpha_i)}I+\frac{s^m}{\prod_{i=1}^m(s+\alpha_i)}\left(\sum_{i=1}^m\frac{1}{s^i}\prod_{\ell=1}^i\hat{k}_\ell\right)N_o(0)^{\mathrm{L}}N_j=:\frac{s^m}{\prod_{i=1}^m(s+\alpha_i)}(I+\hat{K}N_j).\tag{A.9}$$

With $\hat{K}$ and $\hat{C}_o$ related as

$$\frac{s^m}{\prod_{i=1}^m(s+\alpha_i)}\hat{K}=\frac{s^m}{\prod_{i=1}^m(s+\alpha_i)}\left(\sum_{i=1}^m\frac{1}{s^i}\prod_{\ell=1}^i\hat{k}_\ell\right)N_o(0)^{\mathrm{L}}=\frac{N_o(0)^{\mathrm{L}}}{\prod_{i=1}^m(s+\alpha_i)}\sum_{i=1}^m s^{m-i}\prod_{\ell=1}^i\hat{k}_\ell=\hat{C}_o,$$

we write $\hat{M}_o:=\prod_{i=1}^m\hat{M}_{i0}=s^m/[\prod_{i=1}^m(s+\alpha_i)]I+\hat{C}_oN_j$ from (A.9) and obtain

$$\begin{bmatrix}\hat{M}_o^{-1}&\hat{M}_o^{-1}\hat{C}_o\\-N_o\hat{M}_o^{-1}&I-N_o\hat{M}_o^{-1}\hat{C}_o\end{bmatrix}\begin{bmatrix}\frac{s^m}{\prod_{i=1}^m(s+\alpha_i)}I&-\hat{C}_o\\N_o&I\end{bmatrix}=I_{n_u+n_y}.\tag{A.10}$$

By (A.10), the controller C stabilizes

$$P_o=N_o\left(\frac{s^m}{\prod_{i=1}^m(s+\alpha_i)}I\right)^{-1}$$

if and only if

$$C=\frac{s^m}{\prod_{i=1}^m(s+\alpha_i)}(\hat{Q}+\hat{K})(I-N_o\hat{Q})^{-1}=((I-\hat{Q}N_o)\hat{M}_o^{-1})^{-1}(\hat{M}_o^{-1}\hat{C}_o+\hat{Q}(I-N_o\hat{M}_o^{-1}\hat{C}_o)),$$

as claimed in (10). This controller stabilizes all $P_j \in \mathscr{P}$ in addition to P_o if and only if

$$(I - \hat{Q}N_o)\hat{M}_o^{-1} \frac{s^m}{\prod_{i=1}^m (s+\alpha_i)} + (\hat{M}_o^{-1}\hat{C}_o + \hat{Q}(I - N_o\hat{M}_o^{-1}\hat{C}_o))N_j$$
$$= (I - \hat{Q}N_o + \hat{Q}N_j\hat{M}_j^{-1}\hat{M}_o)\hat{M}_o^{-1}\hat{M}_j = [I + \hat{Q}(N_j - N_o)(I + N_j\hat{K})^{-1}]\hat{M}_o^{-1}\hat{M}_j \tag{A.11}$$

is $\mathscr{R}$-unimodular, equivalently, $\hat{D}_j$ in (11) is $\mathscr{R}$-unimodular. Since (A.11) is $\mathscr{R}$-unimodular for $\hat{Q}=0$, $\hat{C}_o$ given in (9) simultaneously stabilizes all $P_j \in \mathscr{P}$. Again, the controller C in (10) is proper if and only if $(I - \hat{Q}N_o)$ is biproper. □

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